5. Slope fields (Notes on Diffy Qs, 1.2)

Jiří Lebl

Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

Consider the general first order equation $y' = f(x, y)$.

Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,

Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y).

Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

E.g., $y' = xy$.

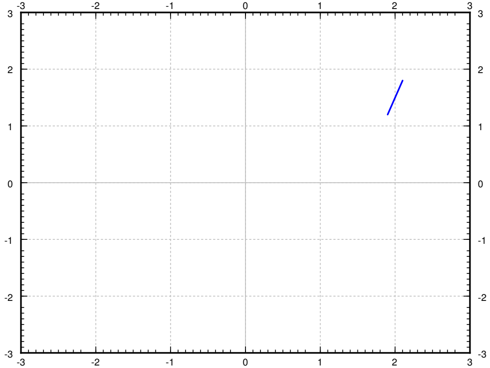
Consider the general first order equation $\quad y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

E.g., $y' = xy$.
At $(2, 1.5)$, draw a line of slope $xy = 2 \times 1.5 = 3$



Consider the general first order equation $y' = f(x, y)$.

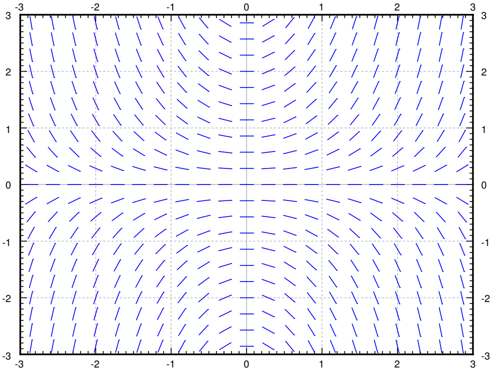
We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

E.g., $y' = xy$.
At $(2, 1.5)$, draw a line of slope $xy = 2 \times 1.5 = 3$

Now do it for a grid of points.



Consider the general first order equation $\quad y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

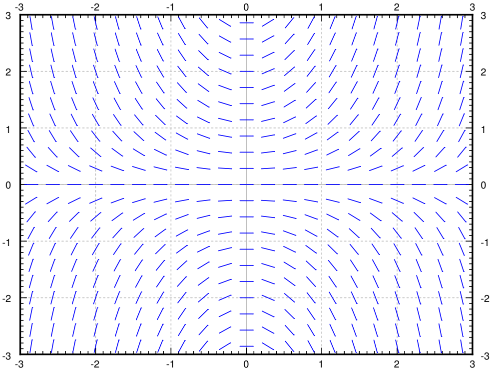
At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

E.g., $y' = xy$.
At $(2, 1.5)$, draw a line of slope $xy = 2 \times 1.5 = 3$

Now do it for a grid of points.

This is called the *slope field*.



Consider the general first order equation $\quad y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

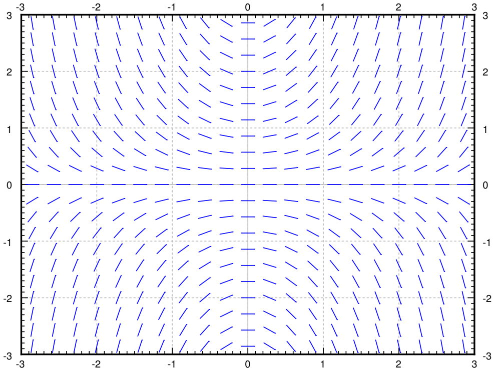
So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

E.g., $y' = xy$.
At $(2, 1.5)$, draw a line of slope $xy = 2 \times 1.5 = 3$

Now do it for a grid of points.

This is called the *slope field*.

Given a specific initial condition,
$y(x_0) = y_0$, we have a solution.



Consider the general first order equation $\quad y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

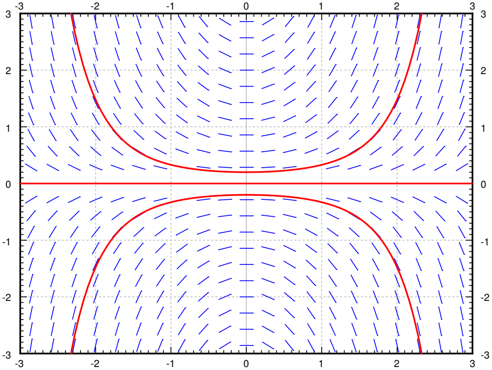
E.g., $y' = xy$.
At $(2, 1.5)$, draw a line of slope $xy = 2 \times 1.5 = 3$

Now do it for a grid of points.

This is called the *slope field*.

Given a specific initial condition,
$y(x_0) = y_0$, we have a solution.

E.g., $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.



Consider the general first order equation $y' = f(x, y)$.

We can't always solve such an equation explicitly,
but we can always figure out the shape and behavior of the solutions.

At each (x, y) in the plane, the solution would have a slope $y' = f(x, y)$.

So why not draw a little line with the slope $f(x, y)$ at the point (x, y). And do it for all (x, y).

E.g., $y' = xy$.
At $(2, 1.5)$, draw a line of slope $xy = 2 \times 1.5 = 3$

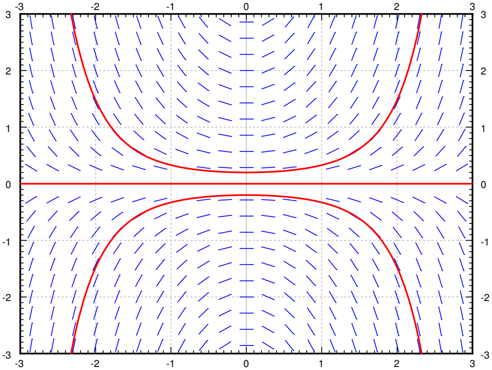
Now do it for a grid of points.

This is called the *slope field*.

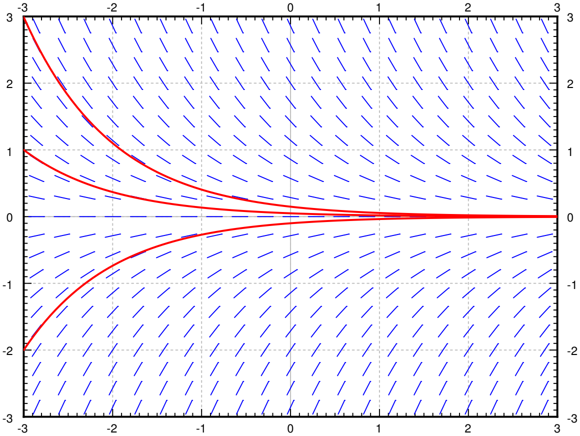
Given a specific initial condition,
$y(x_0) = y_0$, we have a solution.

E.g., $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

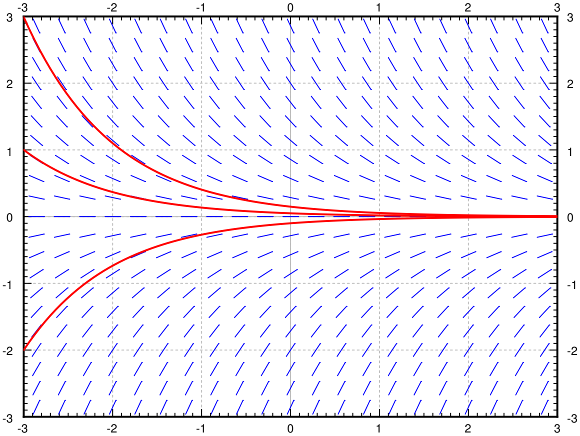
We can tell just by the slope field that $y(0) > 0$
leads to very different behavior from $y(0) < 0$.



Here's another example: $y' = -y$ with a few solutions given:



Here's another example: $y' = -y$ with a few solutions given:



Note how in this case we can tell from the slope field that all solutions just tend to $y = 0$.

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

The answer is usually yes to both.

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

The answer is usually yes to both.
If the answer is no, you probably do not have the right model.

Consider

$$y' = f(x,y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

The answer is usually yes to both.
If the answer is no, you probably do not have the right model.

Example: Answer to (i) can be no:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

The answer is usually yes to both.
If the answer is no, you probably do not have the right model.

Example: Answer to (i) can be no:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

(or the more harmless looking $xy' = 1$)

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

The answer is usually yes to both.
If the answer is no, you probably do not have the right model.

Example: Answer to (i) can be no:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

(or the more harmless looking $xy' = 1$)

Integration yields: $y = \ln|x| + C$.

Consider

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

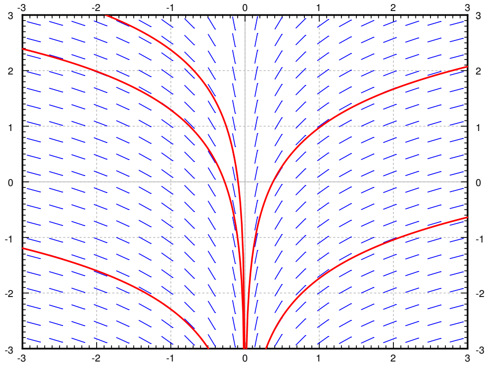
The answer is usually yes to both.
If the answer is no, you probably do not have the right model.

Example: Answer to (i) can be no:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

(or the more harmless looking $xy' = 1$)

Integration yields: $y = \ln |x| + C$.
There is no solution at $x = 0$.



Example:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

Example:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

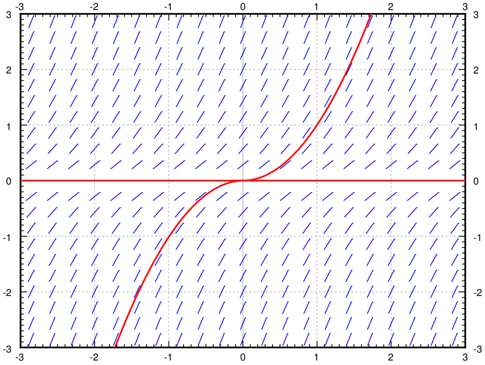
Note that $y = 0$ is a solution.

Example:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$



So answer to (ii) can also be false.

All hope is not lost!

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x, y)$ is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $y' = 1/x,\, y(0) = 0$ and $y' = 2\sqrt{|y|},\, y(0) = 0$ do not satisfy the hypotheses.

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $\quad y' = {}^1\!/_x,\, y(0) = 0 \quad$ and $\quad y' = 2\sqrt{|y|},\, y(0) = 0 \quad$ do not satisfy the hypotheses.

Example: Remember $\quad y' = y^2, \qquad y(0) = A \quad ?$

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $y' = {}^1/_x,\ y(0) = 0$ and $y' = 2\sqrt{|y|},\ y(0) = 0$ do not satisfy the hypotheses.

Example: Remember $y' = y^2, \qquad y(0) = A$?

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x, y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $\quad y' = 1/x, y(0) = 0 \quad$ and $\quad y' = 2\sqrt{|y|}, y(0) = 0 \quad$ do not satisfy the hypotheses.

Example: Remember $\quad y' = y^2, \qquad y(0) = A \quad$?

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

If $A \neq 0$, then solution is $y = \frac{1}{C-x}$.

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $y' = 1/x,\ y(0) = 0$ and $y' = 2\sqrt{|y|},\ y(0) = 0$ do not satisfy the hypotheses.

Example: Remember $y' = y^2, \qquad y(0) = A \qquad ?$

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

If $A \neq 0$, then solution is $y = \frac{1}{C-x}$.

$A = y(0) = \frac{1}{C-0} = 1/C$

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0,y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $\quad y' = {}^1\!/_x,\, y(0) = 0 \quad$ and $\quad y' = 2\sqrt{|y|},\, y(0) = 0 \quad$ do not satisfy the hypotheses.

Example: Remember $\quad y' = y^2, \qquad y(0) = A \quad$?

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

If $A \neq 0$, then solution is $y = \frac{1}{C-x}$.

$$A = y(0) = \tfrac{1}{C-0} = {}^1\!/_C \quad \Longrightarrow \quad C = {}^1\!/_A$$

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $\quad y' = {}^1\!/_x,\ y(0) = 0 \quad$ and $\quad y' = 2\sqrt{|y|},\ y(0) = 0 \quad$ do not satisfy the hypotheses.

Example: Remember $\quad y' = y^2, \qquad y(0) = A \quad$?

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

If $A \neq 0$, then solution is $y = \frac{1}{C-x}$.

$A = y(0) = \frac{1}{C-0} = {}^1\!/_C \quad \Rightarrow \quad C = {}^1\!/_A \quad \Rightarrow \quad$ The solution is $y = \dfrac{1}{{}^1\!/_A - x}$.

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $y' = {}^1/_x,\ y(0) = 0$ and $y' = 2\sqrt{|y|},\ y(0) = 0$ do not satisfy the hypotheses.

Example: Remember $y' = y^2, \qquad y(0) = A$?

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

If $A \neq 0$, then solution is $y = \frac{1}{C-x}$.

$A = y(0) = \frac{1}{C-0} = {}^1/_C \Rightarrow C = {}^1/_A \Rightarrow$ The solution is $y = \dfrac{1}{{}^1/_A - x}$.

If $A = 1$, the solution blows up when $x = 1$. The solution exists for $x < 1$.

All hope is not lost!

Theorem (Picard's theorem on existence and uniqueness)

If $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0), then a solution to

$$y' = f(x,y), \qquad y(x_0) = y_0,$$

exists (for x in some interval) and is unique.

E.g., $y' = {}^1\!/_x,\, y(0) = 0$ and $y' = 2\sqrt{|y|},\, y(0) = 0$ do not satisfy the hypotheses.

Example: Remember $y' = y^2, \quad y(0) = A$?

If $A = 0$, then $y = 0$ is the solution. It exists for all x.

If $A \neq 0$, then solution is $y = \frac{1}{C-x}$.

$A = y(0) = \frac{1}{C-0} = {}^1\!/_C \quad \Rightarrow \quad C = {}^1\!/_A \quad \Rightarrow$ The solution is $y = \dfrac{1}{{}^1\!/_A - x}$.

If $A = 1$, the solution blows up when $x = 1$. The solution exists for $x < 1$.

If $A = 100$, the solution blows up when $x = \frac{1}{100} = 0.01$. The solution exists for $x < 0.01$.