11. Exact equations (part 1)
(Notes on Diffy Qs, 1.8)

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The textbook: `https://www.jirka.org/diffyqs/`

An *exact equation* is an equation that encodes

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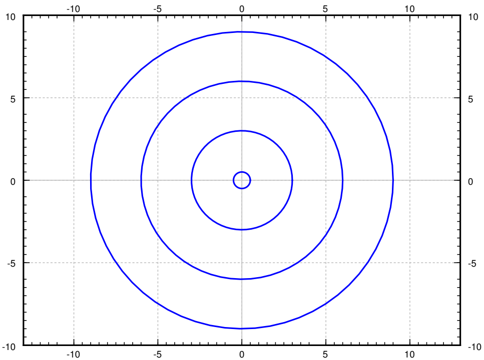
Such equations come up when there is some conservation law at play.
(e.g., conservation of energy)

Example: Let $F(x, y) = x^2 + y^2$

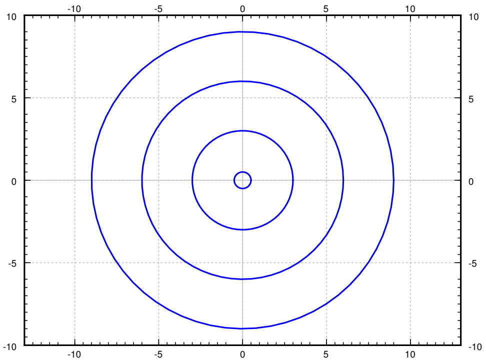


Solutions to $F(x, y) = x^2 + y^2 = C$.

Example: Let $F(x, y) = x^2 + y^2$

Take the *total derivative* of F:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = F_x dx + F_y dy.$$



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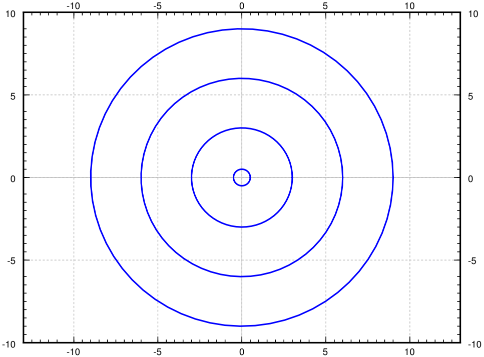

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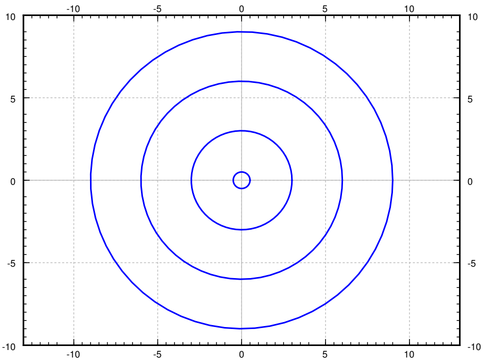

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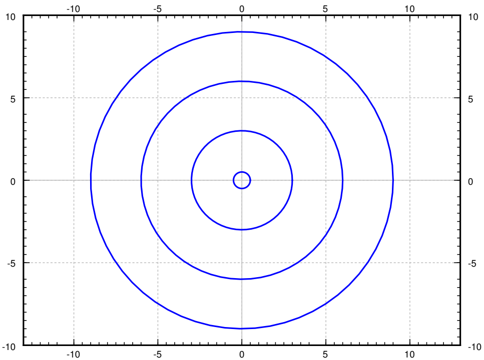
$$2x\,dx + 2y\,dy = 0 \quad \text{or} \quad 2x + 2y\,\frac{dy}{dx} = 0$$



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Normally, we start with $M\,dx + N\,dy = 0$ and we wish to find the unknown $F(x, y)$.

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Let γ be a path in the plane from (x_1, y_1) and ending at (x_2, y_2) and think of $\vec{v}$ as force.

The work required to move along γ is $\displaystyle\int_\gamma \vec{v}(\vec{r}) \cdot d\vec{r} = \int_\gamma M\,dx + N\,dy = F(x_2, y_2) - F(x_1, y_1).$

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Exact equations are conservative vector fields, and their implicit solutions are given by the potential functions.

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If $F(x, y)$ works, $F(x, y) + 3$ or $F(x, y) - 8$ also work.

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Roles of x and y (and so M and N) can be reversed.

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The equation is not exact! F does not exist.

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In fact:

Theorem (Poincaré)

If M and N are continuously differentiable functions of (x, y), and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

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Implicit (general) solution is $x^2 + xy - y = C$

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Differentiate in y and set to N to find $x - 1 = x + A'(y)$

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Implicit (general) solution is $x^2 + xy - y = C$

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Finally, solve $x^2 + xy - y = -1$ for y:

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Explicit solution: $\quad y = \tan(C)x$.
As $y(1) = 2$, we get $\tan(C) = 2$,
so $\quad y = 2x \quad$ is the solution (only for $x > 0$).