BA: 3.1

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Definition

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(v) $\mathbb{N}$ has no cluster points.

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Let $S \subset \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of S if and only if there exists a convergent sequence of numbers $\{x_n\}_{n=1}^{\infty}$ such that $x_n \neq x$ and $x_n \in S$ for all n, and $\lim\limits_{n\to\infty} x_n = x$.

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$\Rightarrow$ $x_M \in (x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$. $\square$

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$\Rightarrow \quad L_1 = L_2.$ $\square$

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By reverse triangle inequality, $\quad |x| - |c| \leq |x-c| < 1.$

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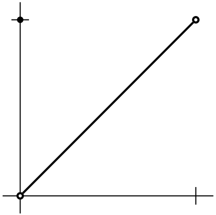

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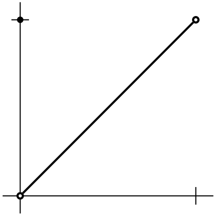

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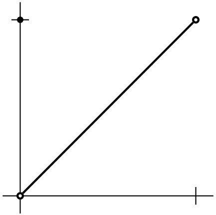

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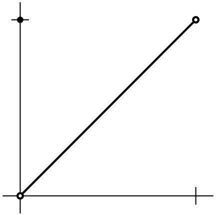

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Exercise: It is possible to strengthen the $\Leftarrow$: It is enough to suppose that $\{f(x_n)\}_{n=1}^{\infty}$ converges for every $\{x_n\}_{n=1}^{\infty}$ without requiring a specific limit.

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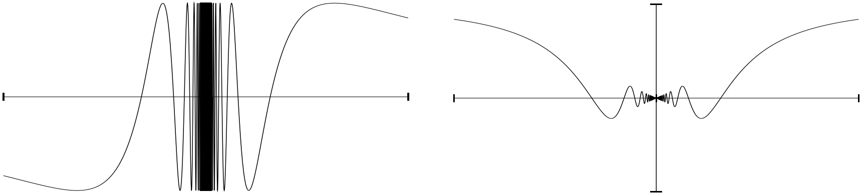

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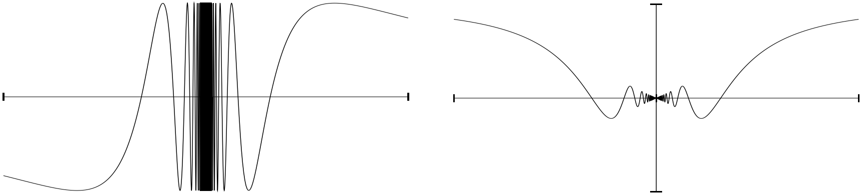

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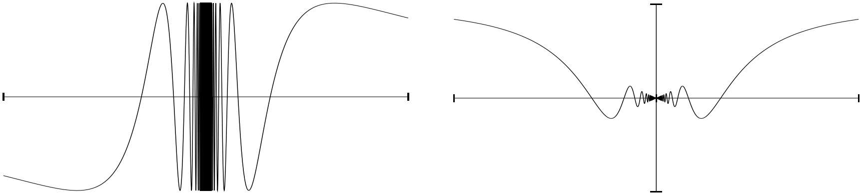

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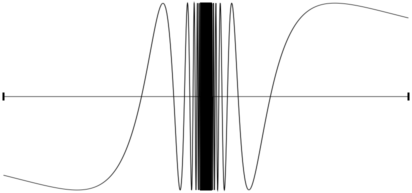
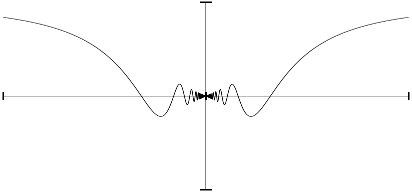

Proof: Define $x_n := \frac{1}{\pi n + \pi/2}$, so that $x_n \to 0 \quad \Rightarrow \quad \sin(1/x_n) = \sin(\pi n + \pi/2)$

Example: $\lim_{x\to 0} \sin(1/x)$ does not exist, but $\lim_{x\to 0} x\sin(1/x) = 0$.

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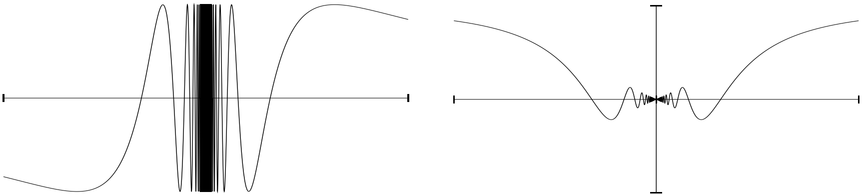

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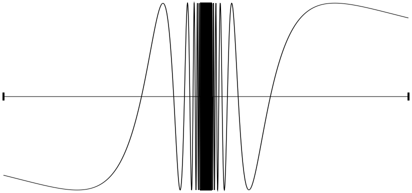

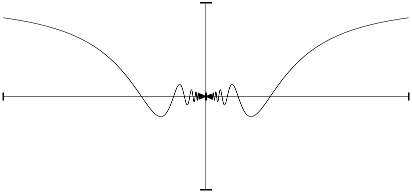

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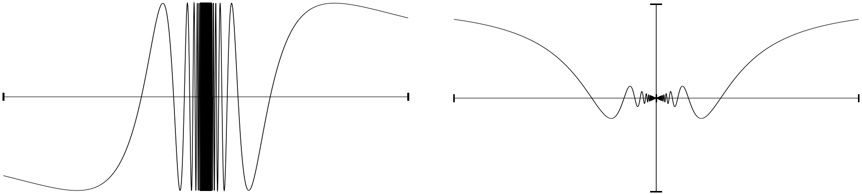

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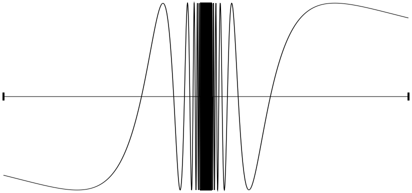

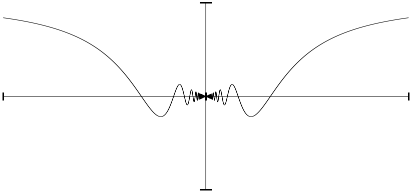

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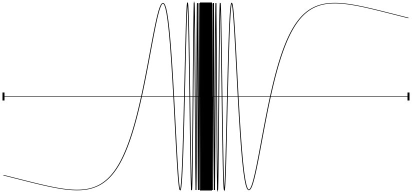

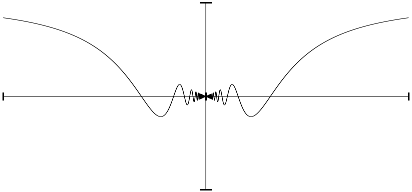

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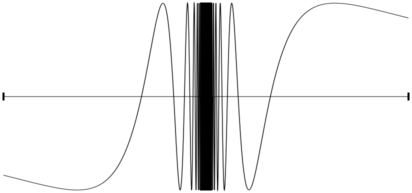

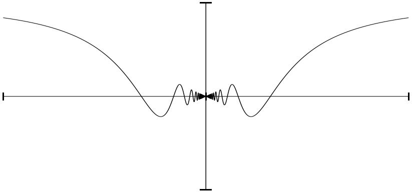

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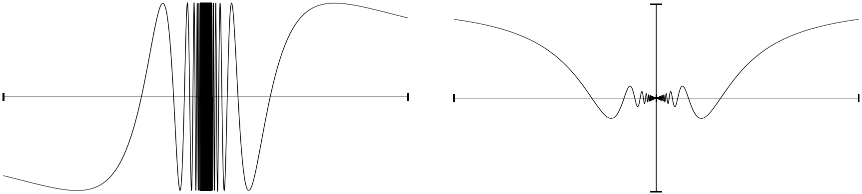

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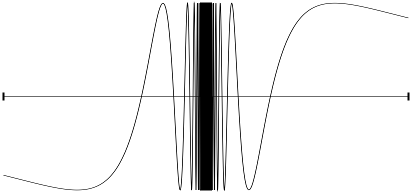

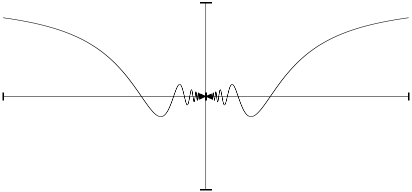

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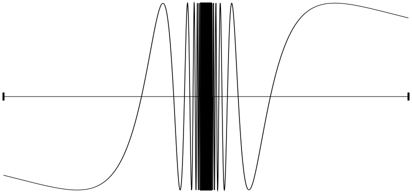
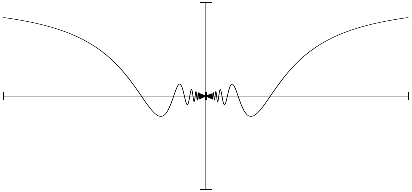

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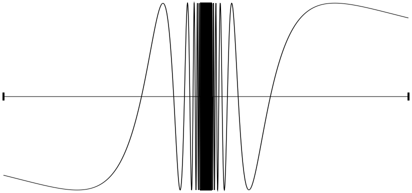
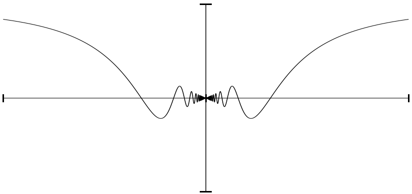

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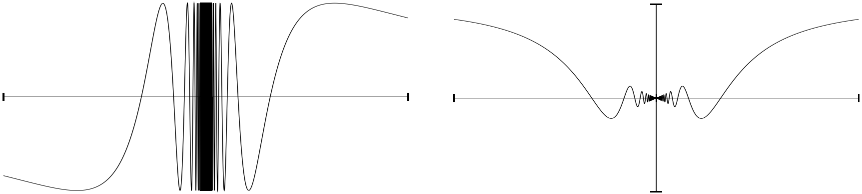

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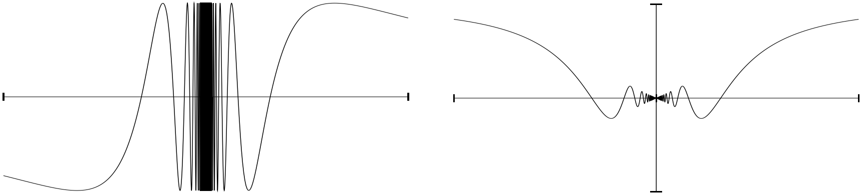

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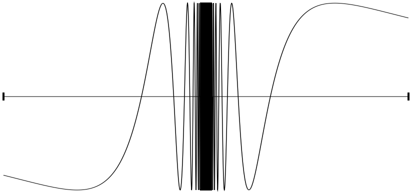

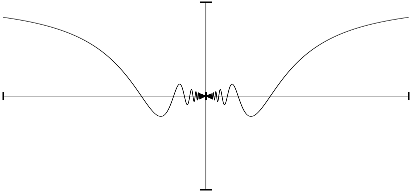

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Remark: Keep in mind the "for every sequence":

If $x_n := 1/\pi n$, then $\{\sin(1/x_n)\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$, but $\lim_{x\to 0} \sin(1/x)$ DNE.

Corollary

Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist, and

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$\Rightarrow \quad L_1 \leq L_2$ (by lemma about sequence limits) $\square$

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Let $S \subset \mathbb{R}$ *and let* c *be a cluster point of* S. *Suppose* $f\colon S \to \mathbb{R}$, $g\colon S \to \mathbb{R}$, *and* $h\colon S \to \mathbb{R}$ *are such that* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in S \setminus \{c\}$.

Proofs of the next corollaries are exercises.

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Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is such that the limit of $f(x)$ as $x \to c$ exists. Suppose $\exists\, a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b$ for all $x \in S \setminus \{c\}$.

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Then the limit of $g(x)$ as x goes to c exists and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x).$

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Notation: $\displaystyle \lim_{\substack{x \to c \\ x \in A}} f(x) := \lim_{x \to c} f|_A(x).$

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Hint: $\big(S \cap (-\infty, c)\big) \cup \big(S \cap (c, \infty)\big) = S \setminus \{c\}$.

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Hint: Note that $f(x)$ could equal c_2 for many $x \in A$.