BA: 1.4

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Proposition

A set $I \subset \mathbb{R}$ is an interval if and only if I contains at least 2 points and for all $a, c \in I$ and $b \in \mathbb{R}$ such that $a < b < c$, we have $b \in I$.

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Proof is an exercise.

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We'll give essentially Cantor's original 1874 proof.

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The two sequences are now defined.

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Claim: *$a_n < b_m$ for all n and m in $\mathbb{N}$.*

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Cantor's theorem shows $\exists$ non-algebraic (transcendental) numbers.