2. Introduction to differential equations (Notes on Diffy Qs, 0.2)

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The textbook: `https://www.jirka.org/diffyqs/`

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It is a *first order differential equation*.

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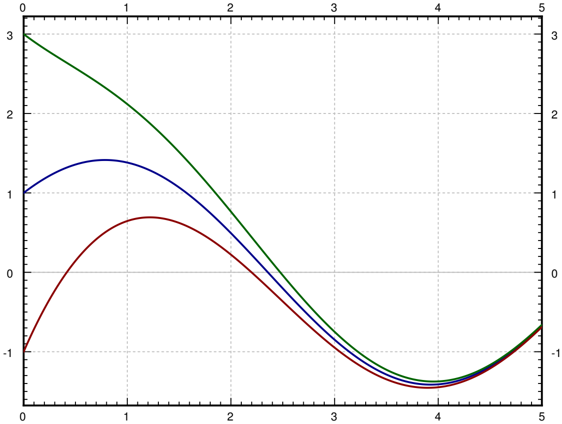

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For simple cases we find exact analytic solutions.

For complicated cases we may have to be satisfied with approximate, numerical solutions.

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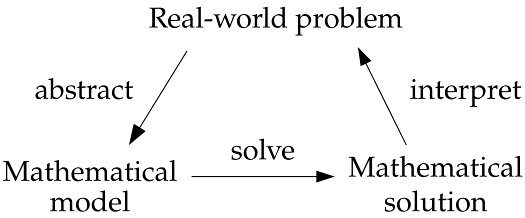

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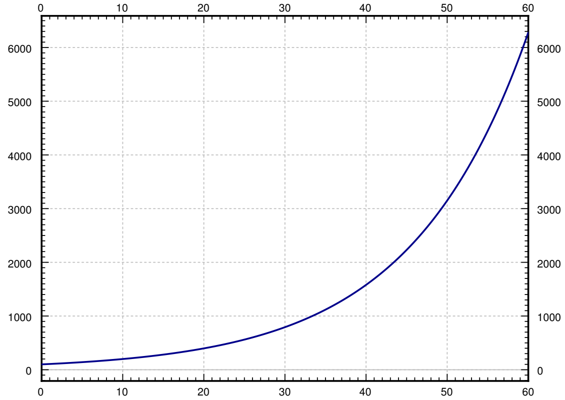

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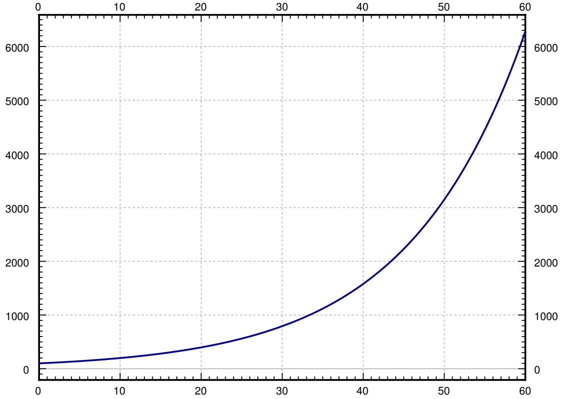

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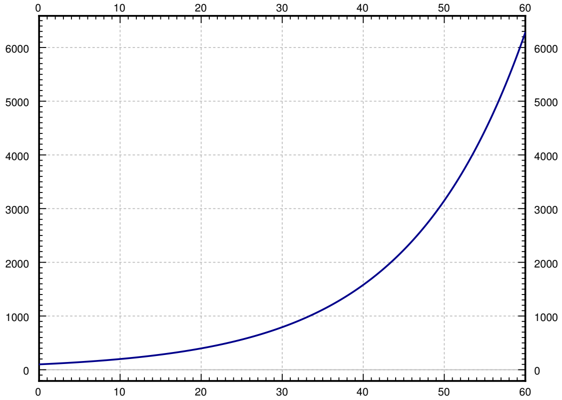

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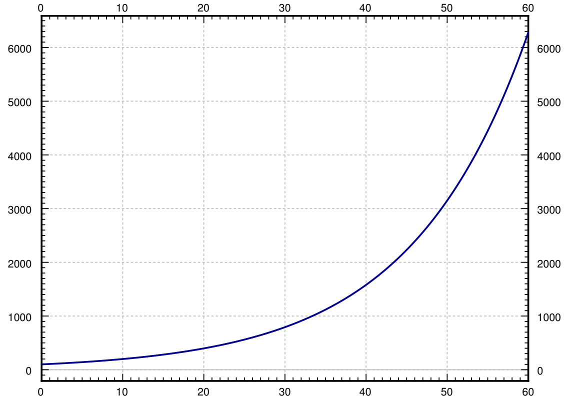

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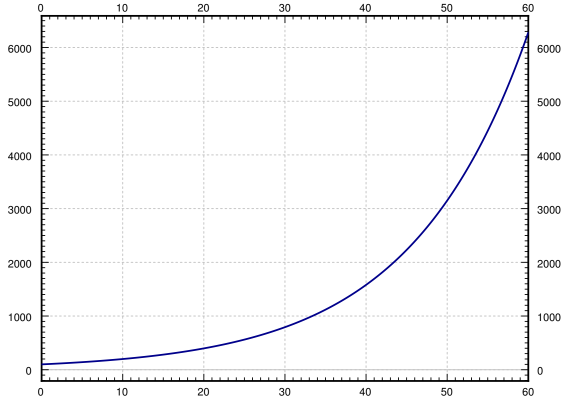

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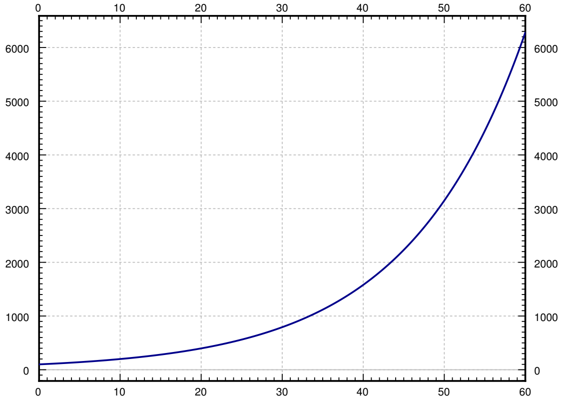
Obviously this is approximate.



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   General solution: $y(x) = C_1 e^{kx} + C_2 e^{-kx}$ or $y(x) = D_1 \cosh(kx) + D_2 \sinh(kx)$

For those that have not seen sinh and cosh, the *hyperbolic sine* and *hyperbolic cosine*:

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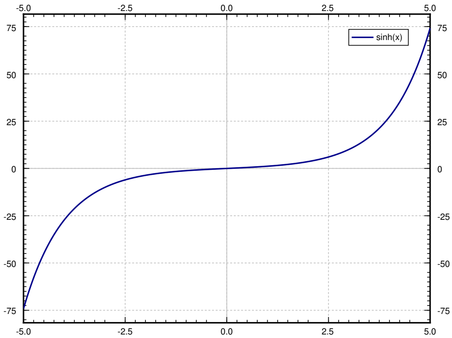
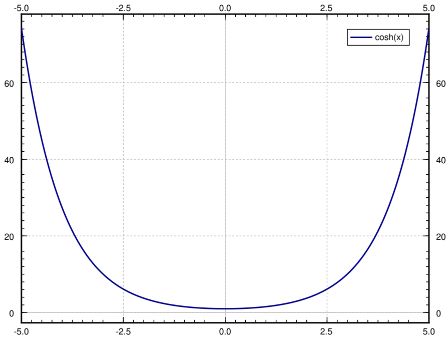
$$\cosh 0 = 1, \quad \sinh 0 = 0, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x.$$

Remark: The shape of the graph of cosh is called a *catenary*. The arch in Saint Louis is an inverted cosh:

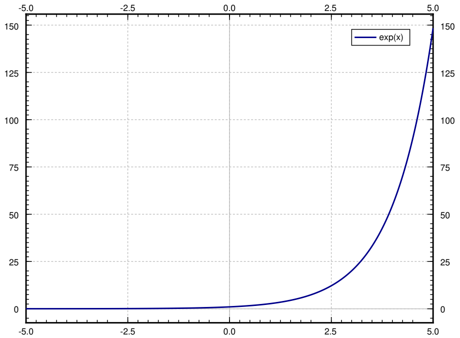
$$y = -127.7\ \text{ft} \cdot \cosh(x/127.7\ \text{ft}) + 757.7\ \text{ft}.$$

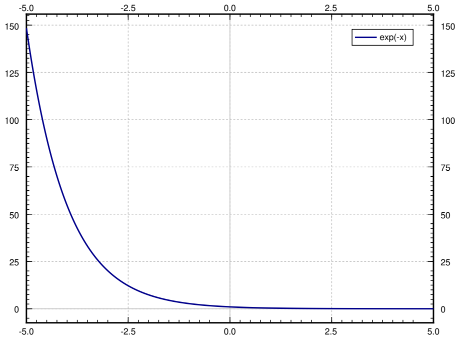
Here are the graphs of sinh and cosh:





Compare with the graphs of exponential growth e^x and exponential decay e^{-x}





Just for completeness here are the graphs of sin and cos: