12. Exact equations (part 2)
(Notes on Diffy Qs, 1.8)

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The textbook: `https://www.jirka.org/diffyqs/`

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(Or reverse the roles of x and y and M and N).

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One could solve for x in terms of y for any initial condition (messy).

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Let $Q(y) = \frac{M_y - N_x}{M} \quad \Rightarrow \quad u(y) = e^{-\int Q(y)\,dy}$.

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Compute $u(x) = e^{\int P(x)\,dx} = e^{\ln(x+1)} = x+1.$

The equation becomes: $x^2 + y^2 + 2y(x+1)\dfrac{dy}{dx} = 0.$

Example: Solve $\dfrac{x^2+y^2}{x+1} + 2y\dfrac{dy}{dx} = 0.$

$M = \frac{x^2+y^2}{x+1}$ and $N = 2y$.

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This is exact and we solved it a moment ago:

$$y = \pm\sqrt{\frac{C - (1/3)x^3}{x+1}}$$

Example: Solve $y^2 + (xy+1)\dfrac{dy}{dx} = 0$.

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$M_y - N_x$

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