BA: 0.3 part 2

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We take this property as an axiom.

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Remark: It may be convenient to start at a number different than 1.

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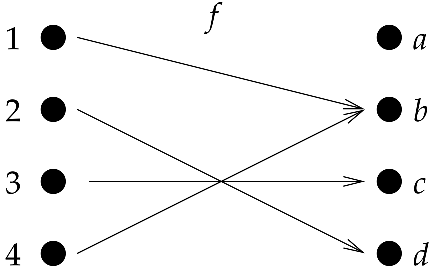
Example: Define $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ by
$f(1) := b, f(2) := d, f(3) := c, f(4) := b$



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So a composition of bijections is a bijection.