

10. Numerical methods: Euler's method (Notes on Diffy Qs, 1.7)

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The textbook: <https://www.jirka.org/diffyqs/>

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We will cover the simplest method of all: Euler's method.

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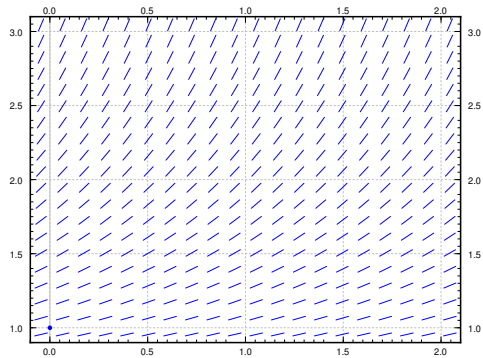
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Essentially, keep repeating : $x_{i+1} = x_i + h$, $y_{i+1} = y_i + hf(x_i, y_i)$.

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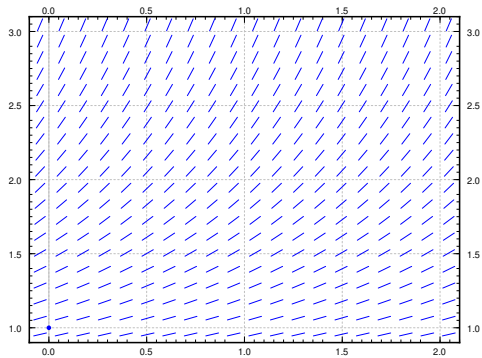
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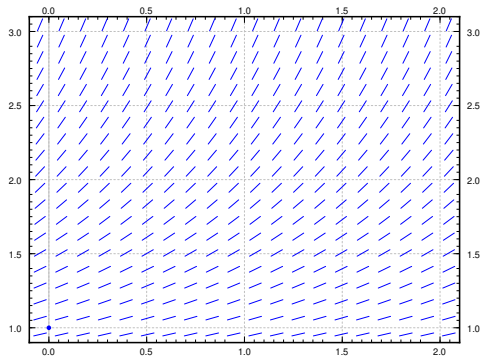


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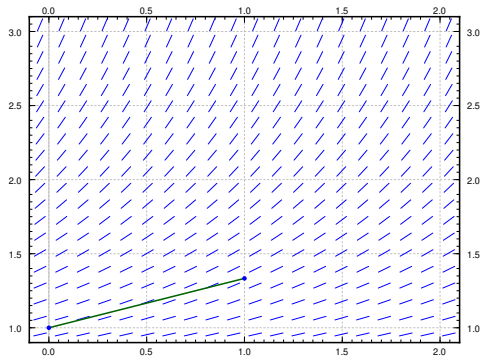
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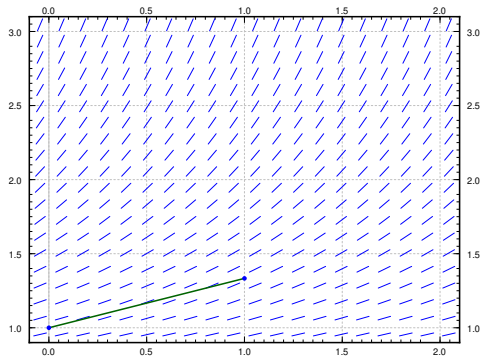
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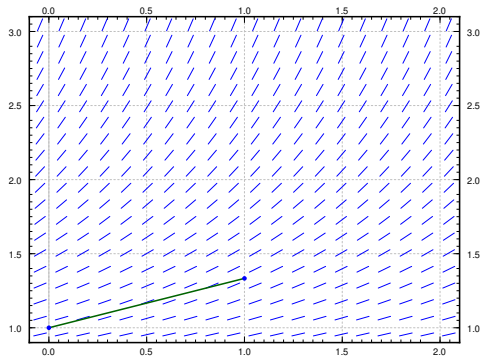
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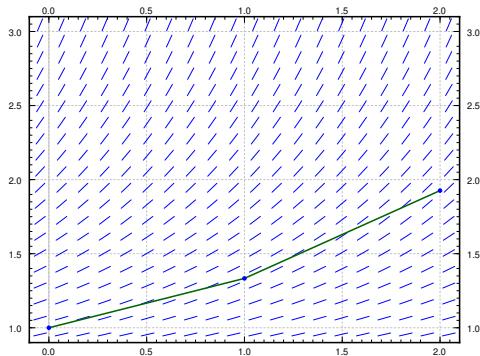
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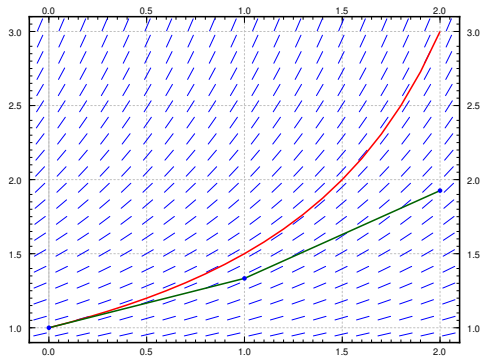
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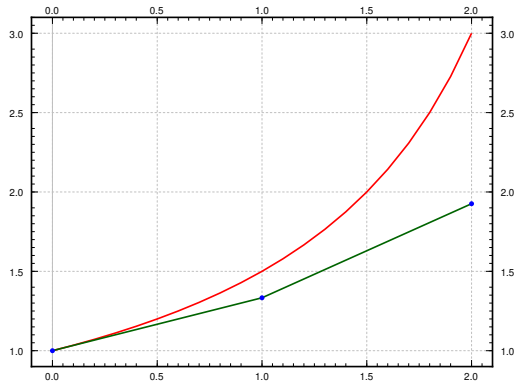
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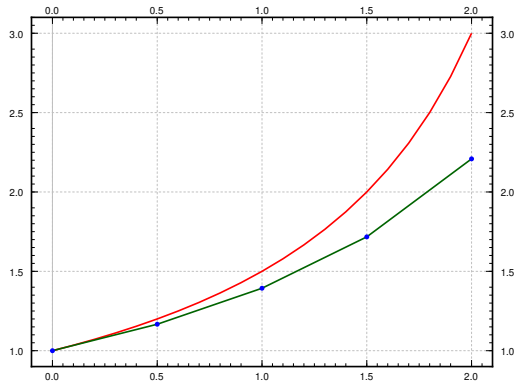


Let's do a few more h s.



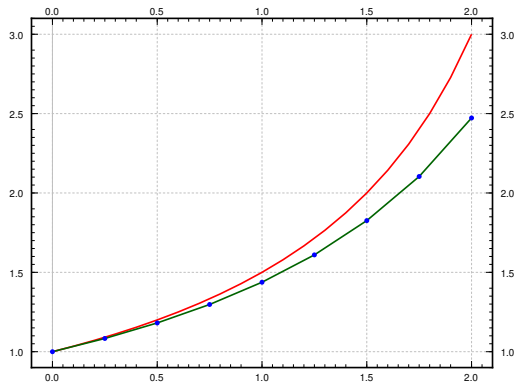
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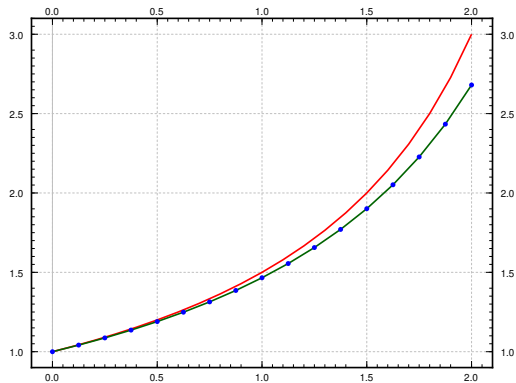
$$h = 1/2 = 0.5$$

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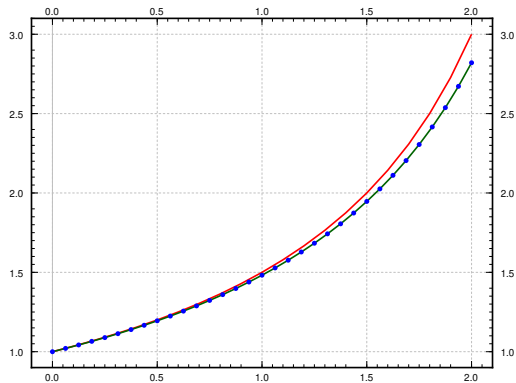
$$h = 1/4 = 0.25$$

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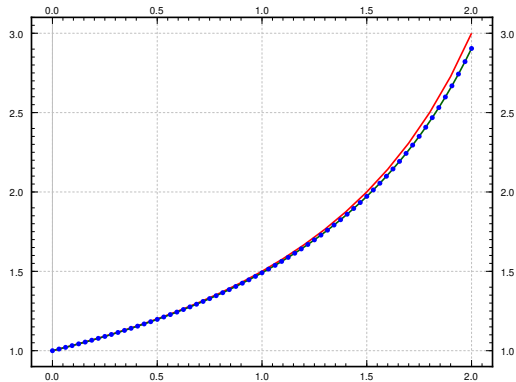
$$h = 1/8 = 0.125$$

Let's do a few more h s.



$$h = 1/16 = 0.0625$$

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$$h = 1/32 = 0.03125$$

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0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
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Note: It seems like error halves when we halve h .

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The trick in practice is to estimate the error so that we can pick the right h .

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Suppose we attempt to find $y(3)$:

h	Approximate $y(3)$
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0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

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0.25	6.86079
0.125	10.80321
0.0625	17.59893
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Numerical methods are still a current research topic.

The simplest method used in practice is the *Runge–Kutta method*.

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The idea is the same as Euler except each step looks like

$$k_1 = f(x_i, y_i),$$

$$k_2 = f(x_i + h/2, y_i + k_1(h/2)),$$

$$k_3 = f(x_i + h/2, y_i + k_2(h/2)),$$

$$k_4 = f(x_i + h, y_i + k_3h).$$

$$x_{i+1} = x_i + h,$$

$$y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} h,$$