Chapter 8

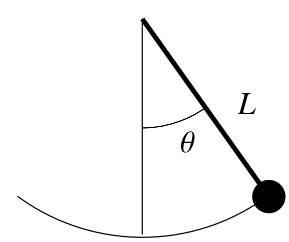
Nonlinear systems

8.1 Linearization, critical points, and equilibria

Note: 1 lecture, §6.1–§6.2 in [EP], §9.2–§9.3 in [BD]

Except for a few brief detours in chapter 1, we considered mostly linear equations. Linear equations suffice in many applications, but in reality most phenomena require nonlinear equations. Nonlinear equations, however, are notoriously more difficult to understand than linear ones, and many strange new phenomena appear when we allow our equations to be nonlinear.

Not to worry, we did not waste all this time studying linear equations. Nonlinear equations can often be approximated by linear ones if we only need a solution "locally," for example, only for a short period of time, or only for certain parameters. Understanding linear equations can also give us qualitative understanding about a more general nonlinear problem. The idea is similar to what you did in calculus in trying to approximate a function by a line with the right slope.



In § 2.4 we looked at the pendulum of length L. The goal was to solve for the angle $\theta(t)$ as a function of the time t. The equation for the setup is the nonlinear equation

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

Instead of solving this equation, we solved the rather easier linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

While the solution to the linear equation is not exactly what we were looking for, it is rather close to the original, as long as the angle θ is small and the time period involved is short.

You might ask: Why don't we just solve the nonlinear problem? Well, it might be very difficult, impractical, or impossible to solve analytically, depending on the equation in question. We may not even be interested in the actual solution, we might only be interested in some qualitative idea of what the solution is doing. For example, what happens as time goes to infinity?

8.1.1 Autonomous systems and phase plane analysis

We restrict our attention to a two dimensional autonomous system

$$x' = f(x, y), \qquad y' = g(x, y),$$

where $f(x, y)$ and $g(x, y)$ are functions of two variables, and the derivatives are taken with respect to time t. Solutions are functions $x(t)$ and $y(t)$ such that

$$x'(t) = f\big(x(t), y(t)\big), \qquad y'(t) = g\big(x(t), y(t)\big).$$

The way we will analyze the system is very similar to § 1.6, where we studied a single autonomous equation. The ideas in two dimensions are the same, but the behavior can be far more complicated.

It may be best to think of the system of equations as the single vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}. \tag{8.1}$$

As in § 3.1 we draw the *phase portrait* (or *phase diagram*), where each point (x, y) corresponds to a specific state of the system. We draw the *vector field* given at each point (x, y) by the vector $\left[\begin{smallmatrix} f(x,y) \\ g(x,y) \end{smallmatrix}\right]$. And as before if we find solutions, we draw the trajectories by plotting all points $\big(x(t), y(t)\big)$ for a certain range of t.

Example 8.1.1: Consider the second order equation $x'' = -x + x^2$. Write this equation as a first order nonlinear system

$$x' = y, \qquad y' = -x + x^2.$$

The phase portrait with some trajectories is drawn in Figure 8.1.

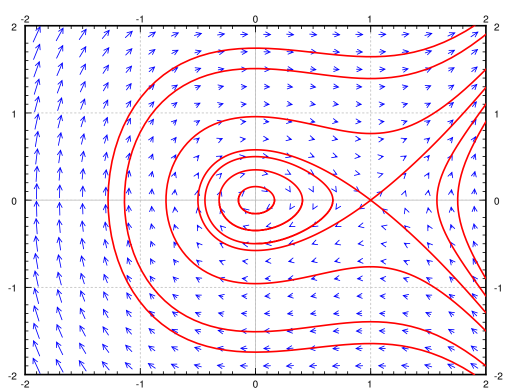


Figure 8.1: Phase portrait with some trajectories of $x' = y$, $y' = -x + x^2$.

From the phase portrait it should be clear that even this simple system has fairly complicated behavior. Some trajectories keep oscillating around the origin, and some go off towards infinity. We will return to this example often, and analyze it completely in this (and the next) section.

Let us concentrate on those points in the phase diagram above where the trajectories seem to start, end, or go around. We see two such points: $(0,0)$ and $(1,0)$. The trajectories seem to go around the point $(0,0)$, and they seem to either go in or out of the point $(1,0)$. These points are precisely those points where the derivatives of both x and y are zero. Let us define the *critical points* as the points (x, y) such that

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{0}.$$

In other words, the points where both $f(x, y) = 0$ and $g(x, y) = 0$.

The critical points are where the behavior of the system is in some sense the most complicated. If $\left[\begin{smallmatrix} f(x,y) \\ g(x,y) \end{smallmatrix}\right]$ is zero, then nearby, the vector can point in any direction whatsoever. Also, the trajectories are either going towards, away from, or around these points, so if we are looking for long term behavior of the system, we should look at what happens there.

Critical points are also sometimes called *equilibria*, since we have so-called *equilibrium solutions* at critical points. If (x_0, y_0) is a critical point, then we have the solutions

$$x(t) = x_0, \quad y(t) = y_0.$$

In Example 8.1.1 on the facing page, there are two equilibrium solutions:

$$x(t) = 0, \quad y(t) = 0, \qquad \text{and} \qquad x(t) = 1, \quad y(t) = 0.$$

Compare this discussion on equilibria to the discussion in § 1.6. The underlying concept is exactly the same.

8.1.2 Linearization

In § 3.5 we studied the behavior of a homogeneous linear system of two equations near a critical point. For a linear system of two variables the only critical point is generally the origin $(0,0)$. Let us put the understanding we gained in that section to good use understanding what happens near critical points of nonlinear systems.

In calculus we learned to estimate a function by taking its derivative and linearizing. We work similarly with nonlinear systems of ODE. Suppose (x_0, y_0) is a critical point. First change variables to (u, v), so that $(u, v) = (0,0)$ corresponds to (x_0, y_0). That is,

$$u = x - x_0, \qquad v = y - y_0.$$

Next we need to find the derivative. In multivariable calculus you may have seen that the several variables version of the derivative is the *Jacobian matrix**. The Jacobian matrix of the vector-valued function $\left[\begin{smallmatrix} f(x,y) \\ g(x,y) \end{smallmatrix}\right]$ at (x_0, y_0) is

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix}.$$

*Named for the German mathematician Carl Gustav Jacob Jacobi (1804 – 1851).

This matrix gives the best linear approximation as u and v (and therefore x and y) vary. We define the *linearization* of the equation (8.1) as the linear system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Example 8.1.2: Let us keep with the same equations as Example 8.1.1: $x' = y$, $y' = -x + x^2$. There are two critical points, $(0, 0)$ and $(1, 0)$. The Jacobian matrix at any point is

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x & 0 \end{bmatrix}.$$

Therefore at $(0, 0)$ the linearization is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

where $u = x$ and $v = y$.

At the point $(1, 0)$, we have $u = x - 1$ and $v = y$, and the linearization is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

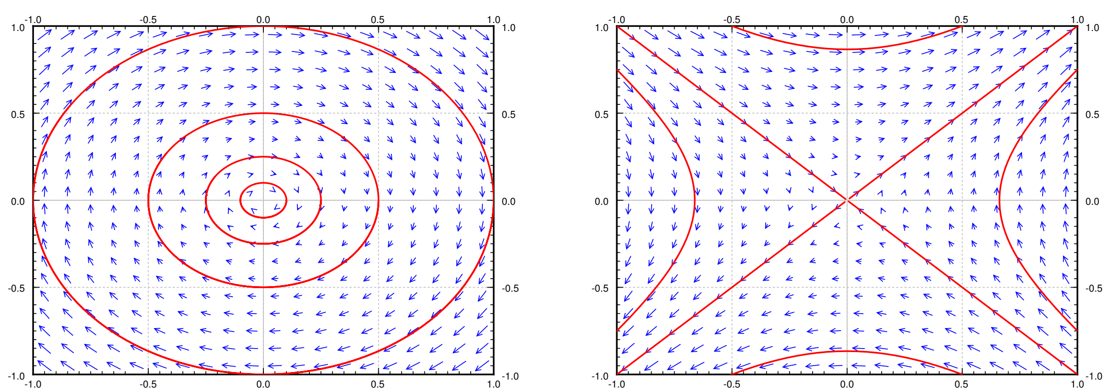


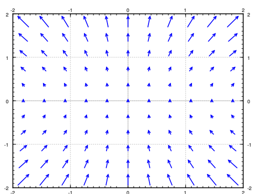
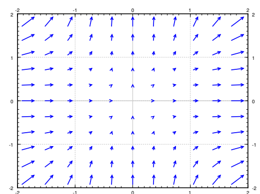
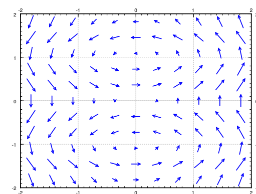
Figure 8.2: Phase diagram with some trajectories of linearizations at the critical points $(0, 0)$ (left) and $(0, 1)$ (right) of $x' = y$, $y' = -x + x^2$.

The phase diagrams of the two linearizations at the point $(0, 0)$ and $(0, 1)$ are given in Figure 8.2. Note that the variables are now u and v. Compare Figure 8.2 with Figure 8.1 on page 302, and look especially at the behavior near the critical points.

8.1.3 Exercises

Exercise* 8.1.1*: *Sketch the phase plane vector field for:*
a) $x' = x^2$, $y' = y^2$,
b) $x' = (x - y)^2$, $y' = -x$,
c) $x' = e^y$, $y' = e^x$.

Exercise* 8.1.2*: *Match systems*
1) $x' = x^2$, $y' = y^2$, *2)* $x' = xy$, $y' = 1 + y^2$, *3)* $x' = \sin(\pi y)$, $y' = x$,
to the vector fields below. Justify.

a)  *b)*  *c)* 

Exercise* 8.1.3*: *Find the critical points and linearizations of the following systems.*
a) $x' = x^2 - y^2$, $y' = x^2 + y^2 - 1$,
b) $x' = -y$, $y' = 3x + yx^2$,
c) $x' = x^2 + y$, $y' = y^2 + x$.

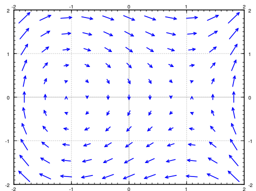
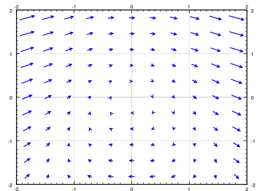
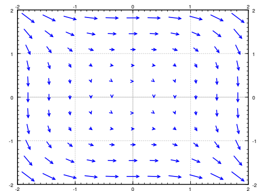
Exercise* 8.1.4*: *For the following systems, verify they have critical point at* $(0,0)$*, and find the linearization at* $(0,0)$.
a) $x' = x + 2y + x^2 - y^2$, $y' = 2y - x^2$
b) $x' = -y$, $y' = x - y^3$
c) $x' = ax + by + f(x,y)$, $y' = cx + dy + g(x,y)$*, where* $f(0,0) = 0$, $g(0,0) = 0$*, and all first partial derivatives of* f *and* g *are also zero at* $(0,0)$*, that is,* $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = \frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0$.

Exercise* 8.1.5*: *Take* $x' = (x - y)^2$, $y' = (x + y)^2$.
a) Find the set of critical points.
b) Sketch a phase diagram and describe the behavior near the critical point(s).
c) Find the linearization. Is it helpful in understanding the system?

Exercise* 8.1.6*: *Take* $x' = x^2$, $y' = x^3$.
a) Find the set of critical points.
b) Sketch a phase diagram and describe the behavior near the critical point(s).
c) Find the linearization. Is it helpful in understanding the system?

Exercise* 8.1.101*: *Find the critical points and linearizations of the following systems.*
a) $x' = \sin(\pi y) + (x - 1)^2$, $y' = y^2 - y$,
b) $x' = x + y + y^2$, $y' = x$,
c) $x' = (x - 1)^2 + y$, $y' = x^2 + y$.

Exercise* 8.1.102*: *Match systems*
1) $x' = y^2$, $y' = -x^2$, *2)* $x' = y$, $y' = (x-1)(x+1)$, *3)* $x' = y + x^2$, $y' = -x$,
to the vector fields below. Justify.

a)  *b)*  *c)* 

Exercise* 8.1.103*: *The idea of critical points and linearization works in higher dimensions as well. You simply make the Jacobian matrix bigger by adding more functions and more variables. For the following system of 3 equations find the critical points and their linearizations:*
$x' = x + z^2$,
$y' = z^2 - y$,
$z' = z + x^2$.

Exercise* 8.1.104*: *Any two-dimensional non-autonomous system* $x' = f(x, y, t)$, $y' = g(x, y, t)$ *can be written as a three-dimensional autonomous system (three equations). Write down this autonomous system using the variables u, v, w.*

8.2 Stability and classification of isolated critical points

Note: 2 lectures, §6.1 – §6.2 in [EP], §9.2–§9.3 in [BD]

8.2.1 Isolated critical points and almost linear systems

A critical point is *isolated* if it is the only critical point in some small "neighborhood" of the point. That is, if we zoom in far enough it is the only critical point we see. In the above example, the critical point was isolated. If on the other hand there would be a whole curve of critical points, then it would not be isolated.

A system is called *almost linear* (at a critical point (x_0, y_0)) if the critical point is isolated and the Jacobian at the point is invertible, or equivalently if the linearized system has an isolated critical point. In such a case, the nonlinear terms will be very small and the system will behave like its linearization, at least if we are close to the critical point.

In particular the system we have just seen in Examples 8.1.1 and 8.1.2 has two isolated critical points $(0, 0)$ and $(0, 1)$, and is almost linear at both critical points as both of the Jacobian matrices $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ are invertible.

On the other hand a system such as $x' = x^2$, $y' = y^2$ has an isolated critical point at $(0, 0)$, however the Jacobian matrix

$$\begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

is zero when $(x, y) = (0, 0)$. Therefore the system is not almost linear. Even a worse example is the system $x' = x$, $y' = x^2$, which does not have an isolated critical point, as x' and y' are both zero whenever $x = 0$, that is, the entire y axis.

Fortunately, most often critical points are isolated, and the system is almost linear at the critical points. So if we learn what happens here, we have figured out the majority of situations that arise in applications.

8.2.2 Stability and classification of isolated critical points

Once we have an isolated critical point, the system is almost linear at that critical point, and we computed the associated linearized system, we can classify what happens to the solutions. We more or less use the classification for linear two-variable systems from § 3.5, with one minor caveat. Let us list the behaviors depending on the eigenvalues of the Jacobian matrix at the critical point in Table 8.1 on the next page. This table is very similar to Table 3.1 on page 113, with the exception of missing "center" points. We will discuss centers later, as they are more complicated.

In the new third column, we have marked points as *asymptotically stable* or *unstable*. Formally, a *stable critical point* (x_0, y_0) is one where given any small distance ϵ to (x_0, y_0), and any initial condition within a perhaps smaller radius around (x_0, y_0), the trajectory of the system will never go further away from (x_0, y_0) than ϵ. An *unstable critical point* is one that is not stable. Informally, a

Eigenvalues of the Jacobian matrix	Behavior	Stability
real and both positive	source / unstable node	unstable
real and both negative	sink / stable node	asymptotically stable
real and opposite signs	saddle	unstable
complex with positive real part	spiral source	unstable
complex with negative real part	spiral sink	asymptotically stable

Table 8.1: Behavior of an almost linear system near an isolated critical point.

point is stable if we start close to a critical point and follow a trajectory we will either go towards, or at least not get away from, this critical point.

A stable critical point (x_0, y_0) is called *asymptotically stable* if given any initial condition sufficiently close to (x_0, y_0) and any solution $\big(x(t), y(t)\big)$ given that condition, then

$$\lim_{t\to\infty} \big(x(t), y(t)\big) = (x_0, y_0).$$

That is, the critical point is asymptotically stable if any trajectory for a sufficiently close initial condition goes towards the critical point (x_0, y_0).

Example 8.2.1: Consider $x' = -y - x^2$, $y' = -x + y^2$. See Figure 8.3 on the next page for the phase diagram. Let us find the critical points. These are the points where $-y - x^2 = 0$ and $-x + y^2 = 0$. The first equation means $y = -x^2$, and so $y^2 = x^4$. Plugging into the second equation we obtain $-x + x^4 = 0$. Factoring we obtain $x(1 - x^3) = 0$. Since we are looking only for real solutions we get either $x = 0$ or $x = 1$. Solving for the corresponding y using $y = -x^2$, we get two critical points, one being $(0, 0)$ and the other being $(1, -1)$. Clearly the critical points are isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} -2x & -1 \\ -1 & 2y \end{bmatrix}.$$

At the point $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$ and so the two eigenvalues are 1 and -1. As the matrix is invertible, the system is almost linear at $(0, 0)$. As the eigenvalues are real and of opposite signs, we get a saddle point, which is an unstable equilibrium point.

At the point $(1, -1)$ we get the matrix $\left[\begin{smallmatrix} -2 & -1 \\ -1 & -2 \end{smallmatrix}\right]$ and computing the eigenvalues we get -1, -3. The matrix is invertible, and so the system is almost linear at $(1, -1)$. As we have real eigenvalues both negative, the critical point is a sink, and therefore an asymptotically stable equilibrium point. That is, if we start with any point (x_i, y_i) close to $(1, -1)$ as an initial condition and plot a trajectory, it will approach $(1, -1)$. In other words,

$$\lim_{t\to\infty} \big(x(t), y(t)\big) = (1, -1).$$

As you can see from the diagram, this behavior is true even for some initial points quite far from $(1, -1)$, but it is definitely not true for all initial points.

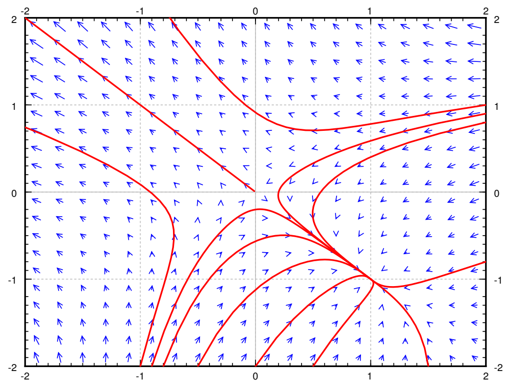


Figure 8.3: The phase portrait with few sample trajectories of $x' = -y - x^2$, $y' = -x + y^2$.

Example 8.2.2: Let us look at $x' = y + y^2 e^x$, $y' = x$. First let us find the critical points. These are the points where $y + y^2 e^x = 0$ and $x = 0$. Simplifying we get $0 = y + y^2 = y(y + 1)$. So the critical points are $(0, 0)$ and $(0, -1)$, and hence are isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} y^2 e^x & 1 + 2ye^x \\ 1 & 0 \end{bmatrix}.$$

At the point $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and so the two eigenvalues are 1 and -1. As the matrix is invertible, the system is almost linear at $(0, 0)$. And, as the eigenvalues are real and of opposite signs, we get a saddle point, which is an unstable equilibrium point.

At the point $(0, -1)$ we get the matrix $\left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ whose eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The matrix is invertible, and so the system is almost linear at $(0, -1)$. As we have complex eigenvalues with positive real part, the critical point is a spiral source, and therefore an unstable equilibrium point.

See Figure 8.4 on the following page for the phase diagram. Notice the two critical points, and the behavior of the arrows in the vector field around these points.

8.2.3 The trouble with centers

Recall, a linear system with a center meant that trajectories travelled in closed elliptical orbits in some direction around the critical point. Such a critical point we would call a *center* or a *stable center*. It would not be an asymptotically stable critical point, as the trajectories would never approach the critical point, but at least if you start sufficiently close to the critical point, you will stay close to the critical point. The simplest example of such behavior is the linear system with a center. Another example is the critical point $(0, 0)$ in Example 8.1.1 on page 302.

The trouble with a center in a nonlinear system is that whether the trajectory goes towards or away from the critical point is governed by the sign of the real part of the eigenvalues of the Jacobian.

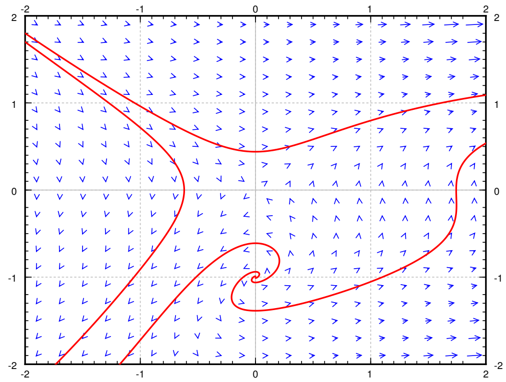


Figure 8.4: The phase portrait with few sample trajectories of $x' = y + y^2 e^x$, $y' = x$.

Since this real part is zero at the critical point itself, it can have either sign nearby, meaning the trajectory could be pulled towards or away from the critical point.

Example 8.2.3: An easy example where such a problematic behavior is exhibited is the system $x' = y, y' = -x + y^3$. The only critical point is the origin $(0,0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 3y^2 \end{bmatrix}.$$

At $(0,0)$ the Jacobian matrix is $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, which has eigenvalues $\pm i$. Therefore, the linearization has a center.

Using the quadratic equation, the eigenvalues of the Jacobian matrix at any point (x, y) are

$$\lambda = \frac{3}{2}y^2 \pm i\frac{\sqrt{4 - 9y^4}}{2}.$$

At any point where $y \neq 0$ (so at most points near the origin), the eigenvalues have a positive real part (y^2 can never be negative). This positive real part will pull the trajectory away from the origin. A sample trajectory for an initial condition near the origin is given in Figure 8.5 on the facing page.

The moral of the example is that further analysis is needed when the linearization has a center. The analysis will in general be more complicated than in the above example, and is more likely to involve case-by-case consideration. Such a complication should not be surprising to you. By now in your mathematical career, you have seen many places where a simple test is inconclusive, perhaps starting with the second derivative test for maxima or minima, and requires more careful, and perhaps ad hoc analysis of the situation.

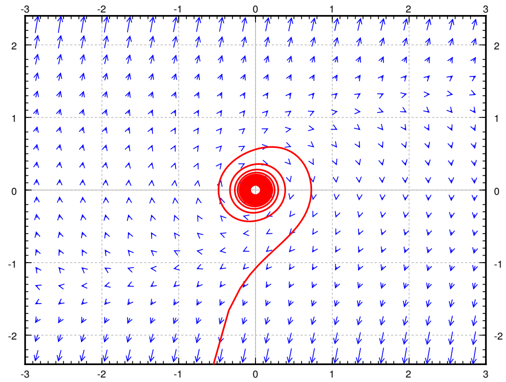


Figure 8.5: An unstable critical point (spiral source) at the origin for $x' = y, y' = -x + y^3$, even if the linearization has a center.

8.2.4 Conservative equations

An equation of the form

$$x'' + f(x) = 0$$

for an arbitrary function $f(x)$ is called a *conservative equation*. For example the pendulum equation is a conservative equation. The equations are conservative as there is no friction in the system so the energy in the system is "conserved." Let us write this equation as a system of nonlinear ODE.

$$x' = y, \qquad y' = -f(x).$$

These types of equations have the advantage that we can solve for their trajectories easily.

The trick is to first think of y as a function of x for a moment. Then use the chain rule

$$x'' = y' = y\frac{dy}{dx},$$

where the prime indicates a derivative with respect to t. We obtain $y\frac{dy}{dx} + f(x) = 0$. We integrate with respect to x to get $\int y\frac{dy}{dx}\,dx + \int f(x)\,dx = C$. In other words

$$\frac{1}{2}y^2 + \int f(x)\,dx = C.$$

We obtained an implicit equation for the trajectories, with different C giving different trajectories. The value of C is conserved on any trajectory. This expression is sometimes called the *Hamiltonian* or the energy of the system

Example 8.2.4: Let us find the trajectories for the equation $x'' + x - x^2 = 0$, which is the equation from Example 8.1.1 on page 302. The corresponding first order system is

$$x' = y, \qquad y' = -x + x^2.$$

Trajectories satisfy

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3 = C.$$

We solve for y

$$y = \pm\sqrt{-x^2 + \frac{2}{3}x^3 + 2C}.$$

Plotting these graphs we get exactly the trajectories in Figure 8.1 on page 302. In particular we notice that near the origin the trajectories are *closed curves*: they keep going around the origin, never spiraling in or out. Therefore we discovered a way to verify that the critical point at $(0, 0)$ is a stable center. The critical point at $(0, 1)$ is a saddle as we already noticed. This example is typical for conservative equations.

Consider an arbitrary conservative equation. The trajectories are given by

$$y = \pm\sqrt{-2\int f(x)\,dx + 2C}.$$

So all trajectories are mirrored across the x-axis. In particular, there can be no spiral sources nor sinks. All critical points occur when $y = 0$ (the x-axis), that is when $x' = 0$. The critical points are simply those points on the x-axis where $f(x) = 0$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -f'(x) & 0 \end{bmatrix}.$$

So the critical point is almost linear if $f'(x) \neq 0$ at the critical point. Let J denote the Jacobian matrix, then the eigenvalues of J are solutions to

$$0 = \det(J - \lambda I) = \lambda^2 + f'(x).$$

Therefore $\lambda = \pm\sqrt{-f'(x)}$. In other words, either we get real eigenvalues of opposite signs, or we get purely imaginary eigenvalues. There are only two possibilities for critical points, either an unstable saddle point, or a stable center. There are never any asymptotically stable points.

8.2.5 Exercises

Exercise* 8.2.1*: *For the systems below, find and classify the critical points, also indicate if the equilibria are stable, asymptotically stable, or unstable.*

a) $x' = -x + 3x^2, y' = -y$ *b)* $x' = x^2 + y^2 - 1,\ y' = x$ *c)* $x' = ye^x,\ y' = y - x + y^2$

Exercise* 8.2.2*: *Find the implicit equations of the trajectories of the following conservative systems. Next find their critical points (if any) and classify them.*
a) $x'' + x + x^3 = 0$ *b)* $\theta'' + \sin\theta = 0$ *c)* $z'' + (z-1)(z+1) = 0$ *d)* $x'' + x^2 + 1 = 0$

Exercise* 8.2.3*: *Find and classify the critical point(s) of* $x' = -x^2$, $y' = -y^2$.

Exercise* 8.2.4*: *Suppose* $x' = -xy$, $y' = x^2 - 1 - y$. *a) Show there are two spiral sinks at* $(-1,0)$ *and* $(1,0)$. *b) For any initial point of the form* $(0,y_0)$*, find what is the trajectory. c) Can a trajectory starting at* (x_0,y_0) *where* $x_0 > 0$ *spiral into the critical point at* $(-1,0)$*? Why or why not?*

Exercise* 8.2.5*: *In the example* $x' = y$, $y' = y^3 - x$ *show that for any trajectory, the distance from the origin is an increasing function. Conclude that the origin behaves like is a spiral source. Hint: Consider* $f(t) = \big(x(t)\big)^2 + \big(y(t)\big)^2$ *and show it has positive derivative.*

Exercise* 8.2.6*: *Suppose* f *is always positive. Find the trajectories of* $x'' + f(x') = 0$*. Are there any critical points?*

Exercise* 8.2.7*: *Suppose that* $x' = f(x,y)$, $y' = g(x,y)$*. Suppose that* $g(x,y) > 1$ *for all* x *and* y*. Are there any critical points? What can we say about the trajectories at* t *goes to infinity?*

Exercise* 8.2.101*: *For the systems below, find and classify the critical points.*
a) $x' = -x + x^2$, $y' = y$ *b)* $x' = y - y^2 - x$, $y' = -x$ *c)* $x' = xy$, $y' = x + y - 1$

Exercise* 8.2.102*: *Find the implicit equations of the trajectories of the following conservative systems. Next find their critical points (if any) and classify them.*
a) $x'' + x^2 = 4$ *b)* $x'' + e^x = 0$ *c)* $x'' + (x+1)e^x = 0$

Exercise* 8.2.103*: *The conservative system* $x'' + x^3 = 0$ *is not almost linear. Classify its critical point(s) nonetheless.*

Exercise* 8.2.104*: *Derive an analogous classification of critical points for equations in one dimension, such as* $x' = f(x)$ *based on the derivative. A point* x_0 *is critical when* $f(x_0) = 0$ *and almost linear if in addition* $f'(x_0) \neq 0$*. Figure out if the critical point is stable or unstable depending on the sign of* $f'(x_0)$*. Explain. Hint: see § 1.6.*

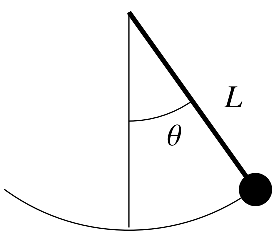
8.3 Applications of nonlinear systems

Note: 2 lectures, §6.3 – §6.4 in [EP], §9.3, §9.5 in [BD]

In this section we will study two very standard examples of nonlinear systems. First, we will look at the nonlinear pendulum equation. We saw the pendulum equation's linearization before, but we noted it was only valid for small angles and short times. Now we will find out what happens for large angles. Next, we will look at the predator-prey equation, which finds various applications in modeling problems in biology, chemistry, economics and elsewhere.

8.3.1 Pendulum

The first example we will study is the pendulum equation $\theta'' + \frac{g}{L}\sin\theta = 0$. Here, θ is the angular displacement, g is the gravitational constant, and L is the length of the pendulum. In this equation we disregard friction, so we are talking about an idealized pendulum.



As we have mentioned before, this equation is a conservative equation, so we will be able to use our analysis of conservative equations from the previous section. Let us change the equation to a two-dimensional system in variables (θ, ω) by introducing the new variable ω:

$$\begin{bmatrix}\theta\\ \omega\end{bmatrix}' = \begin{bmatrix}\omega\\ -\frac{g}{L}\sin\theta\end{bmatrix}.$$

The critical points of this system are when $\omega = 0$ and $-\frac{g}{L}\sin\theta = 0$, or in other words if $\sin\theta = 0$. So the critical points are when $\omega = 0$ and θ is a multiple of π. That is the points are $\ldots(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)\ldots$. While there are infinitely many critical points, they are all isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix}\frac{\partial}{\partial\theta}\big(\omega\big) & \frac{\partial}{\partial\omega}\big(\omega\big)\\ \frac{\partial}{\partial\theta}\big(-\frac{g}{L}\sin\theta\big) & \frac{\partial}{\partial\omega}\big(-\frac{g}{L}\sin\theta\big)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{L}\cos\theta & 0\end{bmatrix}.$$

For conservative equations, there are two types of critical points. Either stable centers, or saddle points. The eigenvalues of the Jacobian are $\lambda = \pm\sqrt{-\frac{g}{L}\cos\theta}$.

The eigenvalues are going to be real when $\cos\theta < 0$. This happens at the odd multiples of π. The eigenvalues are going to be purely imaginary when $\cos\theta > 0$. This happens at the even multiples of π. Therefore the system has a stable center at the points $\ldots(-2\pi, 0), (0, 0), (2\pi, 0)\ldots$, and it has an unstable saddle at the points $\ldots(-3\pi, 0), (-\pi, 0), (\pi, 0), (3\pi, 0)\ldots$. Look at the phase diagram in Figure 8.6 on the next page, where for simplicity we let $\frac{g}{L} = 1$.

In the linearized equation we only had a single critical point, the center at $(0, 0)$. We can now see more clearly what we meant when we said the linearization was good for small angles. The horizontal axis is the deflection angle. The vertical axis is the angular velocity of the pendulum.

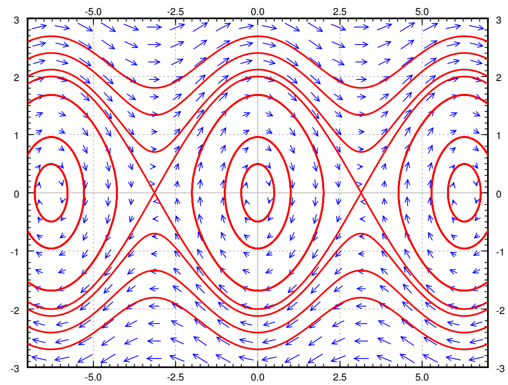


Figure 8.6: Phase plane diagram and some trajectories of the nonlinear pendulum equation.

Suppose we start at $\theta = 0$ (no deflection), and we start with a small angular velocity ω. Then the trajectory keeps going around the critical point $(0, 0)$ in an approximate circle. This corresponds to short swings of the pendulum back and forth. When θ stays small, the trajectories really look like circles and hence are very close to our linearization.

When we give the pendulum a big enough push, it will go across the top and keep spinning about its axis. This behavior corresponds to the wavy curves that do not cross the horizontal axis in the phase diagram. Let us suppose we look at the top curves, when the angular velocity ω is large and positive. Then the pendulum is going around and around its axis. The velocity is going to be large when the pendulum is near the bottom, and the velocity is the smallest when the pendulum is close to the top of its loop.

At each critical point, there is an equilibrium solution. The solution $\theta = 0$ is a stable solution. That is when the pendulum is not moving and is hanging straight down. Clearly this is a stable place for the pendulum to be, hence this is a *stable* equilibrium.

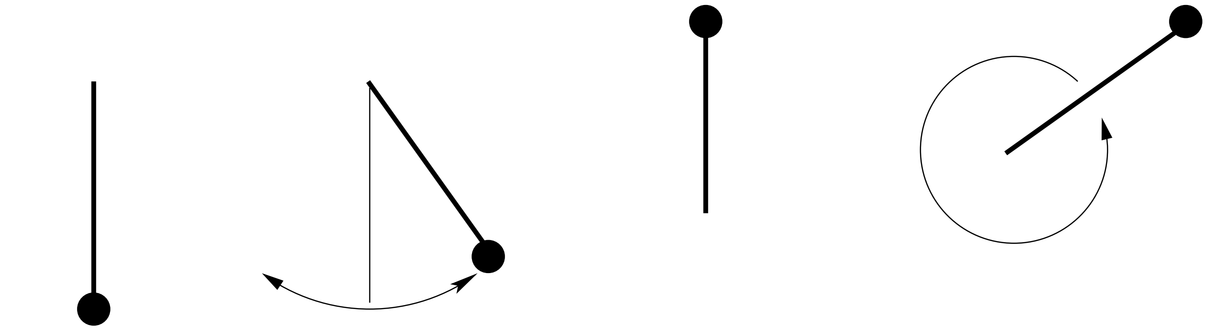
The other type of equilibrium solution is at the unstable point, for example $\theta = \pi$. Here the pendulum is upside down. Sure you can balance the pendulum this way and it will stay, but this is an *unstable* equilibrium. Even the tiniest push will make the pendulum start swinging wildly.

See Figure 8.7 on the following page for a diagram. The first picture is the stable equilibrium $\theta = 0$. The second picture corresponds to those "almost circles" in the phase diagram around $\theta = 0$ when the angular velocity is small. The next picture is the unstable equilibrium $\theta = \pi$. The last picture corresponds to the wavy lines for large angular velocities.

The quantity

$$\frac{1}{2}\omega^2 - \frac{g}{L}\cos\theta$$

is conserved by any solution. This is the energy or the Hamiltonian of the system.



$\theta = 0$ Small angular velocities $\theta = \pi$ Large angular velocities

Figure 8.7: Various possibilities for the motion of the pendulum.

We have a conservative equation and so (exercise) the trajectories are given by

$$\omega = \pm \sqrt{\frac{2g}{L} \cos\theta + C},$$

for various values of C. Let us look at the initial condition of $(\theta_0, 0)$, that is, we take the pendulum to angle θ_0, and just let it go (initial angular velocity 0). We plug the initial conditions into the above and solve for C to obtain

$$C = -\frac{2g}{L} \cos\theta_0.$$

Thus the expression for the trajectory is

$$\omega = \pm \sqrt{\frac{2g}{L}} \sqrt{\cos\theta - \cos\theta_0}.$$

Let us figure out the period. That is, the time it takes for the pendulum to swing back and forth. We notice that the oscillation about the origin in the phase plane is symmetric about both the θ and the ω axis. That is, in terms of θ, the time it takes from θ_0 to $-\theta_0$ is the same as it takes from $-\theta_0$ back to θ_0. Furthermore, the time it takes from $-\theta_0$ to 0 is the same as to go from 0 to θ_0. Therefore, let us find how long it takes for the pendulum to go from angle 0 to angle θ_0, which is a quarter of the full oscillation and then multiply by 4.

We figure out this time by finding $\frac{dt}{d\theta}$ and integrating from 0 to ω_0. The period is four times this integral. Let us stay in the region where ω is positive. Since $\omega = \frac{d\theta}{dt}$, inverting we get

$$\frac{dt}{d\theta} = \sqrt{\frac{L}{2g}} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}.$$

Therefore the period T is given by

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta.$$

The integral is an improper integral, and we cannot in general evaluate it symbolically. We must resort to numerical approximation if we want to compute a particular T.

Recall from § 2.4, the linearized equation $\theta'' + \frac{g}{L}\theta = 0$ has period

$$T_{\text{linear}} = 2\pi\sqrt{\frac{L}{g}}.$$

We plot T, T_{linear}, and the relative error $\frac{T - T_{\text{linear}}}{T}$ in Figure 8.8. The relative error says how far is our approximation from the real period percentage-wise. Note that T_{linear} is simply a constant, it does not change with the initial angle θ_0. The actual period T gets larger and larger as θ_0 gets larger. Notice how the relative error is small when θ_0 is small. It is still only 15% when $\theta_0 = \frac{\pi}{2}$, that is, a 90 degree angle. The error is 3.8% when starting at $\frac{\pi}{4}$, a 45 degree angle. At a 5 degree initial angle, the error is only 0.048%.

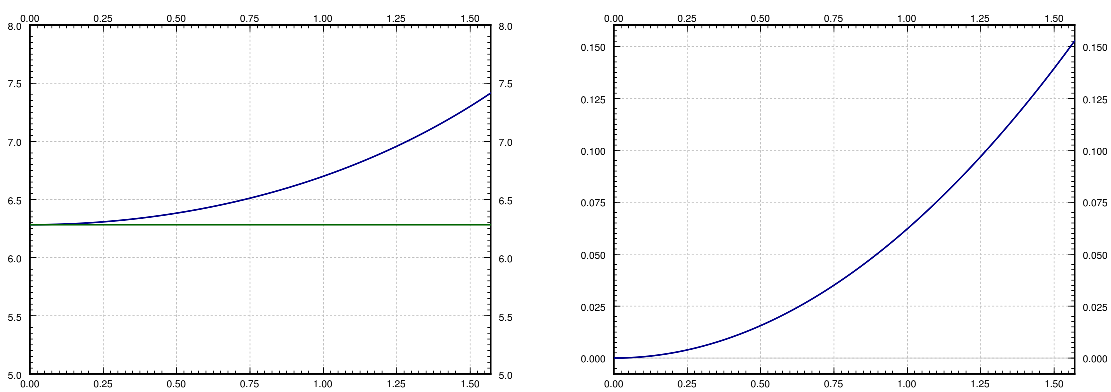


Figure 8.8: The plot of T and T_{linear} with $\frac{g}{L} = 1$ (left), and the plot of the relative error $\frac{T - T_{\text{linear}}}{T}$ (right), for θ_0 between 0 and $\pi/2$.

While it is not immediately obvious from the formula, it is true that

$$\lim_{\theta_0 \uparrow \pi} T = \infty.$$

That is, the period goes to infinity as the initial angle approaches the unstable equilibrium point. So if we put the pendulum almost upside down it may take a very long time before it gets down. This is consistent with the limiting behavior, where the exactly upside down pendulum never makes an oscillation, so we could think of that as infinite period.

8.3.2 Predator-prey or Lotka-Volterra systems

One of the most common simple applications of nonlinear systems are the so-called *predator-prey* or *Lotka-Volterra*[*] systems. For example, these systems arise when two species interact, one as a prey and one as a predator. It is then no surprise that the equations also see applications in economics. This simple system of equations explains the natural periodic variations of populations of different species in nature. Before the application of differential equations, these periodic variations in the population baffled biologists. Another example where the system arises is in chemical reactions.

Let us keep with the classical example of hares and foxes in a forest, as it is the easiest to understand.

$$x = \text{\# of hares (the prey)},$$
$$y = \text{\# of foxes (the predator)}.$$

When there are a lot of hares, there is plenty of food for the foxes, so the fox population grows. However, when the fox population grows, the foxes eat more hares, so when there are lots of foxes, the hare population should go down, and vice versa. The Lotka-Volterra model proposes that this behavior is described by the system of equations

$$x' = (a - by)x,$$
$$y' = (cx - d)y,$$

where a, b, c, d are some parameters that describe the interaction of the foxes and hares[†]. In this model, these are all positive numbers.

Let us analyze the idea behind this model. The model is a slightly more complicated idea based on the exponential population model. First expand,

$$x' = (a - by)x = ax - byx.$$

The hares are expected to simply grow exponentially in the absence of foxes, that is where the ax term comes in, the growth in population is proportional to the population itself. We are assuming the hares will always find enough food and have enough space to reproduce. However, there is another component $-byx$, that is, the population also is decreasing proportionally to the number of foxes. Together we can write the equation as $(a - by)x$, so it is like exponential growth or decay but the constant depends on the number of foxes.

The equation for foxes is very similar, expand again

$$y' = (cx - d)y = cxy - dy.$$

The foxes need food (hares) to reproduce: the more food, the bigger the rate of growth, hence the cxy term. On the other hand, there are natural deaths in the fox population, and hence the $-dy$ term.

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[*]Named for the American mathematician, chemist, and statistician Alfred James Lotka (1880 – 1949) and the Italian mathematician and physicist Vito Volterra (1860 – 1940).

[†]This interaction does not end well for the hare.

Without further delay, let us start with an explicit example. Suppose the equations are

$$x' = (0.4 - 0.01y)x, \qquad y' = (0.003x - 0.3)y.$$

See Figure 8.9 for the phase portrait. In this example it makes sense to also plot x and y as graphs with respect to time. Therefore the second graph in Figure 8.9 is the graph of x and y on the vertical axis (the prey x is the thinner line with taller peaks), against time on the horizontal axis. The particular trajectory graphed was with initial conditions of 20 foxes and 50 hares.

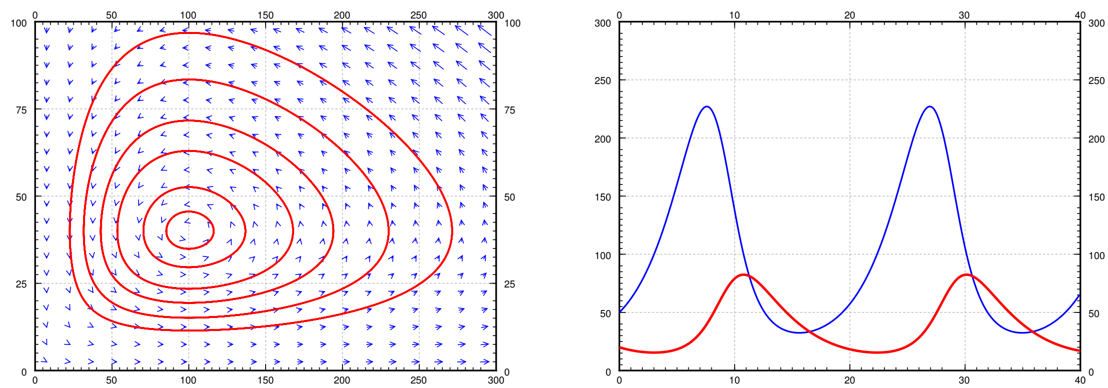

Figure 8.9: The phase portrait (left) and graphs of x and y for a sample trajectory (right).

Let us analyze what we see on the graphs. We work in the general setting rather than putting in specific numbers. We start with finding the critical points. Set $(a - by)x = 0$, and $(cx - d)y = 0$. The first equation is satisfied if either $x = 0$ or $y = a/b$. If $x = 0$, the second equation implies $y = 0$. If $y = a/b$, the second equation implies $x = d/c$. There are two equilibria: at $(0, 0)$ when there are no animals at all, and at $(d/c, a/b)$.

In our specific example $x = d/c = 100$, and $y = a/b = 40$. This is the point where there are 100 hares and 40 foxes.

Let us compute the Jacobian matrix:

$$\begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix}.$$

At the origin $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & -d \end{smallmatrix}\right]$, so the eigenvalues are a and $-d$, hence real and of opposite signs. So the critical point at the origin is a saddle. This makes sense. If you started with some foxes but no hares, then the foxes would go extinct, that is, you would approach the origin. If you started with no foxes and a few hares, then the hares would keep multiplying without check, and so you would go away from the origin.

OK, how about the other critical point at $({}^{d}/_{c}, {}^{a}/_{b})$. Here the Jacobian matrix becomes

$$\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}.$$

Computing the eigenvalues we get the equation $\lambda^2 + ad = 0$. In other words, $\lambda = \pm i\sqrt{ad}$. The eigenvalues being purely imaginary, we are in the case where we cannot quite decide using only linearization. We could have a stable center, spiral sink, or a spiral source. That is, the equilibrium could be asymptotically stable, stable, or unstable. Of course I gave you a picture above that seems to imply it is a stable center. But never trust a picture only. Perhaps the oscillations are getting larger and larger, but only *very* slowly. Of course this would be bad as it would imply something will go wrong with our population sooner or later. And I only graphed a very specific example with very specific trajectories.

How can we be sure we are in the stable situation? As we said before, in the case of purely imaginary eigenvalues, we have to do a bit more work. Previously we found that for conservative systems, there was a certain quantity that was conserved on the trajectories, and hence the trajectories had to go in closed loops. We can use a similar technique here. We just have to figure out what is the conserved quantity. After some trial and error we find the constant

$$C = \frac{y^a x^d}{e^{cx+by}} = y^a x^d e^{-cx-by}$$

is conserved. Such a quantity is called the *constant of motion*. Let us check C really is a constant of motion. How do we check, you say? Well, a constant is something that does not change with time, so let us compute the derivative with respect to time:

$$C' = ay^{a-1}y'x^d e^{-cx-by} + y^a dx^{d-1}x'e^{-cx-by} + y^a x^d e^{-cx-by}(-cx' - by').$$

Our equations give us what x' and y' are so let us plug those in:

$$\begin{aligned} C' &= ay^{a-1}(cx-d)yx^d e^{-cx-by} + y^a dx^{d-1}(a-by)xe^{-cx-by} + y^a x^d e^{-cx-by}(-c(a-by)x - b(cx-d)y) \\ &= y^a x^d e^{-cx-by}\Big(a(cx-d) + d(a-by) + \big(-c(a-by)x - b(cx-d)y\big)\Big) \\ &= 0. \end{aligned}$$

So along the trajectories C is constant. In fact, the expression $C = \frac{y^a x^d}{e^{cx+by}}$ gives us an implicit equation for the trajectories. In any case, once we have found this constant of motion, it must be true that the trajectories are simple curves, that is, the level curves of $\frac{y^a x^d}{e^{cx+by}}$. It turns out, the critical point at $({}^{d}/_{c}, {}^{a}/_{b})$ is a maximum for C (left as an exercise). So $({}^{d}/_{c}, {}^{a}/_{b})$ is a stable equilibrium point, and we do not have to worry about the foxes and hares going extinct or their populations exploding.

One blemish on this wonderful model is that the number of foxes and hares are discrete quantities and we are modeling with continuous variables. Our model has no problem with there being 0.1 fox in the forest for example, while in reality that makes no sense. The approximation is a reasonable one as long as the number of foxes and hares are large, but it does not make much sense for small numbers. One must be careful in interpreting any results from such a model.

An interesting consequence (perhaps counterintuitive) of this model is that adding animals to the forest might lead to extinction, because the variations will get too big, and one of the populations will get close to zero. For example, suppose there are 20 foxes and 50 hares as before, but now we bring in more foxes, bringing their number to 200. If we run the computation, we will find the number of hares will plummet to just slightly more than 1 hare in the whole forest. In reality that will most likely mean the hares die out, and then the foxes will die out as well as they will have nothing to eat.

Showing that a system of equations has a stable solution can be a very difficult problem. In fact, when Isaac Newton put forth his laws of planetary motions, he proved that a single planet orbiting a single sun is a stable system. But any solar system with more than 1 planet proved very difficult indeed. In fact, such a system will behave chaotically (see § 8.5), meaning small changes in initial conditions will lead to very different long term outcomes. From numerical experimentation and measurements, we know the earth will not fly out into the empty space or crash into the sun, for at least some millions of years to go. But we do not know what happens beyond that.

8.3.3 Exercises

***Exercise* 8.3.1:** *Take the* damped nonlinear pendulum equation $\theta'' + \mu\theta' + (g/L)\sin\theta = 0$ *for some* $\mu > 0$ *(that is, there is some friction). a) Suppose* $\mu = 1$ *and* $g/L = 1$ *for simplicity, find and classify the critical points. b) Do the same for any* $\mu > 0$ *and any* g *and* L*, but such that the damping is small, in particular,* $\mu^2 < 4(g/L)$*. c) Explain what your findings mean, and if it agrees with what you expect in reality.*

***Exercise* 8.3.2:** *Suppose the hares do not grow exponentially, but logistically. In particular consider*

$$x' = (0.4 - 0.01y)x - \gamma x^2, \qquad y' = (0.003x - 0.3)y.$$

For the following two values of γ*, find and classify all the critical points in the positive quadrant, that is, for* $x \geq 0$ *and* $y \geq 0$*. Then sketch the phase diagram. Discuss the implication for the long term behavior of the population. a)* $\gamma = 0.001$*, b)* $\gamma = 0.01$*.*

***Exercise* 8.3.3:** *a) Suppose* x *and* y *are positive variables. Show* $\frac{yx}{e^{x+y}}$ *attains a maximum at* $(1, 1)$*. b) Suppose* a, b, c, d *are positive constants, and also suppose* x *and* y *are positive variables. Show* $\frac{y^a x^d}{e^{cx+by}}$ *attains a maximum at* $(d/c, a/b)$*.*

***Exercise* 8.3.4:** *Suppose that for the pendulum equation we take a trajectory giving the spinning-around motion, for example* $\omega = \sqrt{\frac{2g}{L}\cos\theta + \frac{2g}{L} + \omega_0^2}$*. This is the trajectory where the lowest angular velocity is* ω_0^2*. Find an integral expression for how long it takes the pendulum to go all the way around.*

Exercise* 8.3.5** (challenging):*** *Take the pendulum, suppose the initial position is* $\theta = 0$.
a) Find the expression for ω *giving the trajectory with initial condition* $(0, \omega_0)$. *Hint: Figure out what C should be in terms of* ω_0.
b) Find the crucial angular velocity ω_1, *such that for any higher initial angular velocity, the pendulum will keep going around its axis, and for any lower initial angular velocity, the pendulum will simply swing back and forth. Hint: When the pendulum doesn't go over the top the expression for* ω *will be undefined for some* θ*s.*
c) What do you think happens if the initial condition is $(0, \omega_1)$, *that is, the initial angle is 0, and the initial angular velocity is exactly* ω_1.

Exercise* 8.3.101*: *Take the damped nonlinear pendulum equation* $\theta'' + \mu\theta' + (g/L) \sin\theta = 0$ *for some* $\mu > 0$ *(that is, there is friction). Suppose the friction is large, in particular* $\mu^2 > 4(g/L)$. *a) Find and classify the critical points. b) Explain what your findings mean, and if it agrees with what you expect in reality.*

Exercise* 8.3.102*: *Suppose we have the system predator-prey system where the foxes are also killed at a constant rate h (h foxes killed per unit time):* $x' = (a - by)x$, $y' = (cx - d)y - h$. *a) Find the critical points and the Jacobin matrices of the system. b) Put in the constants* $a = 0.4$, $b = 0.01$, $c = 0.003$, $d = 0.3$, $h = 10$. *Analyze the critical points. What do you think it says about the forest?*

Exercise* 8.3.103** (challenging):*** *Suppose the foxes never die. That is, we have the system* $x' = (a - by)x$, $y' = cxy$. *Find the critical points and notice they are not isolated. What will happen to the population in the forest if it starts at some positive numbers. Hint: Think of the constant of motion.*

8.4 Limit cycles

Note: 1 lecture , discussed in §6.1 and §6.4 in [EP] , §9.7 in [BD]

For nonlinear systems, trajectories do not simply need to approach or leave a single point. They may in fact approach a larger set, such as a circle or another closed curve.

Example 8.4.1: The *Van der Pol oscillator** is the following equation

$$x'' - \mu(1 - x^2)x' + x = 0,$$

where μ is some positive constant. The Van der Pol oscillator comes up often in applications, for example in electrical circuits.

For simplicity, let us use $\mu = 1$. A phase diagram is given in the left hand plot in Figure 8.10. Notice how the trajectories seem to very quickly settle on a closed curve. On the right hand plot we have the plot of a single solution for $t = 0$ to $t = 30$ with initial conditions $x(0) = 0.1$ and $x'(0) = 0.1$. Notice how the solution quickly tends to a periodic solution.

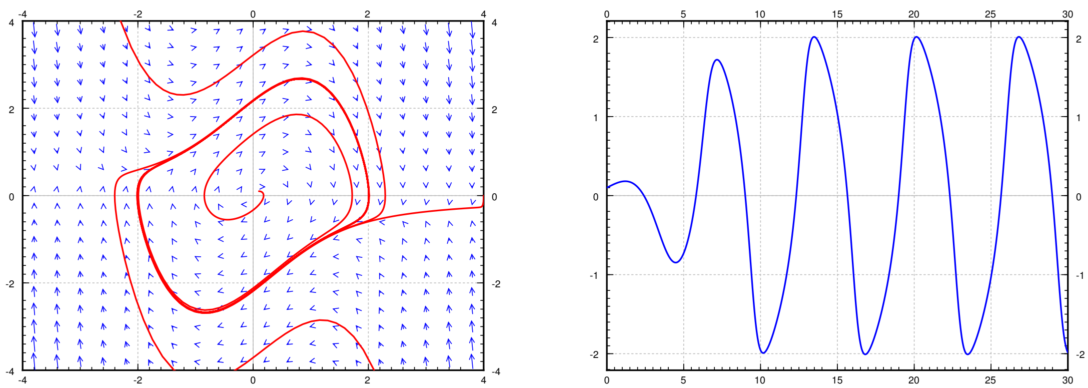

Figure 8.10: The phase portrait (left) and graphs of sample solutions of the Van der Pol oscillator.

The Van der Pol oscillator is an example of so-called *relaxation oscillation*. The word relaxation comes from the sudden jump (the very steep part of the solution). For larger μ the steep part becomes even more pronounced, for small μ the limit cycle looks more like a circle. In fact setting $\mu = 0$, we get $x'' + x = 0$, which is a linear system with a center and all trajectories become circles.

The closed curve in the phase portrait above is called a *limit cycle*. A limit cycle is a closed trajectory such that at least one other trajectory spirals into it (or spirals out of it). If all trajectories that start near the limit cycle spiral into it, the limit cycle is called *asymptotically stable*. The limit cycle in the Van der Pol oscillator is asymptotically stable.

*Named for the Dutch physicist Balthasar van der Pol (1889 – 1959).

Given a limit cycle on an autonomous system, any solution that starts on it is periodic. In fact, this is true for any trajectory that is a closed curve (a so-called *closed trajectory*). Such a curve is called a *periodic orbit*. More precisely, if $(x(t), y(t))$ is a solution such that for some t_0 the point $(x(t_0), y(t_0))$ lies on a periodic orbit, then both $x(t)$ and $y(t)$ are periodic functions (with the same period). That is, there is some number P such that $x(t) = x(t + P)$ and $y(t) = y(t + P)$.

Consider the system

$$x' = f(x, y), \qquad y' = g(x, y), \tag{8.2}$$

where the functions f and g have continuous derivatives.

Theorem 8.4.1 (Poincarè-Bendixson*)**.** *Suppose R is a closed bounded region (a region in the plane that includes its boundary and does not have points arbitrarily far from the origin). Suppose $(x(t), y(t))$ is a solution of* (8.2) *in R that exists for all $t \geq t_0$. Then either the solution is a periodic function, or the solution spirals towards a periodic solution in R.*

The main point of the theorem is that if you find one solution that exists for all t large enough (that is, we can let t go to infinity) and stays within a bounded region, then you have found either a periodic orbit, or a solution that spirals towards a limit cycle. That is, in the long term, the behavior will be very close to a periodic function. We should take the theorem more as a qualitative statement rather than something to help us in computations. In practice it is hard to find solutions and therefore hard to show rigorously that they exist for all time. Another caveat to consider is that the theorem only works in two dimensions. In three dimensions and higher, there is simply too much room.

Let us next look when limit cycles (or periodic orbits) do not exist. We will assume the equation (8.2) is defined on a *simply connected region*, that is, a region with no holes we could go around. For example the entire plane is a simply connected region, and so is the inside of the unit disc. However, the entire plane minus a point is not a simply connected domain as it has a "hole" at the origin.

Theorem 8.4.2 (Bendixson-Dulac†)**.** *Suppose f and g are defined in a simply connected region R. If the expression*‡

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is either always positive or always negative on R (except perhaps a small set such as on isolated points or curves) then the system (8.2) *has no closed trajectory inside R.*

The theorem gives us a way of ruling out the existence of a closed trajectory, and hence a way of ruling out limit cycles. The exception about points or lines really means that we can allow the expression to be zero at a few points, or perhaps on a curve, but not on any larger set.

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*Ivar Otto Bendixson (1861 – 1935) was a Swedish mathematician.

†Henri Dulac (1870 – 1955) was a French mathematician.

‡Sometimes the expression in the Poincarè-Dulac Theorem is $\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y}$ for some continuously differentiable function φ. For simplicity let us just consider the case $\varphi = 1$.

Example 8.4.2: Let us look at $x' = y + y^2 e^x$, $y' = x$ in the entire plane (see Example 8.2.2 on page 309). The entire plane is simply connected and so we can apply the theorem. We compute $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = y^2 e^x + 0$. The function $y^2 e^x$ is always positive except on the line $y = 0$. Therefore, via the theorem, the system has no closed trajectories.

In some books (or the internet) the theorem is not stated carefully and it concludes there are no periodic solutions. That is not quite right. The above example has two critical points and hence it has constant solutions, and constant functions are periodic. The conclusion of the theorem should be that there exist no trajectories that form closed curves. Another way to state the conclusion of the theorem would be to say that there exist no nonconstant periodic solutions that stay in R.

Example 8.4.3: Let us look at a somewhat more complicated example. Take the system $x' = -y - x^2$, $y' = -x + y^2$ (see Example 8.2.1 on page 308). We compute $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2x + 2y$. This expression takes on both signs, so if we are talking about the whole plane we cannot simply apply the theorem. However, we could apply it on the set where $x + y > 0$. Via the theorem, there is no closed trajectory in that set. Similarly, there is no closed trajectory in the set $x + y < 0$. We cannot conclude (yet) that there is no closed trajectory in the entire plane. Perhaps half of it is in the set where $x + y > 0$ and the other half is in the set where $x + y < 0$.

The key is to look at the set $x + y = 0$, or $x = -y$. Let us make a substitution $x = z$ and $y = -z$ (so that $x = -y$). Both equations become $z' = z - z^2$. So any solution of $z' = z - z^2$, gives us a solution $x(t) = z(t)$, $y(t) = -z(t)$. In particular, any solution that starts out on the line $x + y = 0$, stays on the line $x + y = 0$. In other words, there cannot be a closed trajectory that starts on the set where $x + y > 0$ and goes through the set where $x + y < 0$, as it would have to pass through $x + y = 0$.

8.4.1 Exercises

Exercise* 8.4.1*: *Show that the following systems have no closed trajectories.*
a) $x' = x^3 + y, y' = y^3 + x^2$,
b) $x' = e^{x-y}, y' = e^{x+y}$,
c) $x' = x + 3y^2 - y^3, y' = y^3 + x^2$.

Exercise* 8.4.2*: *Formulate a condition for a 2-by-2 linear system* $\vec{x}\,' = A\vec{x}$ *to not be a center using the Bendixson-Dulac theorem. That is, the theorem says something about certain elements of* A.

Exercise* 8.4.3*: *Explain why the Bendixson-Dulac Theorem does not apply for any conservative system* $x'' + h(x) = 0$.

Exercise* 8.4.4*: *A system such as* $x' = x, y' = y$ *has solutions that exist for all time* t*, yet there are no closed trajectories or other limit cycles. Explain why the Poincarè-Bendixson Theorem does not apply.*

Exercise* 8.4.5*: *Differential equations can also be given in different coordinate systems. Suppose we have the system* $r' = 1 - r^2$, $\theta' = 1$ *given in polar coordinates. Find all the closed trajectories and check if they are limit cycles and if so, if they are asymptotically stable or not.*

Exercise* 8.4.101*: *Show that the following systems have no closed trajectories.*
a) $x' = x + y^2$, $y' = y + x^2$, *b)* $x' = -x\sin^2(y)$, $y' = e^x$, *c)* $x' = xy$, $y' = x + x^2$.

Exercise* 8.4.102*: *Suppose an autonomous system in the plane has a solution* $x = \cos(t) + e^{-t}$, $y = \sin(t) + e^{-t}$. *What can you say about the system (in particular about limit cycles and periodic solutions)?*

Exercise* 8.4.103*: *Show that the limit cycle of the Van der Pol oscillator (for* $\mu > 0$*) must not lie completely in the set where* $-\sqrt{\frac{1+\mu}{\mu}} < x < \sqrt{\frac{1+\mu}{\mu}}$.

Exercise* 8.4.104*: *Suppose we have the system* $r' = \sin(r)$, $\theta' = 1$ *given in polar coordinates. Find all the closed trajectories.*

8.5 Chaos

Note: 1 lecture, §6.5 in [EP], §9.8 in [BD]

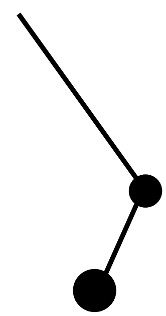
You have surely heard the story about the flap of a butterfly wing in the Amazon causing hurricanes in the North Atlantic. In a prior section, we mentioned that a small change in initial conditions of the planets can lead to very different configuration of the planets in the long term. These are examples of *chaotic systems*. Mathematical chaos is not really chaos, there is precise order behind the scenes. Everything is still deterministic. However a chaotic system is extremely sensitive to initial conditions. This also means even small errors induced via numerical approximation create large errors very quickly, so it is almost impossible to numerically approximate for long times. This is large part of the trouble as chaotic systems cannot be in general solved analytically.

Take the weather for example. As a small change in the initial conditions (the temperature at every point of the atmosphere for example) produces drastically different predictions in relatively short time, we cannot accurately predict weather. This is because we do not actually know the exact initial conditions, we measure temperatures at a few points with some error and then we somehow estimate what is in between. There is no way we can accurately measure the effects of every butterfly wing. Then we will solve numerically introducing new errors. That is why you should not trust weather prediction more than a few days out.

The idea of chaotic behavior was first noticed by Edward Lorenz* in the 1960s when trying to model thermally induced air convection (movement). The equations Lorentz was looking at form the relatively simple looking system:

$$x' = -10x + 10y, \qquad y' = 28x - y - xz, \qquad z' = -\frac{8}{3}z + xy.$$

A small change in the initial conditions yield a very different solution after a reasonably short time.



A very simple example the reader can experiment with, which displays chaotic behavior, is a double pendulum. The equations that govern this system are somewhat complicated and their derivation is quite tedious, so we will not bother to write them down. The idea is to put a pendulum on the end of another pendulum. If you look at the movement of the bottom mass, the movement will appear chaotic. This type of system is a basis for a whole number of office novelty desk toys. It is very simple to build a version. Take a piece of a string, and tie two heavy nuts at different points of the string; one at the end, and one a bit above. Now give the bottom nut a little push, as long as the swings are not too big and the string stays tight, you have a double pendulum system.

8.5.1 Duffing equation and strange attractors

Let us study the so-called *Duffing equation*:

$$x'' + ax' + bx + cx^3 = C\cos(\omega t).$$

*Edward Norton Lorenz (1917–2008) was an American mathematician and meteorologist.

Here a, b, c, C, and ω are constants. You will recognize that except for the cx^3 term, this equation looks like a forced mass-spring system. The cx^3 term comes up when the spring does not exactly obey Hooke's law (which no real-world spring actually does obey exactly). When c is not zero, the equation does not have a nice closed form solution, so we have to resort to numerical solutions as is usual for nonlinear systems. Not all choices of constants and initial conditions will exhibit chaotic behavior. Let us study

$$x'' + 0.05x' + x^3 = 8\cos(t).$$

The equation is not autonomous, so we will not be able to draw the vector field in the phase plane. We can still draw the trajectories however.

In Figure 8.11 we plot trajectories for t going from 0 to 15, for two very close initial conditions $(2,3)$ and $(2,2.9)$, and also the solutions in the (x,t) space. The two trajectories are close at first, but after a while diverge significantly. This sensitivity to initial conditions is precisely what we mean by the system behaving chaotically.

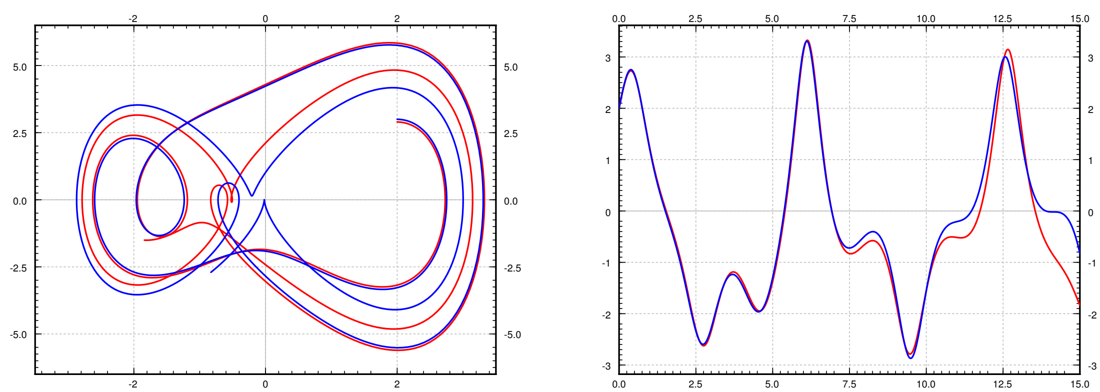


Figure 8.11: On left, two trajectories in phase space for $0 \le t \le 15$, for the Duffing equation one with initial conditions $(2,3)$ and the other with $(2,2.9)$. On right the two solutions in (x,t)-space.

Let us see the long term behavior. In Figure 8.12 on the next page, we plot the behavior of the system for initial conditions $(2,3)$, but for much longer period of time. Note that for this period of time it was necessary to use a ridiculously large number of steps in the numerical algorithm used to produce the graph, as even small errors quickly propagate[*]. From the graph it is hard to see any particular pattern in the shape of the solution except that it seems to oscillate, but each oscillation appears quite unique. The oscillation is expected due to the forcing term.

In general it is very difficult to analyze chaotic systems, or to find the order behind the madness, but let us try to do something that we did for the standard mass-spring system. One way we analyzed what happens is that we figured out what was the long term behavior (not dependent on initial

[*]In fact for reference, 30,000 steps were used with the Runge-Kutta algorithm, see exercises in § 1.7.

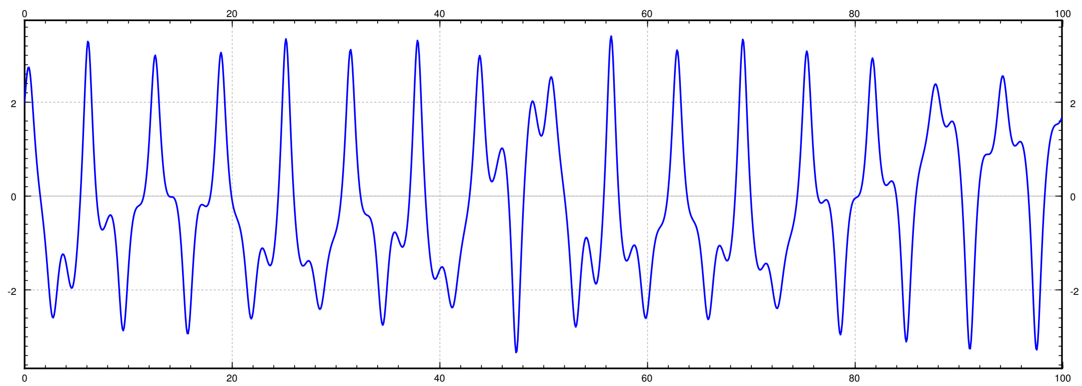


Figure 8.12: The solution to the given Duffing equation for t from 0 to 100.

conditions). From the figure above it is clear that we will not get a nice description of the long term behavior, but perhaps we can figure out some order to what happens on each "oscillation" and what do these oscillations have in common.

The concept we will explore is that of a *Poincarè section**. Instead of looking at t in a certain interval, we will look at where the system is at a certain sequence of points in time. Imagine flashing a strobe at a certain fixed frequency and drawing the points where the solution is during the flashes. The right strobing frequency depends on the system in question. The correct frequency to use for the forced Duffing equation (and other similar systems) is the frequency of the forcing term. For the Duffing equation above, find a solution $(x(t), y(t))$, and look at the points

$$(x(0), y(0)), \quad (x(2\pi), y(2\pi)), \quad (x(4\pi), y(4\pi)), \quad (x(6\pi), y(6\pi)), \quad \ldots$$

As we are really not interested in the transient part of the solution, that is, the part of the solution that depends on the initial condition we skip some number of steps in the beginning. For example, we might skip the first 100 such steps and start plotting points at $t = 100(2\pi)$, that is

$$(x(200\pi), y(200\pi)), \quad (x(202\pi), y(202\pi)), \quad (x(204\pi), y(204\pi)), \quad (x(206\pi), y(206\pi)), \quad \ldots$$

The plot of these points is the Poincarè section. After plotting enough points, a curious pattern emerges in Figure 8.13 on the following page (the left hand picture), a so-called *strange attractor*.

If we have a sequence of points, then an *attractor* is a set towards which the points in the sequence eventually get closer and closer to, that is, they are attracted. The Poincarè section above is not really the attractor itself, but as the points are very close to it, we can see its shape. The strange attractor in the figure is a very complicated set, and it in fact has fractal structure, that is, if you would zoom in as far as you want, you would keep seeing the same complicated structure.

*Named for the French polymath Jules Henri Poincarè (1854 – 1912).

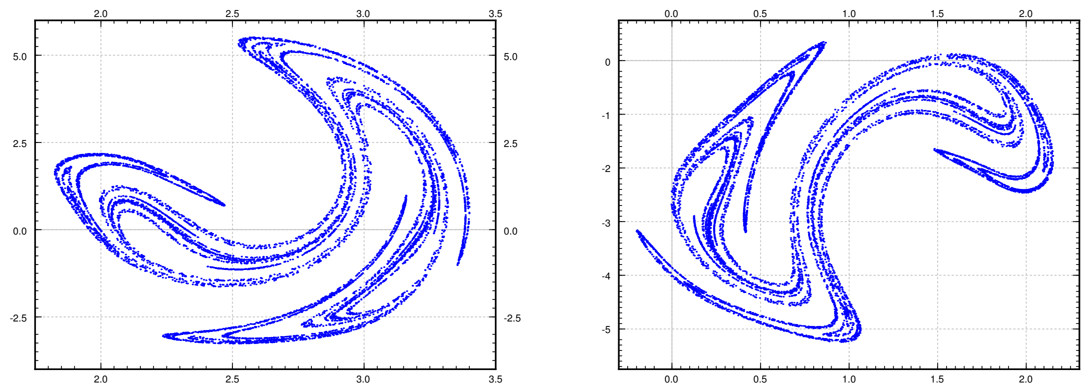


Figure 8.13: Strange attractor. The left plot is with no phase shift, the right plot has phase shift $\pi/4$.

The initial condition does not really make any difference. If we started with different initial condition, the points would eventually gravitate towards the attractor, and so as long as we throw away the first few points, we always get the same picture.

An amazing thing is that a chaotic system such as the Duffing equation is not random at all. There is a very complicated order to it, and the strange attractor says something about this order. We cannot quite say what state the system will be in eventually, but given a fixed strobing frequency we can narrow it down to the points on the attractor.

If you would use a phase shift, for example $\pi/4$, and look at the times

$$\pi/4, \quad 2\pi + \pi/4, \quad 4\pi + \pi/4, \quad 6\pi + \pi/4, \quad \ldots$$

you would obtain a slightly different looking attractor. The picture is the right hand side of Figure 8.13. It is as if we had rotated, distorted slightly, and then moved the original. Therefore for each phase shift you can find the set of points towards which the system periodically keeps coming back to.

You should study the pictures and notice especially the scales—where are these attractors located in the phase plane. Notice the regions where the strange attractor lives and compare it to the plot of the trajectories in Figure 8.11 on page 328.

Let us compare the discussion in this section to the discussion in § 2.6 about forced oscillations. Take the equation

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t).$$

This is like the Duffing equation, but with no x^3 term. The steady periodic solution is of the form

$$x = C\cos(\omega t + \gamma).$$

Strobing using the frequency ω we would obtain a single point in the phase space. So the attractor in this setting is a single point—an expected result as the system is not chaotic. In fact it was the opposite of chaotic. Any difference induced by the initial conditions dies away very quickly, and we settle into always the same steady periodic motion.

8.5.2 The Lorenz system

In two dimensions to have the kind of chaotic behavior we are looking for, we have to study forced, or non-autonomous, systems such as the Duffing equation. Due to the Poincarè-Bendoxson Theorem, if an autonomous two-dimensional system has a solution that exists for all time in the future and does not go towards infinity, then we obtain a limit cycle or a closed trajectory. Hardly the chaotic behavior we are looking for.

Let us very briefly return to the Lorenz system

$$x' = -10x + 10y, \qquad y' = 28x - y - xz, \qquad z' = -\frac{8}{3}z + xy.$$

The Lorenz system is an autonomous system in three dimensions exhibiting chaotic behavior. See the Figure 8.14 for a sample trajectory.

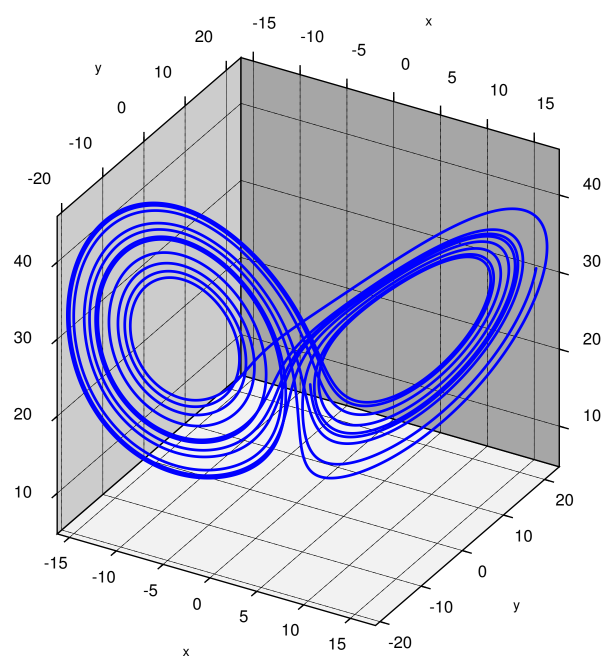


Figure 8.14: A trajectory in the Lorenz system.

The solutions will tend to an *attractor* in space, the so-called *Lorenz attractor*. In this case no strobing is necessary. Again we cannot quite see the attractor itself, but if we try to follow a solution for long enough, as in the figure, we will get a pretty good picture of what the attractor looks like.

The path is not just a repeating figure-eight. The trajectory will spin some seemingly random number of times on the left, then spin a number of times on the right, and so on. As this system arose in weather prediction, one can perhaps imagine a few days of warm weather and then a few days of cold weather, where it is not easy to predict when the weather will change, just as it is not really easy to predict far in advance when the solution will jump onto the other side. See Figure 8.15 for a plot of the x component of the solution drawn above.

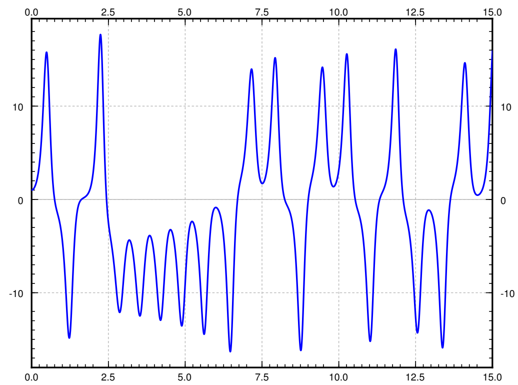


Figure 8.15: Graph of the $x(t)$ component of the solution.

8.5.3 Exercises

Exercise* 8.5.1*: *For the non-chaotic equation $x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$, suppose we strobe with frequency ω as we mentioned above. Use the known steady periodic solution to find precisely the point which is the attractor for the Poincarè section.*

Exercise* 8.5.2** (project):*** *A simple fractal attractor can be drawn via the following chaos game. Draw three points of a triangle (just the vertices) and number them, say p_1, p_2 and p_3. Start with some random point p (does not have to be one of the three points above) and draw it. Roll a die, and use it to pick of the p_1, p_2, or p_3 randomly (for example 1 and 4 mean p_1, 2 and 5 mean p_2, and 3 and 6 mean p_3). Suppose we picked p_2, then let p_{new} be the point exactly halfway between p and p_2. Draw this point and let p now refer to this new point p_{new}. Rinse, repeat. Try to be precise and draw as many iterations as possible. Your points should be attracted to the so-called* Sierpinski triangle. *A computer was used to run the game for 10,000 iterations to obtain the picture in Figure 8.16 on the next page.*

Exercise* 8.5.3** (project):*** *Construct the double pendulum described in the text with a string and two nuts (or heavy beads). Play around with the position of the middle nut, and perhaps use different weight nuts. Describe what you find.*

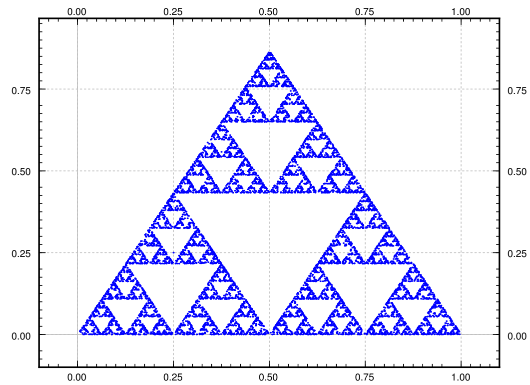


Figure 8.16: 10,000 iterations of the chaos game producing the Sierpinski triangle.

Exercise* 8.5.4** (computer project):*** *If you have access to a computer software such as IODE, try plotting the solution of the given forced Duffing equation with Euler's method. Plotting the solution for t from 0 to 100 with several different (small) step sizes. Discuss.*

Exercise* 8.5.101*: *Find critical points of the Lorenz system and the associated linearizations.*

8.1.101: a) Critical points $(0,0)$ and $(0,1)$. At $(0,0)$ using $u = x$, $v = y$ the linearization is $u' = -2u - (1/\pi)v$, $v' = -v$. At $(0,1)$ using $u = x$, $v = y - 1$ the linearization is $u' = -2u + (1/\pi)v$, $v' = v$.
b) Critical point $(0,0)$. Using $u = x$, $v = y$ the linearization is $u' = u + v$, $v' = u$.
c) Critical point $(1/2, -1/4)$. Using $u = x - 1/2$, $v = y + 1/4$ the linearization is $u' = -u + v$, $v' = u + v$.

8.1.102: 1) is c), 2) is a), 3) is b)

8.1.103: Critical points are $(0,0,0)$, and $(-1,1,-1)$. The linearization at the origin using variables $u = x$, $v = y$, $w = z$ is $u' = u$, $v' = -v$, $z' = w$. The linearization at the point $(-1,1,-1)$ using variables $u = x + 1$, $v = y - 1$, $w = z + 1$ is $u' = u - 2w$, $v' = -v - 2w$, $w' = w - 2u$.

8.1.104: $u' = f(u,v,w)$, $v' = g(u,v,w)$, $w' = 1$.

8.2.101: a) $(0,0)$: saddle (unstable), $(1,0)$: source (unstable), b) $(0,0)$: spiral sink (asymptotically stable), $(0,1)$: saddle (unstable), c) $(1,0)$: saddle (unstable), $(0,1)$: saddle (unstable)

8.2.102: a) $\frac{1}{2}y^2 + \frac{1}{3}x^3 - 4x = C$, critical points $(-2,0)$: an unstable saddle, and $(2,0)$: a stable center. b) $\frac{1}{2}y^2 + e^x = C$, no critical points. c) $\frac{1}{2}y^2 + xe^x = C$, critical point at $(-1,0)$ is a stable center.

8.2.103: Critical point at $(0,0)$. Trajectories are $y = \pm\sqrt{2C + (1/2)x^4}$, for $C > 0$, these give closed curves around the origin, so the critical point is a stable center.

8.2.104: A critical point x_0 is stable if $f'(x_0) < 0$ and unstable when $f'(x_0) > 0$.

8.3.101: a) Critical points are $\omega = 0$, $\theta = k\pi$ for any integer k. When k is odd, we have a saddle point. When k is even we get a sink. b) The findings mean the pendulum will simply go to one of the sinks, for example $(0,0)$ and it will not swing back and forth. The friction is too high for it to oscillate, just like an overdamped mass-spring system.

8.3.102: a) Solving for the critical points we get $(0, -h/d)$ and $(\frac{bh+ad}{ac}, \frac{a}{b})$. The Jacobian at $(0, -h/d)$ is $\begin{bmatrix} a+bh/d & 0 \\ -ch/d & -d \end{bmatrix}$ whose eigenvalues are $a + bh/d$ and $-d$. So the eigenvalues are always real of opposite signs and we get a saddle (In the application however we are only looking at the positive quadrant so this critical point is not relevant). At $(\frac{bh+ad}{ac}, \frac{a}{b})$ we get Jacobian matrix $\begin{bmatrix} 0 & -\frac{b(bh+ad)}{ac} \\ \frac{ac}{b} & \frac{bh+ad}{a} - d \end{bmatrix}$. b) For the specific numbers given, the second critical point is $(\frac{550}{3}, 40)$ the matrix is $\begin{bmatrix} 0 & -11/6 \\ 3/25 & 1/4 \end{bmatrix}$, which has eigenvalues $\frac{5 \pm i\sqrt{327}}{40}$. Therefore there is a spiral source. This means the solution will spiral outwards. The solution will eventually hit one of the axis $x = 0$ or $y = 0$ so something will die out in the forest.

8.3.103: The critical points are on the line $x = 0$. In the positive quadrant the y' is always positive and so the fox population always grows. The constant of motion is $C = y^a e^{-cx-by}$, for any C this curve must hit the y axis (why?), so the trajectory will simply approach a point on the y axis somewhere and the number of hares will go to zero.

8.4.101: Use Bendixson-Dulac Theorem. a) $f_x + g_y = 1 + 1 > 0$, so no closed trajectories. b) $f_x + g_y = -\sin^2(y) + 0 < 0$ for all x, y except the lines given by $y = k\pi$ (where we get zero), so no closed trajectories. c) $f_x + g_y = y + 0 > 0$ for all x, y except the line given by $y = 0$ (where we get zero), so no closed trajectories.

8.4.102: Using Poincarè-Bendixson Theorem, the system has a limit cycle, which is the unit circle centered at the origin as $x = \cos(t) + e^{-t}$, $y = \sin(t) + e^{-t}$ gets closer and closer to the unit circle. Thus we also have that $x = \cos(t)$, $y = \sin(t)$ is the periodic solution.

8.4.103: $f(x, y) = y$, $g(x, y) = \mu(1 - x^2)y - x$. So $f_x + g_y = 1 + \mu(1 - x^2) = 1 + \mu - \mu x^2$. The Bendixson-Dulac Theorem says there is no closed trajectory lying in the set $\frac{1+\mu}{\mu} < x^2$.

8.4.104: The closed trajectories are those where $\sin(r) = 0$, therefore, all the circles with radius a multiple of π are closed trajectories.

8.5.101: Critical points: $(0, 0, 0)$, $(3\sqrt{8}, 3\sqrt{8}, 27)$, $(-3\sqrt{8}, -3\sqrt{8}, 27)$. Linearization at $(0, 0, 0)$ using $u = x$, $v = y$, $w = z$ is $u' = -10u + 10v$, $v' = 28u - v$, $w' = -(8/3)w$. Linearization at $(3\sqrt{8}, 3\sqrt{8}, 27)$ using $u = x - 3\sqrt{8}$, $v = y - 3\sqrt{8}$, $w = z - 27$ is $u' = -10u + 10v$, $v' = u - v - 3\sqrt{8}w$, $w' = 3\sqrt{8}u + 3\sqrt{8}v - (8/3)w$. Linearization at $(-3\sqrt{8}, -3\sqrt{8}, 27)$ using $u = x + 3\sqrt{8}$, $v = y + 3\sqrt{8}$, $w = z - 27$ is $u' = -10u + 10v$, $v' = u - v + 3\sqrt{8}w$, $w' = -3\sqrt{8}u - 3\sqrt{8}v - (8/3)w$.