BA: 3.2

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Definition

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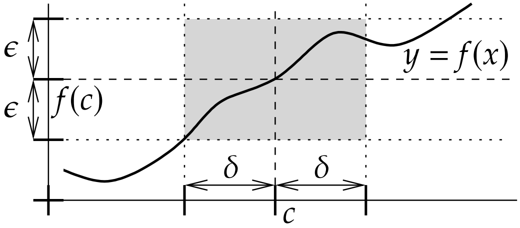

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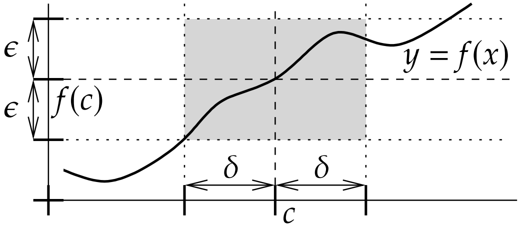

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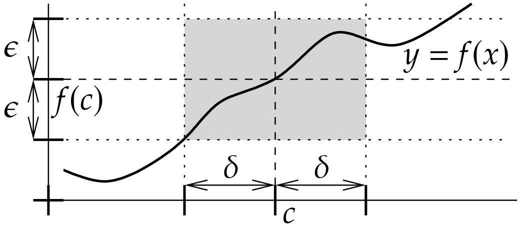

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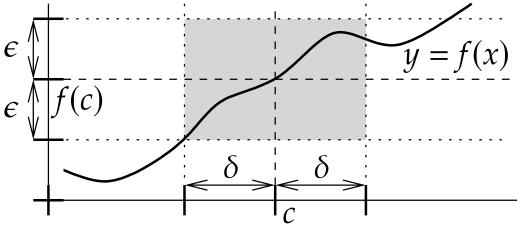

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Remark: If f is continuous on A, then $f|_A$ is continuous (exercise), but the converse does not hold (we'll give an example shortly).

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(ii) *If* c *is a cluster point of* S, *then* f *is continuous at* c *if and only if the limit of* $f(x)$ *as* $x \to c$ *exists and*

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$\Rightarrow \quad \{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. $\square$

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Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $(0,\infty)$ such that $\lim\limits_{n\to\infty} x_n = c$.

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Example: $f\colon (0,\infty) \to \mathbb{R}$ defined by $f(x) := 1/x$ is continuous.

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$\Rightarrow \quad f$ is continuous at c.

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$\Rightarrow$ f is continuous at c.

As f is continuous at all $c \in (0,\infty)$, f is continuous. $\square$

Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ *be a polynomial. That is*

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

for some constants $a_0, a_1, \ldots, a_d$*. Then* f *is continuous.*

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$\Longrightarrow$ f is continuous at c

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$$\begin{aligned} f(c) &= a_d c^d + a_{d-1} c^{d-1} + \cdots + a_1 c + a_0 \\ &= a_d \left(\lim_{n\to\infty} x_n\right)^d + a_{d-1} \left(\lim_{n\to\infty} x_n\right)^{d-1} + \cdots + a_1 \left(\lim_{n\to\infty} x_n\right) + a_0 \\ &= \lim_{n\to\infty} \left(a_d x_n^d + a_{d-1} x_n^{d-1} + \cdots + a_1 x_n + a_0\right) = \lim_{n\to\infty} f(x_n). \end{aligned}$$

$\Rightarrow$ f is continuous at c $\Rightarrow$ f is continuous (c was arbitrary). $\square$

Proposition

Let $f\colon S \to \mathbb{R}$ *and* $g\colon S \to \mathbb{R}$ *be functions continuous at* $c \in S$.

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(i) *The function* $h\colon S \longrightarrow \mathbb{R}$ *defined by* $h(x) := f(x) + g(x)$ *is continuous at* c.

Proposition

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(ii) *The function* $h\colon S \longrightarrow \mathbb{R}$ *defined by* $h(x) := f(x) - g(x)$ *is continuous at* c*.*

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(iii) *The function $h\colon S \to \mathbb{R}$ defined by $h(x) := f(x)g(x)$ is continuous at c.*

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Let $f: S \to \mathbb{R}$ *and* $g: S \to \mathbb{R}$ *be functions continuous at* $c \in S$.

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Proof: Exercise.

Example: $\sin(x)$ and $\cos(x)$ are continuous.

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$$|\sin(x) - \sin(c)| = \left|2\sin\left(\frac{x-c}{2}\right)\cos\left(\frac{x+c}{2}\right)\right|$$

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Details left to student.

Recall $f \circ g$ is defined by $(f \circ g)(x) := f\big(g(x)\big)$.

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Let $A, B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ and $g \colon A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g \colon A \to \mathbb{R}$ is continuous at c.

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Example: $\big(\sin(1/x)\big)^2$ *is a continuous function on* $(0, \infty)$.

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Proof: $1/x$ is continuous on $(0, \infty)$ and $\sin(x)$ is continuous on $(0, \infty)$.

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Example: $\big(\sin(1/x)\big)^2$ *is a continuous function on* $(0, \infty)$.

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Recall $f \circ g$ is defined by $(f \circ g)(x) := f\big(g(x)\big)$.

Proposition

Let $A, B \subset \mathbb{R}$ and $f\colon B \to \mathbb{R}$ and $g\colon A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g\colon A \to \mathbb{R}$ is continuous at c.

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Proof: A restatement of one direction of part (iii) of the proposition above. $\square$

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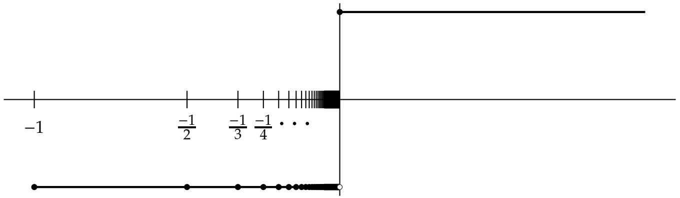

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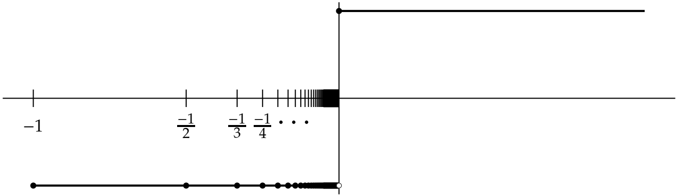

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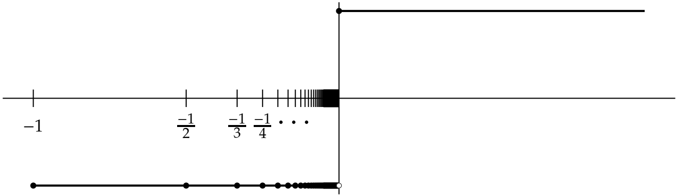

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Example: Consider the *Dirichlet function*.

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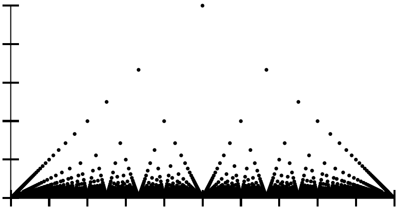
Example: (*Thomae function* or *popcorn function*).
Define $f\colon (0,1) \longrightarrow \mathbb{R}$ as

$$f(x) := \begin{cases} 1/k & \text{if } x = m/k, \text{ where } m,k \in \mathbb{N} \text{ and } m \text{ and } k \text{ have no common divisors,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$



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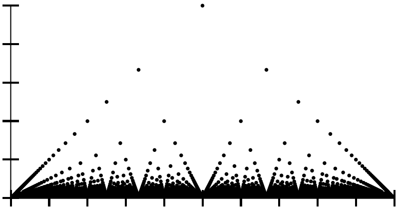

Claim: *f is continuous at every irrational number and discontinuous at every rational number.*

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$\lim\limits_{x\to 0} g(x)$ exists $\quad \lim\limits_{x\to 0} f(x)$ does not.

Example: Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) := 0$ if $x \neq 0$ and $g(0) := 1$.

g is discontinuous at 0, but continuous on $\mathbb{R} \setminus \{0\}$.

$x = 0$ is called a *removable discontinuity*:
We could redefine $g(0)$ and obtain a continuous function.

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$g|_B$ is continuous, and g is continuous on B.