BA: 5.3

Jiří Lebl

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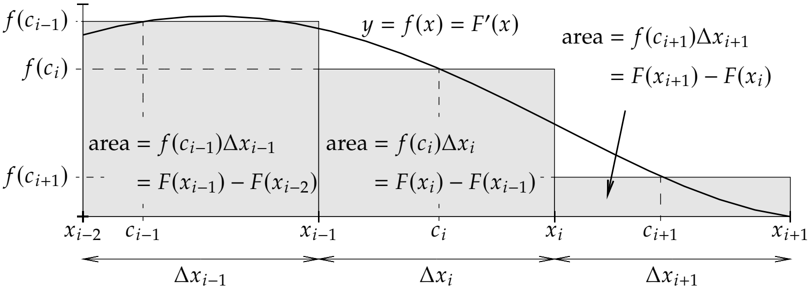

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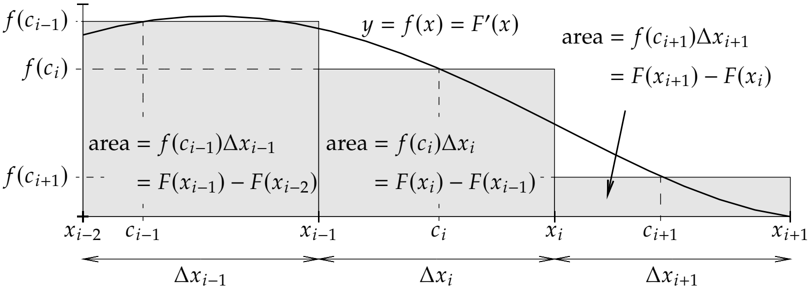

The area of all three shaded rectangles is $F(x_{i+1}) - F(x_{i-2})$.

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3) g is not continuous on $[-1, 1]$, let alone continuously differentiable.

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c) Find g for $f(x) := |x|$, $\epsilon = 1$ (you can assume $[a, b]$ is large enough).