BA: 1.1

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The set of words is an ordered set by using lexicographic ordering.

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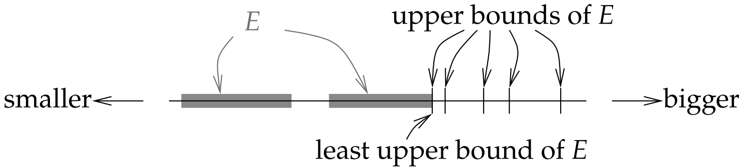
If E is bounded above and bounded below, we say that E is *bounded*.


E
upper bounds of E
smaller
bigger
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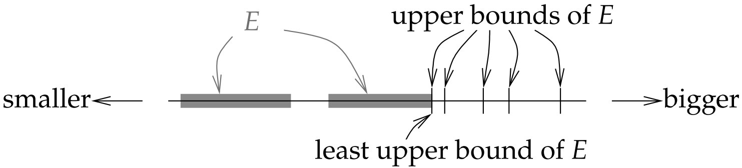
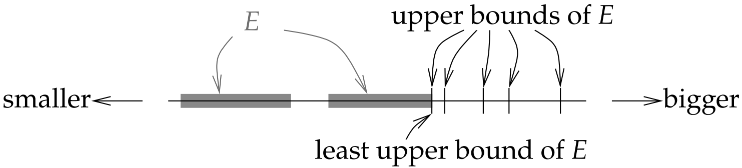


Notation $\sup E$ and $\inf E$ is justified as the supremum (or infimum) is unique (if it exists):



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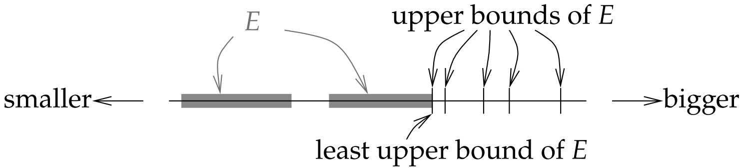

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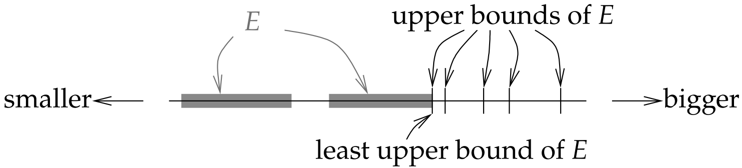

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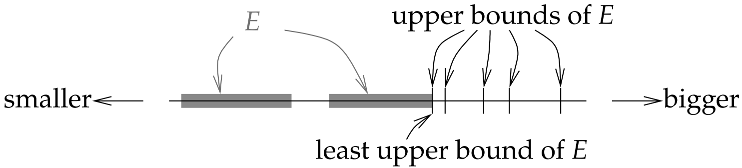

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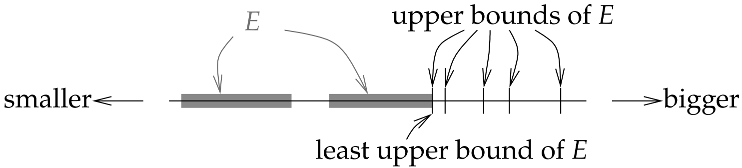

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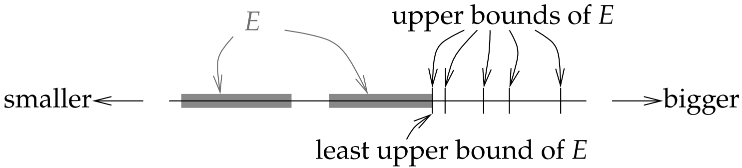

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Example: $P \coloneqq \{x \in \mathbb{Q} : x \geq 0\} \subset \mathbb{Q}$ has no upper bound.

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This is the main reason why analysis needs $\mathbb{R}$ and not just $\mathbb{Q}$.

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(M4) There exists $1 \in F$ (and $1 \neq 0$) such that $1x = x$ for all $x \in F$.

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(M2) *(commutativity of multiplication)* $xy = yx$ for all $x, y \in F$.

(M3) *(associativity of multiplication)* $(xy)z = x(yz)$ for all $x, y, z \in F$.

(M4) There exists $1 \in F$ (and $1 \neq 0$) such that $1x = x$ for all $x \in F$.

(M5) For every $x \in F$ such that $x \neq 0$ there exists $1/x \in F$ such that $x(1/x) = 1$.

Definition

A set F is called a *field* if it has two operations defined on it, addition $x + y$ and multiplication xy, and if it satisfies the following axioms:

(A1) $x \in F$ and $y \in F \Rightarrow x + y \in F$.

(A2) *(commutativity of addition)* $x + y = y + x$ for all $x, y \in F$.

(A3) *(associativity of addition)* $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.

(A4) $\exists 0 \in F$ such that $0 + x = x$ for all $x \in F$.

(A5) For every $x \in F$, there exists $-x \in F$ such that $x + (-x) = 0$.

(M1) $x \in F$ and $y \in F \Rightarrow xy \in F$.

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(D) *(distributive law)* $x(y + z) = xy + xz$ for all $x, y, z \in F$.

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Example: Not hard to check that $\mathbb{Q}$ is an ordered field.

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(ii): $y < z$ implies $0 < z - y$ (item (i) of def).

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Hence $0 < xz - xy$ and by item (i) again, $xy < xz$.

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Thus $(1/x)(1/y) > 0$.

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Thus $({}^1\!/\!{}_x)({}^1\!/\!{}_y) > 0$.
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Example: The complex numbers $\mathbb{C}$ (numbers $x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$) is not an ordered field: In every ordered field $-1 < 0$.

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WLOG $x > 0$ and $y < 0$.
Multiply $y < 0$ by x to get $xy < 0x = 0$. $\square$

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So $-b$ is an upper bound for B.
F has the least-upper-bound property $\Rightarrow c := \sup B$ exists, and $c \leq -b$.
As $y \leq c$ for all $y \in B$, then $-c \leq x$ for all $x \in A$.
So $-c$ is a lower bound for A.
As $-c \geq b$, $-c$ is the greatest lower bound of A. $\square$