13. Second order linear ODEs (Notes on Diffy Qs, 2.1)

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The textbook: `https://www.jirka.org/diffyqs/`

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Examples:

$y'' + k^2y = 0$	Two solutions are: $y_1 = \cos(kx), \quad y_2 = \sin(kx).$
$y'' - k^2y = 0$	Two solutions are: $y_1 = e^{kx}, \quad y_2 = e^{-kx}.$

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Exercise: Verify that

$$\cosh 0 = 1, \qquad \sinh 0 = 0,$$

$$\frac{d}{dt}\Big[\cosh t\Big] = \sinh t, \qquad \frac{d}{dt}\Big[\sinh t\Big] = \cosh t,$$

$$\cosh^2 t - \sinh^2 t = 1.$$

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Similarly, $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

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Solve for $A = \dfrac{y_1}{y_2} = x^3$. That's not a constant! $\quad\Rightarrow\quad$ y_1 and y_2 are linearly independent.

$$\Rightarrow \quad y = C_1x^2 + C_2\frac{1}{x} \quad \text{is the general solution.}$$

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We can even just write down a formula

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\left(y_1(x)\right)^2}\,dx$$

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Equations of the form $ax^2y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy–Euler equations*.