6.4 Dirac delta and impulse response

Note: 1 or 1.5 lecture, §7.6 in [EP], §6.5 in [BD]

6.4.1 Rectangular pulse

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behaviour is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } t < a, \\ M & \text{if } a \le t < b, \\ 0 & \text{if } b \le t. \end{cases}$$

See Figure 6.3 for a graph. Notice that

$$\varphi(t) = M\big(u(t-a) - u(t-b)\big),$$

where $u(t)$ is the unit step function.

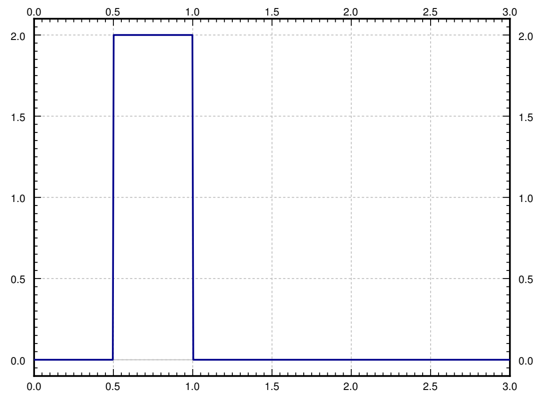

Figure 6.3: Sample square pulse with $a = 0.5$, $b = 1$ and $M = 2$.

Let us take the Laplace transform of a square pulse,

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\big\{M\big(u(t-a) - u(t-b)\big)\big\} = M\frac{e^{-as} - e^{-bs}}{s}.$$

For simplicity let us take $a = 0$, and it is convenient to require

$$\int_0^\infty \varphi(t)\, dt = 1.$$

To achieve this we let $M = {}^{1}\!/\!{}_{b}$. For such a pulse we compute

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\{M\big(u(t-a) - u(t-b)\big)\} = \frac{1-e^{-bs}}{bs}.$$

We generally want that b is very small. That is, we wish to have the pulse be very short and very tall. By letting b go to zero we come to the concept of the Dirac delta function.

6.4.2 The delta function

The *Dirac delta function** is not exactly a function, it is sometimes called a *generalized function*. We avoid unnecessary details and simply say that it is an object that does not really make sense unless you integrate it. The motivation is that we would like a "function" $\delta(t)$ such that for any continuous function $f(t)$ we have

$$\boxed{\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).}$$

The formula should hold if we integrate over any interval that contains 0, not just $(-\infty,\infty)$. So $\delta(t)$ is a "function" with all its "mass" at the single point $t = 0$. In other words, for any interval $[c,d]$

$$\int_c^d \delta(t)\, dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains } 0, \text{ i.e. } c \leq 0 \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

A good way to think about $\delta(t)$ is as a limit of short pulses whose integral is 1. For example, suppose that we have a square pulse $\varphi(t)$ as above with $a = 0$, $M = {}^{1}\!/\!{}_{b}$, that is $\varphi(t) = \frac{u(t)-u(t-b)}{b}$. Compute

$$\int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \int_{-\infty}^{\infty} \frac{u(t)-u(t-b)}{b} f(t)\, dt = \frac{1}{b}\int_0^b f(t)\, dt.$$

If $f(t)$ is continuous at $t = 0$, then for small b, the function $f(t)$ is approximately equal to $f(0)$ on the interval $[0,b]$. We approximate the integral

$$\frac{1}{b}\int_0^b f(t)\, dt \approx \frac{1}{b}\int_0^b f(0)\, dt = f(0).$$

Therefore,

$$\lim_{b\to 0}\int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \lim_{b\to 0}\frac{1}{b}\int_0^b f(t)\, dt = f(0).$$

*Named after the English physicist and mathematician Paul Adrien Maurice Dirac (1902–1984).

Let us therefore accept $\delta(t)$ as an object that is possible to integrate. We often want to shift δ to another point, for example $\delta(t-a)$. In that case we have

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a).$$

Note that $\delta(a-t)$ is the same object as $\delta(t-a)$. In other words, the convolution of $\delta(t)$ with $f(t)$ is again $f(t)$,

$$(f * \delta)(t) = \int_0^t \delta(t-s) f(s)\, ds = f(t).$$

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$\boxed{\mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} e^{-st} \delta(t-a)\, dt = e^{-as}.}$$

In particular,

$$\mathcal{L}\{\delta(t)\} = 1.$$

Remark 6.4.1: Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\, ds = u(t-a).$$

So in a certain sense

$$\text{“} \quad \frac{d}{dt}\big[u(t-a)\big] = \delta(t-a) \quad \text{”}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

Example 6.4.1: Let us compute $\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\}$. So far we have always looked at proper rational functions in the s variable. That is, the numerator was always of lower degree than the denominator. We write,

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\} = \mathcal{L}^{-1}\left\{1 + \frac{1}{s}\right\} = \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \delta(t) + 1.$$

The resulting object is a generalized function and only makes sense when put underneath an integral.

6.4.3 Impulse response

As we said before, in the differential equation $Lx = f(t)$, we think of $f(t)$ as input, and $x(t)$ as the output. Often it is important to find the response to an impulse, and then we use the delta function in place of $f(t)$. The solution to

$$Lx = \delta(t)$$

is called the *impulse response*.

Example 6.4.2: Solve (find the impulse response)

$$x'' + \omega_0^2 x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0. \tag{6.3}$$

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$.

$$s^2X(s) + \omega_0^2 X(s) = 1, \qquad \text{and so} \qquad X(s) = \frac{1}{s^2 + \omega_0^2}.$$

Taking the inverse Laplace transform we obtain

$$x(t) = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Let us notice something about the above example. We have proved before that when the input was $f(t)$, then the solution to $Lx = f(t)$ was given by

$$x(t) = \int_0^t f(\tau)\frac{\sin(\omega_0(t-\tau))}{\omega_0}\, d\tau.$$

Notice that the solution for an arbitrary input is given as convolution with the impulse response. Let us see why. The key is to notice that for functions $x(t)$ and $f(t)$,

$$(x * f)''(t) = \frac{d^2}{dt^2}\left[\int_0^t f(\tau)x(t-\tau)\, d\tau\right] = \int_0^t f(\tau)x''(t-\tau)\, d\tau = (x'' * f)(t).$$

We simply differentiate twice under the integral*, the details are left as an exercise. And so if we convolve the entire equation (6.3), the left hand side becomes

$$(x'' + \omega_0^2 x) * f = (x'' * f) + \omega_0^2(x * f) = (x * f)'' + \omega_0^2(x * f).$$

The right hand side becomes

$$(\delta * f)(t) = f(t).$$

Therefore $y(t) = (x * f)(t)$ is the solution to

$$y'' + \omega_0^2 y = f(t).$$

This procedure works in general for other linear equations $Lx = f(t)$. If you determine the impulse response, you also know how to obtain the output $x(t)$ for any input $f(t)$ by simply convolving the impulse response and the input $f(t)$.

*You should really think of the integral going over $(-\infty, \infty)$ rather than over $[0, t]$ and simply assume that $f(t)$ and $x(t)$ are continuous and zero for negative t.

6.4.4 Three-point beam bending

Let us give another quite different example where delta functions turn up. In this case representing point loads on a steel beam. That is suppose we have a beam of length L, resting on two simple supports at the ends. Let x denote the position on the beam, and let $y(x)$ denote the deflection of the beam in the vertical direction. The deflection $y(x)$ satisfies the *Euler-Bernoulli equation*[*],

$$EI\frac{d^4y}{dx^4} = F(x),$$

where E and I are constants[†] and $F(x)$ is the force applied per unit length at position x. The situation we are interested in is when the force is applied at a single point as in Figure 6.4.

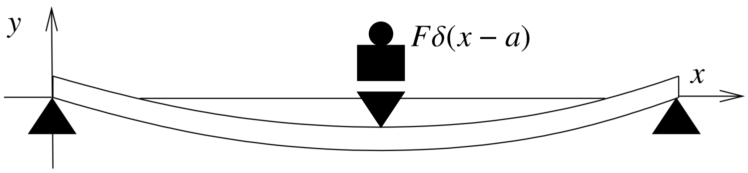

Figure 6.4: Three-point bending.

In this case the equation becomes

$$EI\frac{d^4y}{dx^4} = -F\delta(x-a),$$

where $x = a$ is the point where the mass is applied. F is the force applied and the minus sign indicates that the force is downward in the negative y direction. The end points of the beam satisfy the conditions,

$$\begin{aligned} y(0) &= 0, \qquad & y''(0) &= 0, \\ y(L) &= 0, \qquad & y''(L) &= 0. \end{aligned}$$

See § 5.2, for further information about endpoint conditions applied to beams.

Example 6.4.3: Suppose that length of the beam is 2, and suppose that $EI = 1$ for simplicity. Further suppose that the force $F = 1$ is is applied at $x = 1$. That is, we have the equation

$$\frac{d^4y}{dx^4} = -\delta(x-1),$$

[*]Named for the Swiss mathematicians Jacob Bernoulli (1654 – 1705), Daniel Bernoulli —nephew of Jacob— (1700 – 1782), and Leonhard Paul Euler (1707 – 1783).

[†]E is the elastic modulus and I is the second moment of area. Let us not worry about the details and simply think of these as constants.

and the endpoint conditions are

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(2) = 0, \qquad y''(2) = 0.$$

We could integrate, but using the Laplace transform is even easier. We apply the transform in the x variable rather than the t variable. Let us again denote the transform of $y(x)$ as $Y(s)$.

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property (6.1) to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) + C_1 x + \frac{C_2}{6} x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6} 2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3} C_2.$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6} 3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = \frac{-1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

6.4.5 Exercises

Exercise* 6.4.1*: *Solve (find the impulse response)* $x'' + x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.2*: *Solve (find the impulse response)* $x'' + 2x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.3*: *A pulse can come later and can be bigger. Solve* $x'' + 4x = 4\delta(t-1)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.4*: *Suppose that* $f(t)$ *and* $g(t)$ *are differentiable functions and suppose that* $f(t) = g(t) = 0$ *for all* $t \leq 0$. *Show that*

$$(f * g)'(t) = (f' * g)(t) = (f * g')(t).$$

Exercise* 6.4.5*: *Suppose that* $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, *has the solution* $x = e^{-t}$ *for* $t > 0$. *Find the solution to* $Lx = t^2$, $x(0) = 0$, $x'(0) = 0$ *for* $t > 0$.

Exercise* 6.4.6*: *Compute* $\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^2}\right\}$.

Exercise* 6.4.7** (challenging):*** *Solve Example 6.4.3 via integrating 4 times in the x variable.*

Exercise* 6.4.8*: *Suppose we have a beam of length* 1 *simply supported at the ends and suppose that force* $F = 1$ *is applied at* $x = \frac{3}{4}$ *in the downward direction. Suppose that* $EI = 1$ *for simplicity. Find the beam deflection* $y(x)$.

Exercise* 6.4.101*: *Solve (find the impulse response)* $x'' = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.102*: *Solve (find the impulse response)* $x' + ax = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.103*: *Suppose that* $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, *has the solution* $x(t) = \cos(t)$ *for* $t > 0$. *Find (in closed form) the solution to* $Lx = \sin(t)$, $x(0) = 0$, $x'(0) = 0$ *for* $t > 0$.

Exercise* 6.4.104*: *Compute* $\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\}$.

Exercise* 6.4.105*: *Compute* $\mathcal{L}^{-1}\left\{\frac{3s^2e^{-s}+2}{s^2}\right\}$.

6.3.101: $\frac{1}{2}(\cos t + \sin t - e^{-t})$

6.3.102: $5t - 5\sin t$

6.3.103: $\frac{1}{2}(\sin t - t\cos t)$

6.3.104: $\int_0^t f(\tau)(1 - \cos(t - \tau))\, d\tau$

6.4.101: $x(t) = t$

6.4.102: $x(t) = e^{-at}$

6.4.103: $x(t) = (\cos * \sin)(t) = \frac{1}{2} t \sin(t)$

6.4.104: $\delta(t) - \sin(t)$

6.4.105: $3\delta(t - 1) + 2t$

7.1.101: Yes. Radius of convergence is 10.

7.1.102: Yes. Radius of convergence is e.

7.1.103: $\frac{1}{1-x} = -\frac{1}{1-(2-x)}$ so $\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1}(x - 2)^n$, which converges for $1 < x < 3$.

7.1.104: $\sum_{n=7}^{\infty} \frac{1}{(n-7)!} x^n$

7.1.105: $f(x) - g(x)$ is a polynomial. Hint: Use Taylor series.

7.2.101: $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, recurrence relation (for $k \geq 5$): $a_k = -2a_{k-5}$, so:
$y(x) = a_0 + a_1 x - 2a_0 x^5 - 2a_1 x^6 + 4a_0 x^{10} + 4a_1 x^{11} - 8a_0 x^{15} - 8a_1 x^{16} + \cdots$

7.2.102: a) $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = a_{k-3} + 1$, so
$y(x) = a_0 + a_1 x + \frac{1}{2}x^2 + (a_0+1)x^3 + (a_1+1)x^4 + \frac{3}{2}x^5 + (a_0+2)x^6 + (a_1+2)x^7 + \frac{5}{2}x^8 + (a_0+3)x^9 + (a_1+3)x^{10} + \cdots$
b) $y(x) = \frac{1}{2}x^2 + x^3 + x^4 + \frac{3}{2}x^5 + 2x^6 + 2x^7 + \frac{5}{2}x^8 + 3x^9 + 3x^{10} + \cdots$

7.2.103: Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.