BA: 2.4

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Then $|x_n| \leq B$ for all $n \in \mathbb{N}$. $\square$

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By a theorem from 2.3, $\exists$ subsequences $\{x_{n_i}\}_{i=1}^{\infty}$ and $\{x_{m_i}\}_{i=1}^{\infty}$, such that $\lim_{i\to\infty} x_{n_i} = a \quad \text{and} \quad \lim_{i\to\infty} x_{m_i} = b.$

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So $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ and $\{x_n\}_{n=1}^{\infty}$ converges. $\square$

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In fact, $\lim\limits_{n\to\infty} \left|x_{n+j} - x_n\right| = 0$ for every $j \in \mathbb{N}$.
The key point in the definition of Cauchy is that n and k vary independently and can be arbitrarily far apart.