BA: 4.2

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Definition

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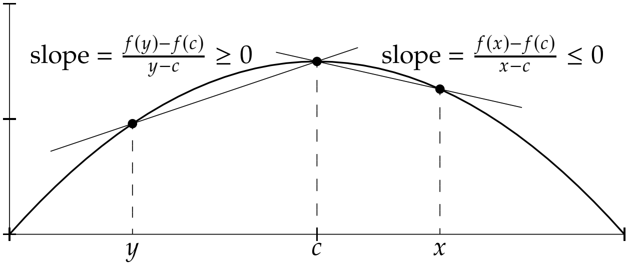
Idea of proof:



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$\Rightarrow \quad f'(c) = 0.$ $\square$

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Let $f : [a, b] \longrightarrow \mathbb{R}$ *be continuous and differentiable on* (a, b) *such that* $f(a) = f(b)$*. Then there exists a* $c \in (a, b)$ *such that* $f'(c) = 0$.

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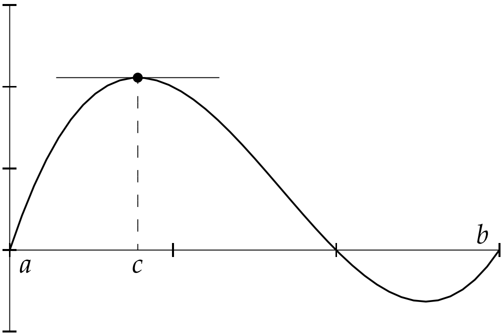

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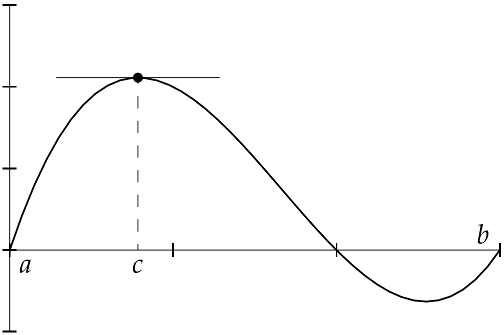

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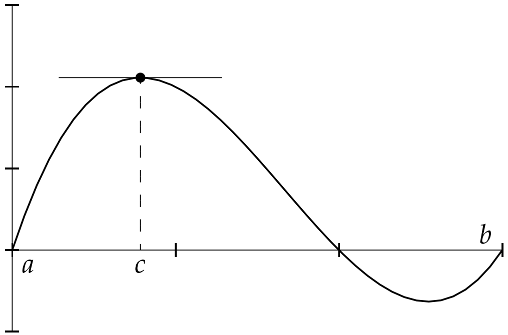

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Consider $f(x) := |x|$ on $[-1, 1]$. $f(-1) = f(1)$, but at no c does $f'(c) = 0$.

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In any case, the lemma applies, and $f'(c) = 0$. $\square$

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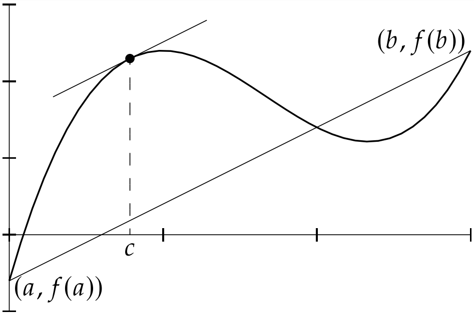

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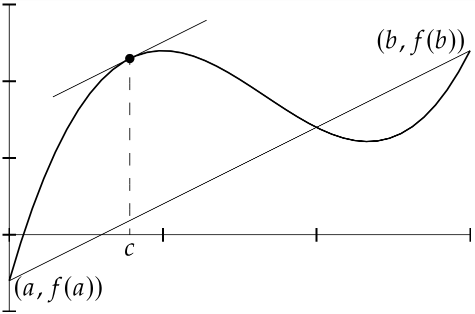

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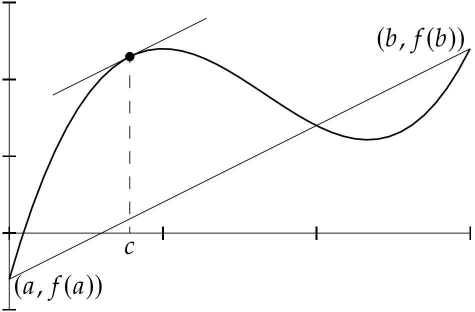

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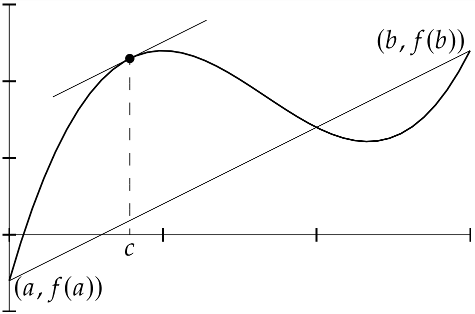
So the average derivative is attained at c:

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$



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Example: Converse not true: $f(x) := x^3$ is strictly increasing, but $f'(0) = 0$.

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Let $f : (a, b) \to \mathbb{R}$ *be continuous. Let* $c \in (a, b)$ *and suppose* f *is differentiable on* (a, c) *and* (c, b).

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Let $f : (a, b) \to \mathbb{R}$ be continuous. Let $c \in (a, b)$ and suppose f is differentiable on (a, c) and (c, b).

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The converse of the proposition does not hold (example later).

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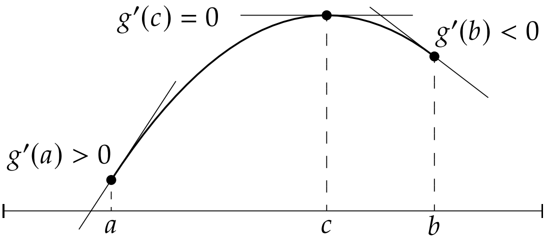

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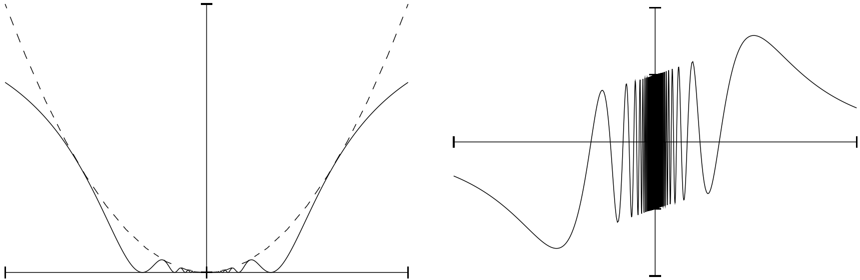

The graph of f and f'. The dashed line represents $f(x) \leq x^2$.

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$f(x) = \left(x \sin(1/x)\right)^2$ if $x \neq 0$, and $f(0) = 0$.

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Claim 2: f has a min at 0, but f' changes sign infinitely often near 0.

Proof of claims: f clearly has min at 0 (as $f(x) \geq 0$ for all x).

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I.e., between any two consecutive roots of f' is at most one root of f.