BA: 2.2

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Lemma (Squeeze lemma)

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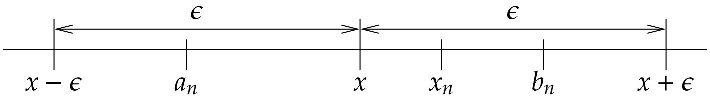

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Proof: Let $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$.

Let $\epsilon > 0$ be given.

Find an M_1 such that $|x_n - x| < \epsilon/2$ for all $n \geq M_1$,
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So for some $n \geq \max\{M_1, M_2\}$, $\quad x - x_n < \epsilon/2$ and $y_n - y < \epsilon/2$.

Add these to get $\quad y_n - x_n + x - y < \epsilon \quad$ or $\quad y_n - x_n < y - x + \epsilon$.

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(i) *If* $\{x_n\}_{n=1}^{\infty}$ *is convergent such that* $x_n \geq 0 \ \forall \ n \in \mathbb{N}$, *then* $\lim\limits_{n\to\infty} x_n \geq 0.$

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Strict inequalities may become non-strict inequalities when limits are applied.

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Note that you can also use constant sequences.
E.g., if $c \in \mathbb{R}$ and $\{x_n\}_{n=1}^{\infty}$ converges, then

$$\lim_{n\to\infty} cx_n = c\left(\lim_{n\to\infty} x_n\right), \qquad \lim_{n\to\infty}(c + x_n) = c + \lim_{n\to\infty} x_n, \qquad \text{etc.}$$

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Proof of (ii) (subtraction) is almost identical (exercise).

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Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent and $x := \lim_{n\to\infty} x_n$, $y := \lim_{n\to\infty} y_n$.

Let $\epsilon > 0$ be given. Let $K := \max\{|x|, |y|, \epsilon/3, 1\}$.
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(iv) follows from (iii) and the claim:

Claim: *If* $\{y_n\}_{n=1}^{\infty}$ *is convergent,* $\lim_{n\to\infty} y_n \neq 0$*, and* $y_n \neq 0$ *for all* $n \in \mathbb{N}$*, then* $\{1/y_n\}_{n=1}^{\infty}$ *converges and* $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{\lim_{n\to\infty} y_n}$.

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Exercise: (induction) $\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k$ for all $k \in \mathbb{N}$.

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Proposition

Let $\{x_n\}_{n=1}^{\infty}$ *be convergent and* $x_n \geq 0$ *for all* $n \in \mathbb{N}$. *Then* $\lim\limits_{n\to\infty} \sqrt{x_n} = \sqrt{\lim\limits_{n\to\infty} x_n}$.

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The rest is an exercise. $\square$

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Proof sketch: The reverse triangle inequality: $\big||x_n| - |x|\big| \le |x_n - x|$. $\square$

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If we show that $x_n^2 - 2 \geq 0$ for all n, then $x_{n+1} \leq x_n$ for all n.

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Write

$$2x_n x_{n+1} = x_n^2 + 2.$$

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You may have noticed the above is *Newton's method* for finding $\sqrt{2}$,
a common and practical way to find roots of equations.

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More general moral: Before you prove that a limit actually exists do not do any calculations with

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Just don't write "$\lim\limits_{n\to\infty}$" anywhere before you prove the limit exists.

Proposition

Let $\{x_n\}_{n=1}^{\infty}$ *be a sequence. Suppose* $\exists\, x \in \mathbb{R}$ *and a convergent* $\{a_n\}_{n=1}^{\infty}$ *such that* $\lim_{n\to\infty} a_n = 0$ *and* $|x_n - x| \le a_n$ *for all* $n \in \mathbb{N}$.

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So for all $n \ge M$, $|x_n - x| \le a_n < \epsilon$. □

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As B was arbitrary, $\{c^n\}_{n=1}^{\infty}$ is unbounded. $\square$

Lemma (Ratio test for sequences)

Let $\{x_n\}_{n=1}^{\infty}$ *be such that* $x_n \neq 0$ *for all* n *and* $L := \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$ *exists.*

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If $L = 1$, no conclusion. E.g.,

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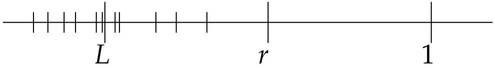

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