BA: 7.2

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Note the difference with $B_{\mathbb{R}}(0, 1/2) = (-1/2, 1/2)$.

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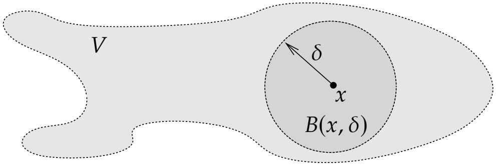

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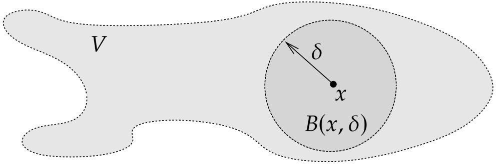

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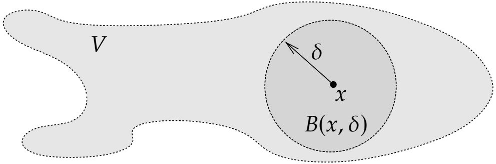

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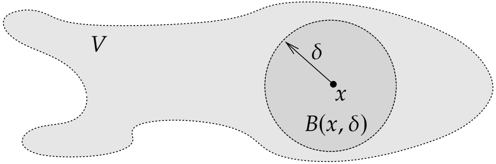

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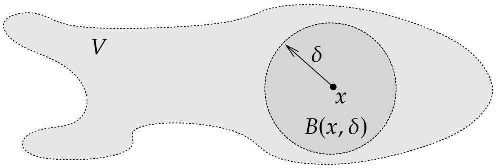

E is closed if everything not in E is some distance away from E.

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In any metric space (X,d), if $x \in X$, then $\{x\}$ is closed (exercise).

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(iii) *If* $\{V_\lambda\}_{\lambda \in I}$ *is an arbitrary collection of open subsets of* X*, then*

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Example: (ii) is not true for an arbitrary intersection:
$\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$, which is not open.

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Proof: Exercise.

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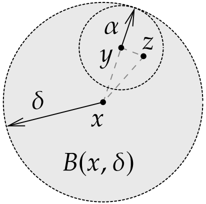

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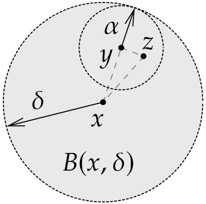

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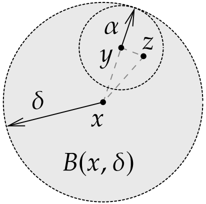

That $C(x, \delta)$ is closed is an exercise. □

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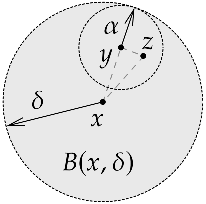

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Careful: $[0, 1/2)$ is an open ball in $[0, 1]$, and so $[0, 1/2)$ is an open set in $[0, 1]$.

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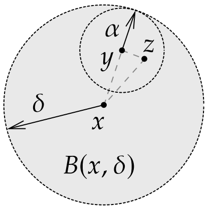

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$[0, 1/2)$ is neither open nor closed in $\mathbb{R}$.

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The opposite direction is an exercise. $\square$

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Now suppose U is open in X.

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(i) *$U \subset V$ is open in the subspace topology* $\Leftrightarrow$ *U is open in X.*

(ii) *$F \subset E$ is closed in the subspace topology* $\Leftrightarrow$ *F is closed in X.*

Proof: (i): Suppose $U \subset V$ is open in the subspace topology.

By the previous proposition, $\exists\, W \subset X$, open in X, such that $U = W \cap V$

Intersection of open sets is open $\Rightarrow$ U is open in X.

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(ii): Exercise. □

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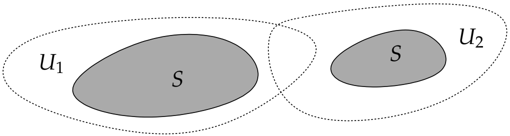

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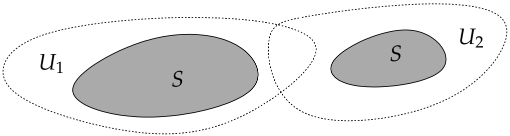

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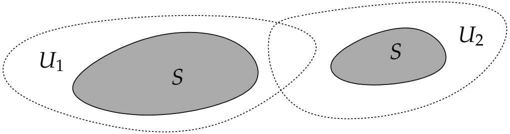

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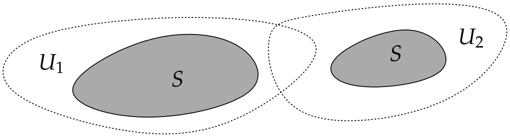

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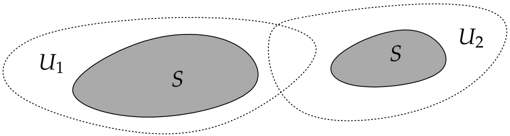

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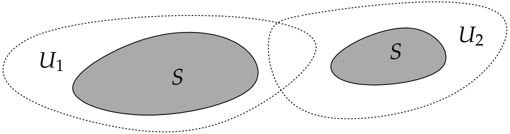

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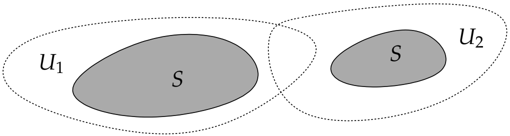

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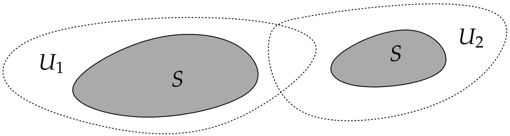

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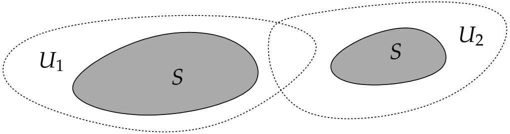

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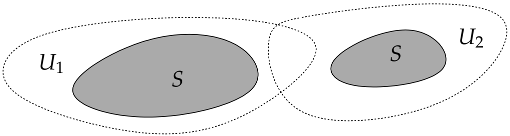

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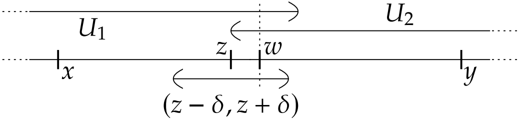

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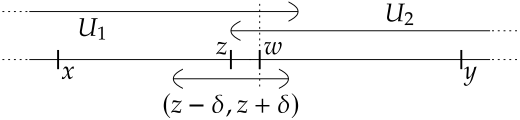

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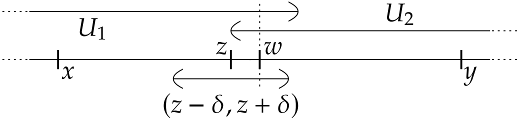

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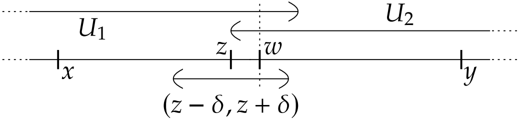

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Definition

Let (X, d) be a metric space and $A \subset X$. The *interior* of A is the set

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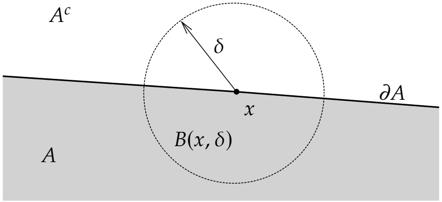

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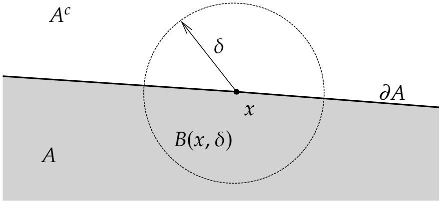

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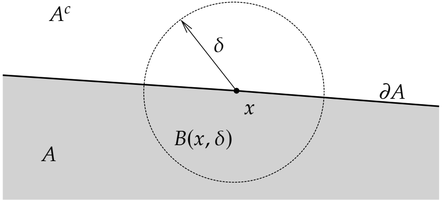

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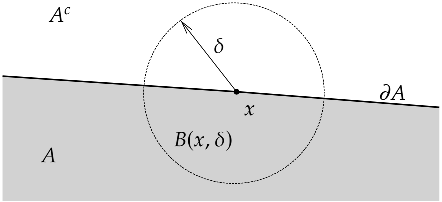

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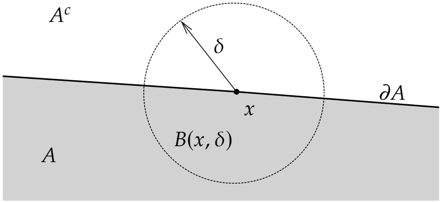

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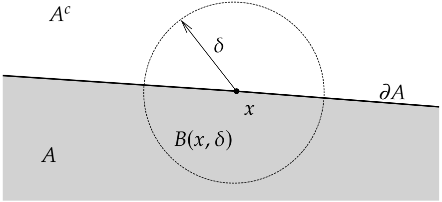

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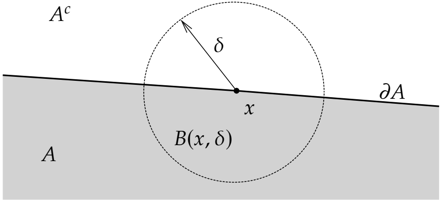

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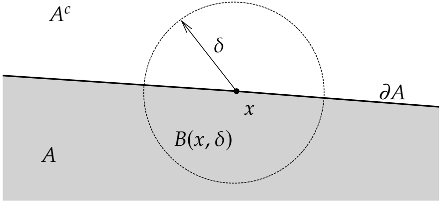

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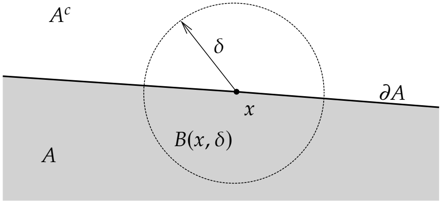

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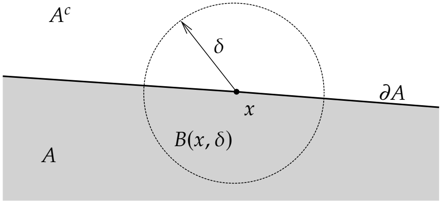

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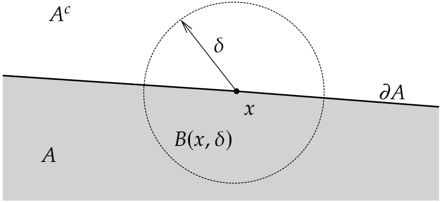

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If $x \in A^\circ$, then $B(x,\delta) \subset A$ for some $\delta > 0$, so $B(x,\delta) \cap A^c = \emptyset$. $\square$

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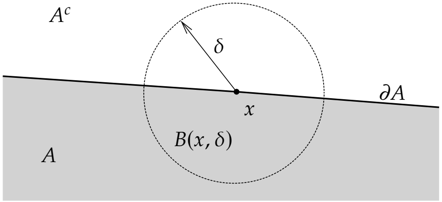

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Corollary

Let (X, d) be a metric space and $A \subset X$. Then $\partial A = \overline{A} \cap \overline{A^c}$.

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Exercise: Let (X, d) be a metric space and $A \subset X$. Show that $A^\circ = \bigcup\{V : V \text{ is open and } V \subset A\}$.