BA: 2.3

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lim sup is called *limit superior* and lim inf is called *limit inferior*.

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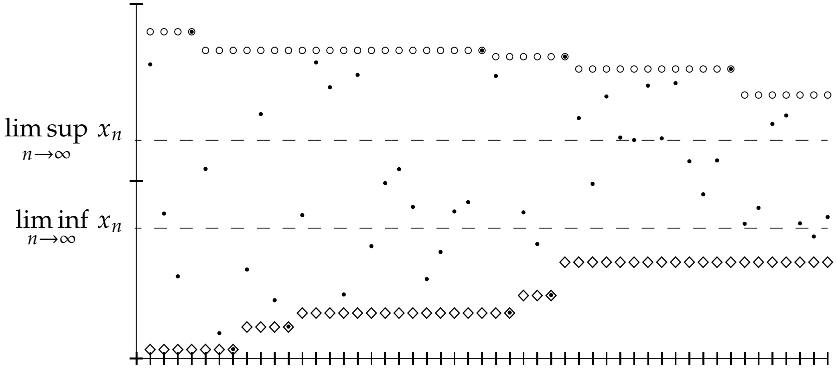

x_n are marked with dots ($\bullet$),
a_n are marked with circles ($\circ$),
b_n are marked with diamonds ($\diamond$).

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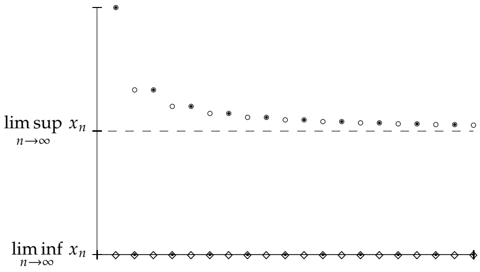

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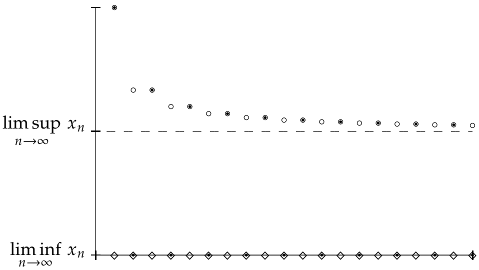

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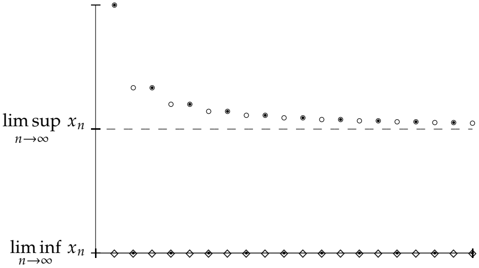

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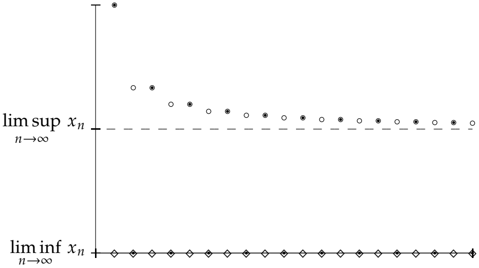

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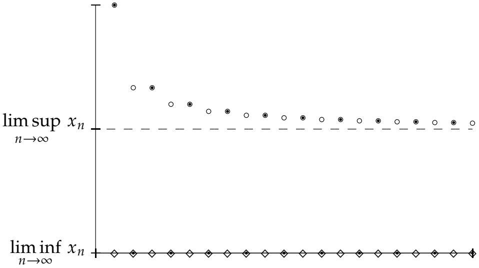

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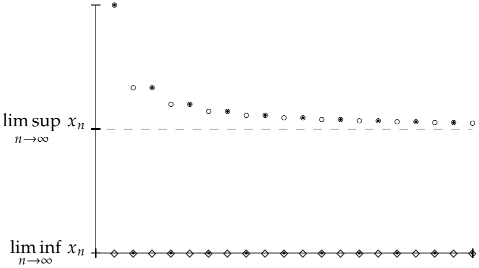

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Let $n_1 := 1$ and suppose we defined $n_1, \ldots, n_{k-1}$.

$\exists\, m \geq n_{k-1} + 1$ such that $a_{(n_{k-1}+1)} - x_m < \dfrac{1}{k}$ $\quad (a_{(n_{k-1}+1)} = \sup\{x_\ell : \ell \geq n_{k-1} + 1\})$.

Let $n_k := m$. The subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ is defined.

For $k \geq 2$, $a_{(n_{k-1}+1)} \geq a_{n_k}$ (why?) and $a_{n_k} \geq x_{n_k}$.

Remark: $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are **not** subsequences of $\{x_n\}_{n=1}^{\infty}$.
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Next we show $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x.

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lim inf is an exercise. $\square$

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By the squeeze lemma, $\{x_n\}_{n=1}^{\infty}$ converges and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n$.

For the other direction suppose $\{x_n\}_{n=1}^{\infty}$ converges to x.

By the theorem, $\exists$ a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to $\limsup_{n\to\infty} x_n$.

As $\{x_n\}_{b=1}^{\infty}$ converges to x, $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x
$\Rightarrow \limsup_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} = x$.

Similarly, $\liminf_{n\to\infty} x_n = x$. □

Proposition

Suppose $\{x_n\}_{n=1}^{\infty}$ *is a bounded sequence and* $\{x_{n_k}\}_{k=1}^{\infty}$ *is a subsequence. Then*

$$\liminf_{n\to\infty} x_n \le \liminf_{k\to\infty} x_{n_k} \le \limsup_{k\to\infty} x_{n_k} \le \limsup_{n\to\infty} x_n.$$

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Remark: lim sup and lim inf are the largest and smallest subsequential limits.

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Remark: lim sup and lim inf are the largest and smallest subsequential limits.
If $\{x_{n_k}\}_{k=1}^{\infty}$ converges, then $\quad \liminf\limits_{n\to\infty} x_n \leq \lim\limits_{k\to\infty} x_{n_k} \leq \limsup\limits_{n\to\infty} x_n$.

We get the following useful convergence test.

Proposition

A bounded sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and converges to x if and only if every convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x.

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Proof: Exercise.

There is another version of this result that we also leave as an exercise.

Exercise: Suppose $\{x_n\}_{n=1}^{\infty}$ is such that every subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ has a subsequence $\{x_{n_{m_i}}\}_{i=1}^{\infty}$ that converges to x. Prove that $\{x_n\}_{n=1}^{\infty}$ converges to x.

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Why call this a theorem?

1) It is a very useful result.

2) It generalizes to contexts other than $\mathbb{R}$, where $\limsup / \liminf$ may not make sense.

Definition

If for every $K \in \mathbb{R}$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n > K$, we say $\{x_n\}_{n=1}^{\infty}$ *diverges to infinity* and write $\lim\limits_{n\to\infty} x_n := \infty$.

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If for every $K \in \mathbb{R}$ there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n < K$, we say $\{x_n\}_{n=1}^{\infty}$ *diverges to minus infinity* and write $\lim\limits_{n\to\infty} x_n := -\infty$.

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Examples: $\lim\limits_{n\to\infty} n = \infty$, $\lim\limits_{n\to\infty} n^2 = \infty$, $\lim\limits_{n\to\infty} -n = -\infty$.

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If $\{a_n\}_{n=1}^{\infty}$ is a sequence of reals, then $\lim\limits_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

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Proposition

Let $\{x_n\}_{n=1}^{\infty}$ be unbounded. Define $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as above. Then $\{a_n\}_{n=1}^{\infty}$ is decreasing, and $\{b_n\}_{n=1}^{\infty}$ is increasing. If $a_n \in \mathbb{R}$ for every n, then $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$. If $b_n \in \mathbb{R}$ for every n, then $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n$.

Proof: $a_n = \sup\{x_k : k \geq n\} \geq \sup\{x_k : k \geq n+1\} = a_{n+1}$. Same for $\{b_n\}_{n=1}^{\infty}$.
If $\{a_n\}_{n=1}^{\infty}$ is a sequence of reals, then $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.
If $\{b_n\}_{n=1}^{\infty}$ is a sequence of reals, then $\lim_{n\to\infty} b_n = \sup\{b_n : n \in \mathbb{N}\}$. $\square$

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$$\lim_{n\to\infty} x_n \quad \text{does not exist,} \qquad \limsup_{n\to\infty} x_n = \infty, \qquad \liminf_{n\to\infty} x_n = 0.$$

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Exercise: Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences.

a) Show that $\{x_n + y_n\}_{n=1}^{\infty}$ is bounded.

b) Show that $\left(\liminf\limits_{n\to\infty} x_n\right) + \left(\liminf\limits_{n\to\infty} y_n\right) \leq \liminf\limits_{n\to\infty}(x_n + y_n)$.

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c) Find an example where $\left(\liminf\limits_{n\to\infty} x_n\right) + \left(\liminf\limits_{n\to\infty} y_n\right) < \liminf\limits_{n\to\infty}(x_n + y_n)$.

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