6. Separable equations (Notes on Diffy Qs, 1.3)

Jiří Lebl

Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

Given $y' = f(x)$ we integrate to solve: $y = \int f(x)\, dx + C$.

Given $y' = f(x)$ we integrate to solve: $y = \int f(x)\,dx + C$.

Similarly we solved $y' = f(y)$ by writing x in terms of y.

Given $y' = f(x)$ we integrate to solve: $y = \int f(x)\,dx + C$.

Similarly we solved $y' = f(y)$ by writing x in terms of y.

But the strategy doesn't work for the general $y' = f(x, y)$.

Given $y' = f(x)$ we integrate to solve: $y = \int f(x)\,dx + C$.

Similarly we solved $y' = f(y)$ by writing x in terms of y.

But the strategy doesn't work for the general $y' = f(x, y)$.

Integrating yields

$$y = \int f(x, y)\,dx + C.$$

Given $y' = f(x)$ we integrate to solve: $y = \int f(x)\,dx + C$.

Similarly we solved $y' = f(y)$ by writing x in terms of y.

But the strategy doesn't work for the general $y' = f(x, y)$.

Integrating yields

$$y = \int f(x, y)\,dx + C.$$

But if the equation is so-called "separable," then we can still integrate.

A differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

A differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

Using the Leibniz notation $\frac{dy}{dx} = f(x)g(y)$, rewrite it as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

A differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

Using the Leibniz notation $\dfrac{dy}{dx} = f(x)g(y)$, rewrite it as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

A differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

Using the Leibniz notation $\dfrac{dy}{dx} = f(x)g(y)$, rewrite it as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Integrate both sides:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

Solve for y (if you can).

Example: Consider $y' = xy$.

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C$$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$\Rightarrow \quad \ln|y| = \dfrac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C}$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\quad \dfrac{dy}{dx} = xy \quad$ as $\quad \dfrac{1}{y}\,dy = x\,dx \qquad$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $$\frac{dy}{dx} = xy \quad \text{as} \quad \frac{1}{y}\,dy = x\,dx \qquad \text{Integrate:} \int \frac{dy}{y} = \int x\,dx + C$$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.
And $y = 0$ is also a solution.

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.
And $y = 0$ is also a solution.

So the general solution is $y = De^{\frac{x^2}{2}}$ (D a constant).

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\quad \dfrac{dy}{dx} = xy \quad$ as $\quad \dfrac{1}{y}\,dy = x\,dx \qquad$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.
And $y = 0$ is also a solution.

So the general solution is $\quad y = De^{\frac{x^2}{2}} \quad$ (D a constant).

Check: y'

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\quad \dfrac{dy}{dx} = xy \quad$ as $\quad \dfrac{1}{y}\,dy = x\,dx \qquad$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.
And $y = 0$ is also a solution.

So the general solution is $\quad y = De^{\frac{x^2}{2}} \quad$ (D a constant).

Check: $y' = Dxe^{\frac{x^2}{2}}$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.
And $y = 0$ is also a solution.

So the general solution is $y = De^{\frac{x^2}{2}}$ (D a constant).

Check: $y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right)$

Example: Consider $y' = xy$.

$y = 0$ is a solution, so remember that and assume $y \neq 0$ from now.

Write: $\dfrac{dy}{dx} = xy$ as $\dfrac{1}{y}\,dy = x\,dx$ Integrate: $\displaystyle\int \frac{dy}{y} = \int x\,dx + C$

$$\Rightarrow \quad \ln|y| = \frac{x^2}{2} + C \quad \Rightarrow \quad |y| = e^{\frac{x^2}{2}+C} = e^C e^{\frac{x^2}{2}}$$

e^C is an arbitrary positive constant.
Because of the absolute value we could replace it with a negative.
And $y = 0$ is also a solution.

So the general solution is $y = De^{\frac{x^2}{2}}$ (D a constant).

Check: $y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy$. ✓

It appears as if we are integrating with two different variables.

It appears as if we are integrating with two different variables.

So why does it work?

It appears as if we are integrating with two different variables.

So why does it work?

Note that $y = y(x)$ and $\dfrac{dy}{dx}$ are functions of x.

It appears as if we are integrating with two different variables.

So why does it work?

Note that $y = y(x)$ and $\dfrac{dy}{dx}$ are functions of x.

Write $\dfrac{dy}{dx} = f(x)g(y)$ as $\dfrac{1}{g(y)}\dfrac{dy}{dx} = f(x)$.

It appears as if we are integrating with two different variables.

So why does it work?

Note that $y = y(x)$ and $\dfrac{dy}{dx}$ are functions of x.

Write $\dfrac{dy}{dx} = f(x)g(y)$ as $\dfrac{1}{g(y)}\dfrac{dy}{dx} = f(x)$.

Now integrate both sides wrt x: $\displaystyle\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx + C.$

It appears as if we are integrating with two different variables.

So why does it work?

Note that $y = y(x)$ and $\dfrac{dy}{dx}$ are functions of x.

Write $\quad \dfrac{dy}{dx} = f(x)g(y) \quad$ as $\quad \dfrac{1}{g(y)}\dfrac{dy}{dx} = f(x).$

Now integrate both sides wrt x: $\quad \displaystyle\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx + C.$

Substitution formula from calculus says $\quad \displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C. \qquad \checkmark$

Solving for y can be difficult.

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2 + 1}.$

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}$. $\qquad$ $y = 0$ is a solution, so assume $y \neq 0$ from now.

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}.$ $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)\,dy = x\,dx.$

Solving for y can be difficult.

Example: Consider $y' = \dfrac{xy}{y^2+1}$. $y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx$. Integrate: $\dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}$. $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx.$ $\qquad$ Integrate: $\quad \dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

Solving for y can be difficult.

Example: Consider $y' = \dfrac{xy}{y^2+1}$. $y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx$. Integrate: $\dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C$,

Simplify $y^2 + 2\ln|y| = x^2 + D \quad (D = 2C)$.

We can't solve for y in a "nice" expression.

Solving for y can be difficult.

Example: Consider $y' = \dfrac{xy}{y^2+1}$. $\quad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx.$ Integrate: $\dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

We can't solve for y in a "nice" expression.
Just leave it as is.

Solving for y can be difficult.

Example: Consider $y' = \dfrac{xy}{y^2+1}$. $y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)\,dy = x\,dx$. Integrate: $\dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C$,

Simplify $y^2 + 2\ln|y| = x^2 + D \quad (D = 2C)$.

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

Solving for y can be difficult.

Example: Consider $y' = \dfrac{xy}{y^2+1}$. $y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx$. Integrate: $\dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C$,

Simplify $y^2 + 2\ln|y| = x^2 + D \quad (D = 2C)$.

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}$. $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx$. $\qquad$ Integrate: $\quad \dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C)$.

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.
Differentiate remembering that $y = y(x)$ is a function of x:

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}.$ $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx.$ $\qquad$ Integrate: $\quad \dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.
Differentiate remembering that $y = y(x)$ is a function of x:

$$y'\,(2y + {}^2\!/_y) = 2x$$

Solving for y can be difficult.

Example: Consider $y' = \dfrac{xy}{y^2+1}$. $\quad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right) dy = x\,dx.$ Integrate: $\dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.
Differentiate remembering that $y = y(x)$ is a function of x:

$$y'\,(2y + {}^2\!/_y) = 2x \quad \Rightarrow \quad y'\,(2y + {}^2\!/_y)\,\frac{y}{2(y^2+1)} = 2x\frac{y}{2(y^2+1)}$$

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}$. $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx.$ $\qquad$ Integrate: $\quad \dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.
Differentiate remembering that $y = y(x)$ is a function of x:

$$y'\,(2y + {}^{2}/_{y}) = 2x \quad \Rightarrow \quad y'\,(2y + {}^{2}/_{y})\,\frac{y}{2(y^2+1)} = 2x\frac{y}{2(y^2+1)} \quad \Rightarrow \quad y' = \frac{xy}{y^2+1} \quad \checkmark$$

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}$. $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right) dy = x\,dx. \qquad$ Integrate: $\quad \dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.
Differentiate remembering that $y = y(x)$ is a function of x:

$$y'\,(2y + {}^2\!/\!_y) = 2x \quad \Rightarrow \quad y'\,(2y + {}^2\!/\!_y)\,\frac{y}{2(y^2+1)} = 2x\frac{y}{2(y^2+1)} \quad \Rightarrow \quad y' = \frac{xy}{y^2+1} \quad \checkmark$$

The general solution is

$$y^2 + 2\ln|y| = x^2 + C, \qquad \text{and} \qquad y = 0.$$

Solving for y can be difficult.

Example: Consider $\quad y' = \dfrac{xy}{y^2+1}$. $\qquad y = 0$ is a solution, so assume $y \neq 0$ from now.

Separate variables: $\quad \dfrac{y^2+1}{y}\,dy = \left(y + \dfrac{1}{y}\right)dy = x\,dx. \qquad$ Integrate: $\quad \dfrac{y^2}{2} + \ln|y| = \dfrac{x^2}{2} + C,$

Simplify $\quad y^2 + 2\ln|y| = x^2 + D \quad (D = 2C).$

We can't solve for y in a "nice" expression.
Just leave it as is. It is called an *implicit solution*.

We can still check that it satisfies the equation.
Differentiate remembering that $y = y(x)$ is a function of x:

$$y'\,(2y + {}^2\!/\!{}_y) = 2x \quad \Rightarrow \quad y'\,(2y + {}^2\!/\!{}_y)\,\frac{y}{2(y^2+1)} = 2x\frac{y}{2(y^2+1)} \quad \Rightarrow \quad y' = \frac{xy}{y^2+1} \quad \checkmark$$

The general solution is

$$y^2 + 2\ln|y| = x^2 + C, \qquad \text{and} \qquad y = 0.$$

Solutions such as $y = 0$ are sometimes called *singular solutions*.

Computing values of y given x is tricky.

Computing values of y given x is tricky. You might even get multiple values.

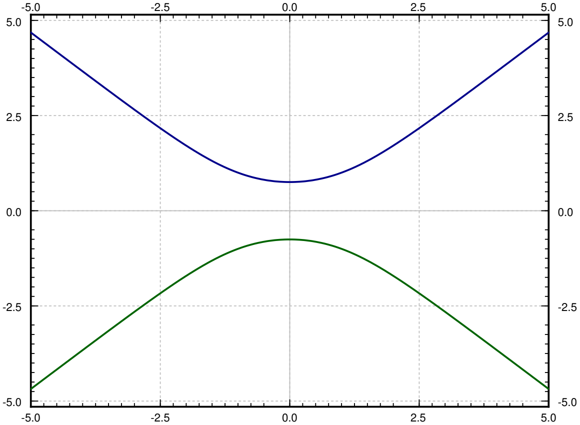
Computing values of y given x is tricky. You might even get multiple values.

Often one uses computers to find values.

Computing values of y given x is tricky. You might even get multiple values.

Often one uses computers to find values.

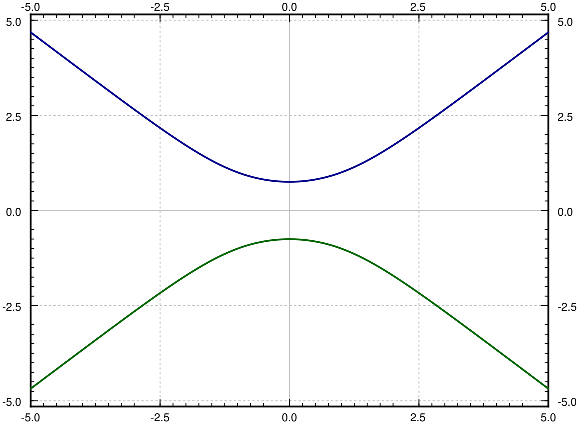
Here is what the set of points (x, y) satisfying $y^2 + 2\ln|y| = x^2$ looks like:



Computing values of y given x is tricky. You might even get multiple values.

Often one uses computers to find values.

Here is what the set of points (x, y) satisfying $y^2 + 2\ln|y| = x^2$ looks like:

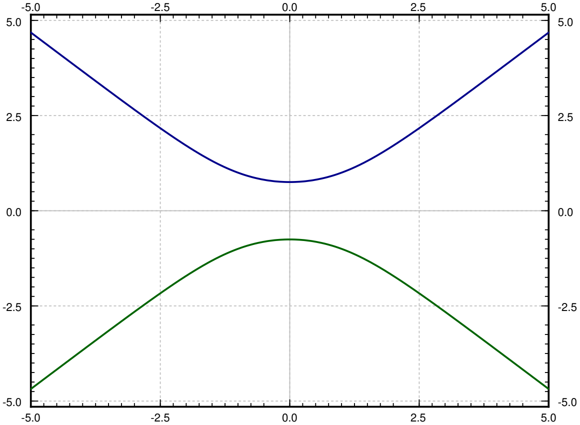


The initial condition tells you which solution to take.

Computing values of y given x is tricky. You might even get multiple values.

Often one uses computers to find values.

Here is what the set of points (x, y) satisfying $y^2 + 2 \ln |y| = x^2$ looks like:



The initial condition tells you which solution to take. E.g., the top curve satisfies $y(1) = 1$.

A couple more examples of separable equations:

A couple more examples of separable equations:

Example: Solve $\quad x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1 + y^2} = \frac{1 - x^2}{x^2}$$

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1$$

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1 \quad \Rightarrow \quad \arctan(y) = \frac{-1}{x} - x + C$$

A couple more examples of separable equations:

Example: Solve $\quad x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $\quad x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1 \quad \Rightarrow \quad \arctan(y) = \frac{-1}{x} - x + C$$

$$\Rightarrow \qquad y = \tan\left(\frac{-1}{x} - x + C\right)$$

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1 \quad \Rightarrow \quad \arctan(y) = \frac{-1}{x} - x + C$$

$$\Rightarrow \qquad y = \tan\left(\frac{-1}{x} - x + C\right)$$

Solve for IC: $0 = \tan(-2 + C)$

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1 \quad \Rightarrow \quad \arctan(y) = \frac{-1}{x} - x + C$$

$$\Rightarrow \qquad y = \tan\left(\frac{-1}{x} - x + C\right)$$

Solve for IC: $0 = \tan(-2 + C) \quad \Rightarrow \quad C = 2$

A couple more examples of separable equations:

Example: Solve $\quad x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $\quad x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1 \quad \Rightarrow \quad \arctan(y) = \frac{-1}{x} - x + C$$

$$\Rightarrow \qquad y = \tan\left(\frac{-1}{x} - x + C\right)$$

Solve for IC: $\quad 0 = \tan(-2 + C) \quad \Rightarrow \quad C = 2 \quad$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.)

A couple more examples of separable equations:

Example: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2, \quad y(1) = 0.$

Factor: $x^2y' = (1 - x^2)(1 + y^2).$

Separate variables, integrate, and solve for y:

$$\frac{y'}{1+y^2} = \frac{1-x^2}{x^2} \quad \Rightarrow \quad \frac{y'}{1+y^2} = \frac{1}{x^2} - 1 \quad \Rightarrow \quad \arctan(y) = \frac{-1}{x} - x + C$$

$$\Rightarrow \qquad y = \tan\left(\frac{-1}{x} - x + C\right)$$

Solve for IC: $0 = \tan(-2 + C) \quad \Rightarrow \quad C = 2 \quad$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.)

The particular solution we seek is $y = \tan\left(\frac{-1}{x} - x + 2\right).$

Example: Bob wants to drink coffee at 60°C.

Example: Bob wants to drink coffee at $60°C$.
Initially (time $t = 0$), the temperature was $89°C$.

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $$\frac{dT}{dt} = k(A - T) \quad \text{for some } k > 0.$$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\displaystyle \frac{dT}{dt} = k(A - T)$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $$\frac{dT}{dt} = k(A - T) \quad \text{for some } k > 0.$$

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\dfrac{dT}{dt} = k(A - T)$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T - A} \frac{dT}{dt} = -k$$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \frac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T - A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T - A) = -kt + C$$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T - A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T - A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt}$$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T-A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T - A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T - A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

First condition: $\quad 89 = T(0)$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T-A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

First condition: $\quad 89 = T(0) = 22 + D$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T-A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

First condition: $\quad 89 = T(0) = 22 + D \quad \Rightarrow \quad D = 67$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad\Rightarrow\quad \ln(T-A) = -kt + C \quad\Rightarrow\quad T - A = D\,e^{-kt} \quad\Rightarrow\quad T = A + D\,e^{-kt}$$

$$\Rightarrow\quad T = 22 + D\,e^{-kt}$$

First condition: $\quad 89 = T(0) = 22 + D \quad\Rightarrow\quad D = 67 \quad\Rightarrow\quad T = 22 + 67\,e^{-kt}$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\,\frac{dT}{dt} = -k \quad\Rightarrow\quad \ln(T-A) = -kt + C \quad\Rightarrow\quad T - A = D\,e^{-kt} \quad\Rightarrow\quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

First condition: $\quad 89 = T(0) = 22 + D \quad\Rightarrow\quad D = 67 \quad\Rightarrow\quad T = 22 + 67\,e^{-kt}$

Second condition $\quad 85 = T(1)$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\dfrac{dT}{dt} = k(A - T)$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T - A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

First condition: $89 = T(0) = 22 + D \quad \Rightarrow \quad D = 67 \quad \Rightarrow \quad T = 22 + 67\,e^{-kt}$

Second condition $85 = T(1) = 22 + 67\,e^{-k}$

Example: Bob wants to drink coffee at 60°C.
Initially (time $t = 0$), the temperature was 89°C.
One minute later, the temperature was 85°C.
The room has temperature 22°C.
When can Bob start drinking?

Let T = the temperature of the coffee in °C.
Let A = the ambient temperature in °C

Newton's law of cooling states $\quad \dfrac{dT}{dt} = k(A - T) \quad$ for some $k > 0$.

We have $A = 22$, $T(0) = 89$, $T(1) = 85$.

Separate variables, integrate (note that $T - A > 0$):

$$\frac{1}{T-A}\frac{dT}{dt} = -k \quad \Rightarrow \quad \ln(T-A) = -kt + C \quad \Rightarrow \quad T - A = D\,e^{-kt} \quad \Rightarrow \quad T = A + D\,e^{-kt}$$

$$\Rightarrow \quad T = 22 + D\,e^{-kt}$$

First condition: $\quad 89 = T(0) = 22 + D \quad \Rightarrow \quad D = 67 \quad \Rightarrow \quad T = 22 + 67\,e^{-kt}$

Second condition $\quad 85 = T(1) = 22 + 67\,e^{-k} \quad \Rightarrow \quad k = -\ln\frac{85-22}{67} \approx 0.0616$

So approximately $T = 22 + 67e^{-0.0616t}$

So approximately $T = 22 + 67e^{-0.0616t}$

Solve $T = 60$ for time t: $60 = 22 + 67e^{-0.0616t}$

So approximately $T = 22 + 67e^{-0.0616t}$

Solve $T = 60$ for time t: $60 = 22 + 67e^{-0.0616t} \quad \Rightarrow \quad t = -\dfrac{\ln \frac{60-22}{67}}{0.0616}$

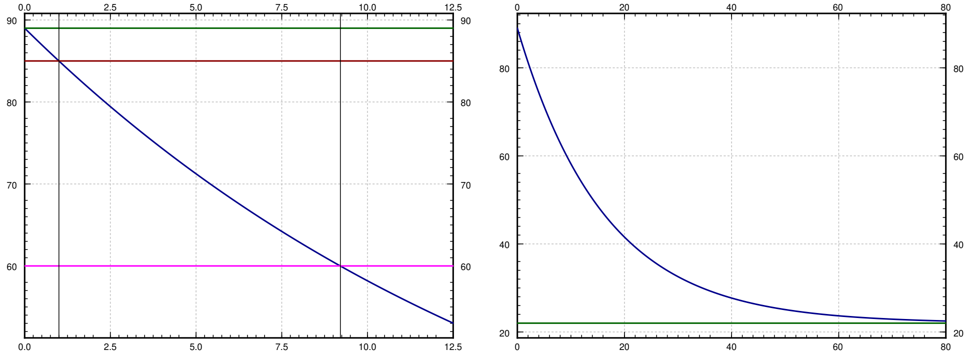
So approximately $\quad T = 22 + 67e^{-0.0616t}$

Solve $T = 60$ for time t: $\quad 60 = 22 + 67e^{-0.0616t} \quad \Longrightarrow \quad t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$ minutes.

So approximately $\quad T = 22 + 67e^{-0.0616t}$

Solve $T = 60$ for time t: $\quad 60 = 22 + 67e^{-0.0616t} \quad \Longrightarrow \quad t = -\dfrac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$ minutes.

Bob can start drinking a little over 9 minutes from when the coffee was made.



Graphs of the coffee temperature function $T(t)$ over different time frames.
89, 85, 60 lines are marked on the left, and $T = A$ line is marked on the right.

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

$y = 0$ is a solution (a singular solution).

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

$y = 0$ is a solution (a singular solution).

Assume $y \neq 0$ and solve:

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

$y = 0$ is a solution (a singular solution).

Assume $y \neq 0$ and solve:

$$\frac{-3}{y^2} y' = x$$

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

$y = 0$ is a solution (a singular solution).

Assume $y \neq 0$ and solve:

$$\frac{-3}{y^2}y' = x \quad \Rightarrow \quad \frac{3}{y} = \frac{x^2}{2} + C$$

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

$y = 0$ is a solution (a singular solution).

Assume $y \neq 0$ and solve:

$$\frac{-3}{y^2}y' = x \quad \Rightarrow \quad \frac{3}{y} = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}$$

Example: Find the general solution to $y' = \frac{-xy^2}{3}$.

$y = 0$ is a solution (a singular solution).

Assume $y \neq 0$ and solve:

$$\frac{-3}{y^2}y' = x \quad \Rightarrow \quad \frac{3}{y} = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}$$

The general solution is

$$y = \frac{6}{x^2 + 2C} \qquad \text{and} \qquad y = 0.$$