3. Classification of differential equations (Notes on Diffy Qs, 0.3)

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The textbook: https://www.jirka.org/diffyqs/

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  Only one independent variable.
- *Partial differential equations* (PDE):
  Several independent variables, using partial derivatives.

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$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{(Wave equation in 2 dimensions)}$$

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Maxwell's equations for electromagnetics are a *system of partial differential equations* (system of PDE):

$$\nabla \cdot \vec{D} = \rho, \qquad \nabla \cdot \vec{B} = 0,$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

(Note: divergence $\nabla\cdot$ and curl $\nabla\times$ are written in partial derivatives in x, y, z.)

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Remark: The most common equations in physics are first and second order.

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An ODE of order n can be put into the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

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Remark: Nonlinear equations are notoriously difficult to handle.

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We solve an inhomogeneous equation using the solution to the corresponding homogeneous equation.

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Autonomous equations often appear when the setup is independent of time.

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First order, linear, inhomogeneous, constant coefficient PDE system.