BA: 2.1

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Definition

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Be careful to distinguish sets and sequences:
$\{(-1)^n\}_{n=1}^{\infty}$ is the sequence $-1, 1, -1, 1, -1, 1, \ldots$, whereas the set of its values, the *range of the sequence*, is the set $\{-1, 1\}$.

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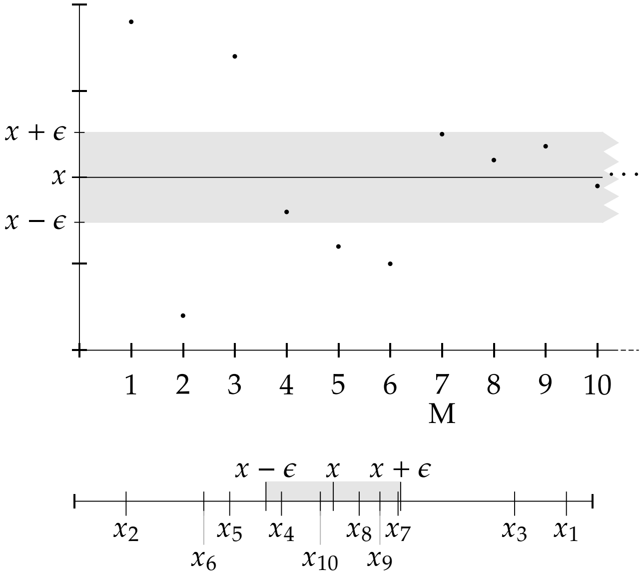
Note the dependence: M may depend on ϵ. We only need to pick M once we know ϵ.


$x+\epsilon$
x
$x-\epsilon$
1 2 3 4 5 6 7 8 9 10
M
$x-\epsilon$ x $x+\epsilon$
x_2 x_5 x_4 x_8 x_7 x_3 x_1
x_6 x_{10} x_9

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A contradiction. $\square$

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A convergent sequence has a unique limit.

Proof: Suppose $\{x_n\}_{n=1}^{\infty}$ has limits x and y.
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Remark: Note the technique. A quantity to be shown as zero is written as a sum of two things that are small.

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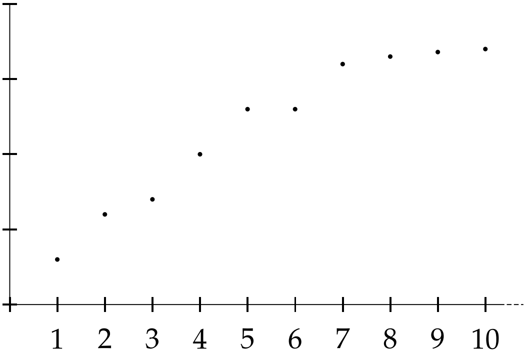
Example: Monotone increasing sequence:



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Note: monotone increasing $\{x_n\}_{n=1}^{\infty}$ is bounded from below by x_1, so enough to check if bounded from above.

Similarly for monotone decreasing, enough to check if bounded from below.

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So $b = 0$ is the greatest lower bound $\quad\Rightarrow\quad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

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However, we'll see later that it does not converge. It is unbounded!

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Proof is an exercise.

Definition

For a sequence $\{x_n\}_{n=1}^{\infty}$, the K-*tail* (where $K \in \mathbb{N}$), or just the *tail*, of $\{x_n\}_{n=1}^{\infty}$ is the sequence starting at $K + 1$, usually written as

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The 0-tail of a sequence is the sequence itself.

The reason for studying tails is that convergence only depends on the tail.

Proposition

Let $\{x_n\}_{n=1}^{\infty}$ *be a sequence. Then the following statements are equivalent:*

(i) *The sequence* $\{x_n\}_{n=1}^{\infty}$ *converges.*

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Furthermore, if any (and hence all) of the limits exist, then for all $K \in \mathbb{N}$

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We will also show that the limits are equal.

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Example:

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