BA: 3.4

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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Remark: See how closed and bounded is important again.

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The lemma gives extension to endpoints for uniformly continuous functions:

Proposition

$f\colon (a,b) \to \mathbb{R}$ is uniformly continuous if and only if the limits

$$L_a := \lim_{x \to a} f(x) \qquad \text{and} \qquad L_b := \lim_{x \to b} f(x)$$

exist and $\widetilde{f}\colon [a,b] \to \mathbb{R}$ defined by

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f is the restriction of $\widetilde{f}$ to (a,b), so f is also uniformly continuous (exercise).

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As $\lim\limits_{x\to a} f(x) = L_a$ exists, $\lim\limits_{x\to a} \widetilde{f}(x)$ exists and equals L_a (Proposition 3.1.15)

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$\Rightarrow$ $\widetilde{f}$ is continuous at a and b.

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Take another $\{y_n\}_{n=1}^{\infty}$ in (a,b) s.t. $y_n \to a$. Similarly, $L_2 := \lim\limits_{n\to\infty} f(y_n)$ exists. WTS $L_1 = L_2$.
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$\exists\, M \in \mathbb{N}$ s.t. for $n \geq M$, $|a-x_n| < \delta/2$, $|a-y_n| < \delta/2$, $|f(x_n)-L_1| < \epsilon/3$, and $|f(y_n)-L_2| < \epsilon/3$.

$$\Rightarrow \quad \text{for } n \geq M,\ |x_n - y_n| = |x_n - a + a - y_n| \leq |x_n - a| + |a - y_n| < \delta/2 + \delta/2 = \delta.$$

$$\begin{aligned} \Rightarrow \quad |L_1 - L_2| &= |L_1 - f(x_n) + f(x_n) - f(y_n) + f(y_n) - L_2| \\ &\leq |L_1 - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - L_2| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

$\Rightarrow \quad L_1 = L_2. \quad \Rightarrow \quad L_a = \lim_{x\to a} f(x)$ exists. $\quad L_b$ is almost the same.

As $\lim\limits_{x\to a} f(x) = L_a$ exists, $\lim\limits_{x\to a} \widetilde{f}(x)$ exists and equals L_a (Proposition 3.1.15) (similarly for b).
$\Rightarrow \quad \widetilde{f}$ is continuous at a and b.
As f is continuous at any $c \in (a,b)$, then $\widetilde{f}$ is continuous at $c \in (a,b)$ (Proposition 3.1.15). $\square$

$\Rightarrow$) Suppose f is uniformly continuous. WTS that L_a and L_b exist.
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Typical application: if $f \colon (-1,0) \cup (0,1) \to \mathbb{R}$ is uniformly continuous, then $\lim\limits_{x\to 0} f(x)$ exists and f has a *removable singularity*: it can be continuously extended to $(-1,1)$.

Definition

A function $f : S \to \mathbb{R}$ is *Lipschitz continuous*, if there exists a $K \in \mathbb{R}$, such that

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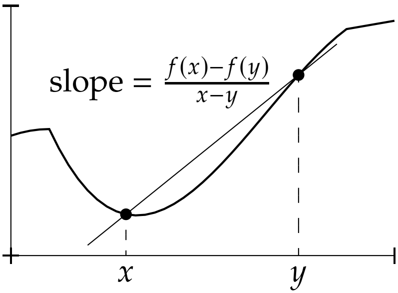

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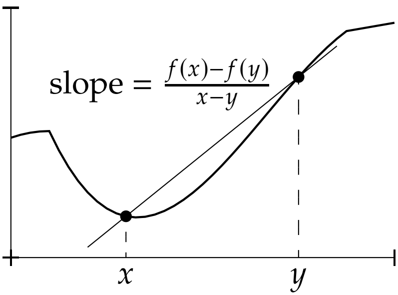

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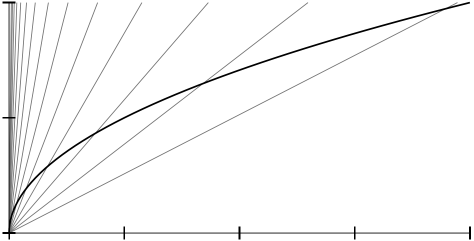


Example: $f\colon [1,\infty) \to \mathbb{R}$ given by $f(x) := \sqrt{x}$ is Lipschitz continuous.

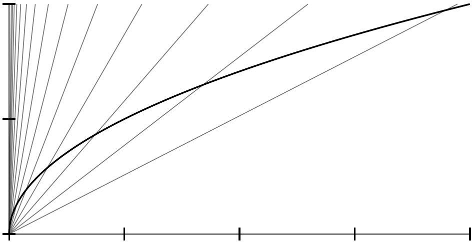
Proof: $\left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$. $\qquad x \geq 1$ and $y \geq 1 \quad \Rightarrow \quad \frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$.

Therefore, $\quad \left|\sqrt{x} - \sqrt{y}\right| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leq \frac{1}{2}\,|x-y|$. $\square$

Example: $g\colon [0,\infty) \to \mathbb{R}$ given by $g(x) := \sqrt{x}$ is **not** Lipschitz continuous.

Proof: Suppose K exists (assume $K > 0$), s.t. $\left|\sqrt{x} - \sqrt{y}\right| \leq K\,|x-y|\ \forall\, x, y \geq 0$.

Set $y = 0 \quad \Rightarrow \quad \sqrt{x} \leq Kx \qquad$ in other words, $1/K \leq \sqrt{x}\quad$ or $\quad 1/K^2 \leq x$.
Cannot be true for all $x > 0 \quad \Rightarrow \quad$ no K exists $\quad \Rightarrow \quad g$ not Lipschitz. $\square$



Remark: g is uniformly continuous (exercise), but not Lipschitz.