16. Higher order linear ODEs
(Notes on Diffy Qs, 2.3)

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The textbook: `https://www.jirka.org/diffyqs/`

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Theorem (Superposition)

If $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0,$$

then the linear combination

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

is also a solution for arbitrary constants $C_1, C_2, \ldots, C_n$.

Theorem (Existence and uniqueness)

Suppose p_0 through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. Then the equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

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We solve these to find $C_1 = -1/4$, $C_2 = 1$, and $C_3 = 1/4$.

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$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}.$$

Initial conditions say

$$1 = y(0) = C_1 + C_2 + C_3, \quad 2 = y'(0) = -C_1 + C_2 + 3C_3, \quad 3 = y''(0) = C_1 + C_2 + 9C_3.$$

We solve these to find $C_1 = -1/4$, $C_2 = 1$, and $C_3 = 1/4$.

So the particular solution to our problem is:

$$y = \frac{-1}{4} e^{-x} + e^x + \frac{1}{4} e^{3x}.$$

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So $3 = (-1)(1)(-r_3)$ or $r_3 = 3$.

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The general solution is

$$y = \underbrace{(C_1 + C_2x + C_3x^2)\,e^x}_{\text{terms coming from } r=1} + \underbrace{\quad C_4 \quad}_{\text{from } r=0}.$$

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If we have such a pair each repeated k times, the corresponding solution is

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$$y = (C_1 + C_2 x)\, e^x \cos x + (C_3 + C_4 x)\, e^x \sin x.$$