8. Substitution
(Notes on Diffy Qs, 1.5)

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The textbook: `https://www.jirka.org/diffyqs/`

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Note: $D = 0$ gives $y = x + 2$, but no D gives $y = x$.

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Some patterns to look for:

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Nothing wrong with making many guesses.

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Substitute $v = y^{1-5} = y^{-4}$

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We just need to solve this linear equation.

$v' - \frac{4(x+1)}{x}v = 4 \quad (v' + P(x)v = F(x) \text{ where } P(x) = -\frac{4(x+1)}{x} \text{ and } F(x) = 4)$

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In the solution, we assume $x > 0$ or $x < 0$ depending on the initial condition.

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Solve: $\displaystyle \int \frac{1}{F(v) - v}\,dv = \int \frac{1}{v^2}\,dv = \ln|x| + C \quad \Rightarrow \quad \frac{-1}{v} = \ln x + C \quad \Rightarrow \quad v = \frac{-1}{\ln x + C}$

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