BA: 7.1

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

We wish to generalize convergence, continuity, etc.

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d \colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. *(identity of indiscernibles)*

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d \colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. *(identity of indiscernibles)*

(iii) $d(x, y) = d(y, x)$. *(symmetry)*

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. (*nonnegativity*)

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. (*identity of indiscernibles*)

(iii) $d(x, y) = d(y, x)$. (*symmetry*)

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. (*triangle inequality*)

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. *(identity of indiscernibles)*

(iii) $d(x, y) = d(y, x)$. *(symmetry)*

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. *(triangle inequality)*

(X, d) is called a *metric space*.

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. *(identity of indiscernibles)*

(iii) $d(x, y) = d(y, x)$. *(symmetry)*

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. *(triangle inequality)*

(X, d) is called a *metric space*.

d is called the *metric* or the *distance function*.

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. (*nonnegativity*)

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. (*identity of indiscernibles*)

(iii) $d(x, y) = d(y, x)$. (*symmetry*)

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. (*triangle inequality*)

(X, d) is called a *metric space*.

d is called the *metric* or the *distance function*.

Sometimes write just X instead of (X, d) if the metric is clear from context.

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. *(identity of indiscernibles)*

(iii) $d(x, y) = d(y, x)$. *(symmetry)*

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. *(triangle inequality)*

(X, d) is called a *metric space*.

d is called the *metric* or the *distance function*.

Sometimes write just X instead of (X, d) if the metric is clear from context.

Think of d as a distance between points.

We wish to generalize convergence, continuity, etc.

Definition

Let X be a set, and let $d\colon X \times X \to \mathbb{R}$ be such that for all $x, y, z \in X$

(i) $d(x, y) \geq 0$. *(nonnegativity)*

(ii) $d(x, y) = 0 \quad \Leftrightarrow \quad x = y$. *(identity of indiscernibles)*

(iii) $d(x, y) = d(y, x)$. *(symmetry)*

(iv) $d(x, z) \leq d(x, y) + d(y, z)$. *(triangle inequality)*

(X, d) is called a *metric space*.

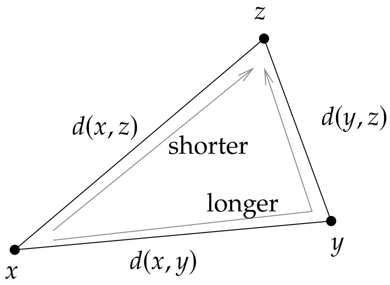
d is called the *metric* or the *distance function*.

Sometimes write just X instead of (X, d) if the metric is clear from context.

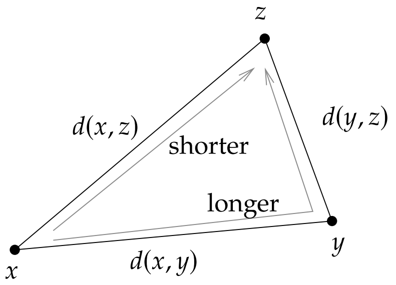
Think of d as a distance between points.

(i)–(iii) have obvious geometric interpretation.

The interpretation of triangle inequality:



The interpretation of triangle inequality:



Warning: It is convenient to draw figures and diagrams in the plane, but that is just one particular metric space!

Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.

Example: $\mathbb{R}$ is a metric space with the metric $\quad d(x,y) := |x-y|$.
(i)–(iii) are easy to verify.

Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z)$$

Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z|$$

Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z|$$

Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z|$$

Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

Example: $\mathbb{R}$ is a metric space with the metric $\quad d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

d is the *standard metric on* $\mathbb{R}$.

Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$.

Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \le |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy.

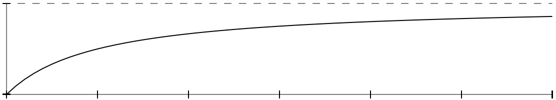
Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x,y) = \varphi(|x-y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



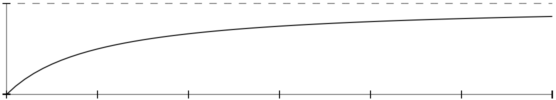
Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x, y) = \varphi(|x - y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$d(x, z)$

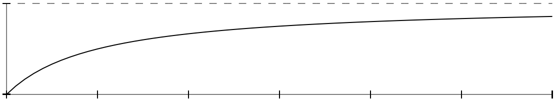
Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \le |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x,y) = \varphi(|x-y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$d(x,z) = \varphi(|x-z|)$

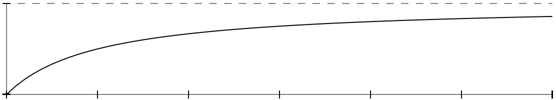
Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x,y) = \varphi(|x-y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$d(x,z) = \varphi(|x-z|) = \varphi(|x-y+y-z|)$

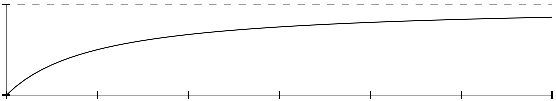
Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z| \le |x - y| + |y - z| = d(x, y) + d(y, z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x, y) := \dfrac{|x - y|}{|x - y| + 1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x, y) = \varphi(|x - y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$$d(x, z) = \varphi(|x - z|) = \varphi(|x - y + y - z|) \le \varphi(|x - y| + |y - z|)$$

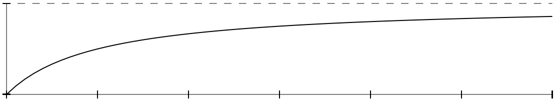
Example: $\mathbb{R}$ is a metric space with the metric $\quad d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \le |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x,y) = \varphi(|x-y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$$d(x,z) = \varphi(|x-z|) = \varphi(|x-y+y-z|) \le \varphi(|x-y| + |y-z|)$$

$$= \frac{|x-y| + |y-z|}{|x-y| + |y-z| + 1}$$

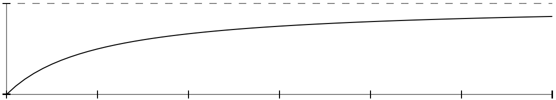
Example: $\mathbb{R}$ is a metric space with the metric $d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x, y) := \dfrac{|x - y|}{|x - y| + 1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x, y) = \varphi(|x - y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$$d(x, z) = \varphi(|x - z|) = \varphi(|x - y + y - z|) \leq \varphi(|x - y| + |y - z|)$$

$$= \frac{|x - y| + |y - z|}{|x - y| + |y - z| + 1} = \frac{|x - y|}{|x - y| + |y - z| + 1} + \frac{|y - z|}{|x - y| + |y - z| + 1}$$

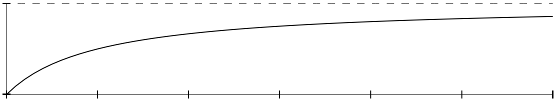
Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x,y) = \varphi(|x-y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$$d(x,z) = \varphi(|x-z|) = \varphi(|x-y+y-z|) \leq \varphi(|x-y|+|y-z|)$$

$$= \frac{|x-y|+|y-z|}{|x-y|+|y-z|+1} = \frac{|x-y|}{|x-y|+|y-z|+1} + \frac{|y-z|}{|x-y|+|y-z|+1}$$

$$\leq \frac{|x-y|}{|x-y|+1} + \frac{|y-z|}{|y-z|+1}$$

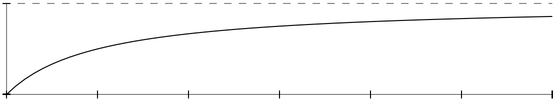
Example: $\mathbb{R}$ is a metric space with the metric $\quad d(x, y) := |x - y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x, z) = |x - z| = |x - y + y - z| \le |x - y| + |y - z| = d(x, y) + d(y, z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x, y) := \dfrac{|x - y|}{|x - y| + 1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x, y) = \varphi(|x - y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$$\begin{aligned} d(x, z) &= \varphi(|x - z|) = \varphi(|x - y + y - z|) \le \varphi(|x - y| + |y - z|) \\ &= \frac{|x - y| + |y - z|}{|x - y| + |y - z| + 1} = \frac{|x - y|}{|x - y| + |y - z| + 1} + \frac{|y - z|}{|x - y| + |y - z| + 1} \\ &\le \frac{|x - y|}{|x - y| + 1} + \frac{|y - z|}{|y - z| + 1} = d(x, y) + d(y, z). \end{aligned}$$

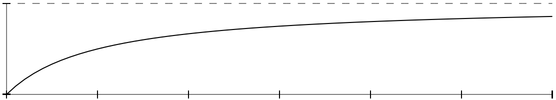
Example: $\mathbb{R}$ is a metric space with the metric $d(x,y) := |x-y|$.
(i)–(iii) are easy to verify. For triangle inequality, (iv):

$$d(x,z) = |x-z| = |x-y+y-z| \leq |x-y| + |y-z| = d(x,y) + d(y,z).$$

d is the *standard metric on* $\mathbb{R}$. (It's the default one if none mentioned.)

Example: We can also have a different metric on $\mathbb{R}$. E.g., $d(x,y) := \dfrac{|x-y|}{|x-y|+1}$.

(i)–(iii) are again easy. For triangle inequality, (iv), note $d(x,y) = \varphi(|x-y|)$, where $\varphi(t) = \frac{t}{t+1}$ and φ is increasing.



$$d(x,z) = \varphi(|x-z|) = \varphi(|x-y+y-z|) \leq \varphi(|x-y|+|y-z|)$$

$$= \frac{|x-y|+|y-z|}{|x-y|+|y-z|+1} = \frac{|x-y|}{|x-y|+|y-z|+1} + \frac{|y-z|}{|x-y|+|y-z|+1}$$

$$\leq \frac{|x-y|}{|x-y|+1} + \frac{|y-z|}{|y-z|+1} = d(x,y) + d(y,z).$$

With this metric, $d(x,y) < 1$ for all $x, y \in \mathbb{R}$.

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $\quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

Lemma (Cauchy–Schwarz inequality)

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

Lemma (Cauchy–Schwarz inequality)

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, *then*

$$\left(\sum_{k=1}^{n} x_k y_k \right)^2 \leq \left(\sum_{k=1}^{n} x_k^2 \right) \left(\sum_{k=1}^{n} y_k^2 \right).$$

Proof: A square of a real number is nonnegative.

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

Lemma (Cauchy–Schwarz inequality)

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, *then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Proof: A square of a real number is nonnegative. Hence,

$0 \leq \sum_{k=1}^{n} \sum_{\ell=1}^{n} (x_k y_\ell - x_\ell y_k)^2$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

Lemma (Cauchy–Schwarz inequality)

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Proof: A square of a real number is nonnegative. Hence,

$0 \leq \sum_{k=1}^{n} \sum_{\ell=1}^{n} (x_k y_\ell - x_\ell y_k)^2 = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(x_k^2 y_\ell^2 + x_\ell^2 y_k^2 - 2 x_k x_\ell y_k y_\ell\right)$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

Lemma (Cauchy–Schwarz inequality)

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Proof: A square of a real number is nonnegative. Hence,

$$0 \leq \sum_{k=1}^{n} \sum_{\ell=1}^{n} (x_k y_\ell - x_\ell y_k)^2 = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(x_k^2 y_\ell^2 + x_\ell^2 y_k^2 - 2 x_k x_\ell y_k y_\ell\right)$$

$$= \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{\ell=1}^{n} y_\ell^2\right) + \left(\sum_{k=1}^{n} y_k^2\right)\left(\sum_{\ell=1}^{n} x_\ell^2\right) - 2\left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{\ell=1}^{n} x_\ell y_\ell\right).$$

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ is the n-dimensional *euclidean space*.
Notation for points: $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. (We don't write $\vec{x}$ nor $\mathbf{x}$)
We also write $0 \in \mathbb{R}^n$ to mean $(0, 0, \ldots, 0)$.

Lemma (Cauchy–Schwarz inequality)

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, *then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Proof: A square of a real number is nonnegative. Hence,

$$0 \leq \sum_{k=1}^{n} \sum_{\ell=1}^{n} (x_k y_\ell - x_\ell y_k)^2 = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \left(x_k^2 y_\ell^2 + x_\ell^2 y_k^2 - 2 x_k x_\ell y_k y_\ell\right)$$

$$= \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{\ell=1}^{n} y_\ell^2\right) + \left(\sum_{k=1}^{n} y_k^2\right)\left(\sum_{\ell=1}^{n} x_\ell^2\right) - 2\left(\sum_{k=1}^{n} x_k y_k\right)\left(\sum_{\ell=1}^{n} x_\ell y_\ell\right).$$

$$\Rightarrow \quad 0 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \left(\sum_{k=1}^{n} x_k y_k\right)^2. \qquad \square$$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Note: If $n = 1$ agrees with above.

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$\big(d(x, z)\big)^2$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$\big(d(x,z)\big)^2 = \sum_{k=1}^{n} (x_k - z_k)^2$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} = \sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$\big(d(x,z)\big)^2 = \sum_{k=1}^{n}(x_k-z_k)^2 = \sum_{k=1}^{n}(x_k-y_k+y_k-z_k)^2$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$\big(d(x,z)\big)^2 = \sum_{k=1}^{n} (x_k - z_k)^2 = \sum_{k=1}^{n} (x_k - y_k + y_k - z_k)^2$

$\quad = \sum_{k=1}^{n} \Big((x_k - y_k)^2 + (y_k - z_k)^2 + 2(x_k - y_k)(y_k - z_k)\Big)$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} = \sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$\big(d(x,z)\big)^2 = \sum_{k=1}^{n}(x_k-z_k)^2 = \sum_{k=1}^{n}(x_k-y_k+y_k-z_k)^2$

$\quad = \sum_{k=1}^{n}\Big((x_k-y_k)^2+(y_k-z_k)^2+2(x_k-y_k)(y_k-z_k)\Big)$

$\quad = \sum_{k=1}^{n}(x_k-y_k)^2+\sum_{k=1}^{n}(y_k-z_k)^2+2\sum_{k=1}^{n}(x_k-y_k)(y_k-z_k)$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$$\big(d(x,z)\big)^2 = \textstyle\sum_{k=1}^n (x_k - z_k)^2 = \sum_{k=1}^n (x_k - y_k + y_k - z_k)^2$$

$$= \textstyle\sum_{k=1}^n \Big((x_k - y_k)^2 + (y_k - z_k)^2 + 2(x_k - y_k)(y_k - z_k)\Big)$$

$$= \textstyle\sum_{k=1}^n (x_k - y_k)^2 + \sum_{k=1}^n (y_k - z_k)^2 + 2\sum_{k=1}^n (x_k - y_k)(y_k - z_k)$$

$$\leq \textstyle\sum_{k=1}^n (x_k - y_k)^2 + \sum_{k=1}^n (y_k - z_k)^2 + 2\sqrt{\sum_{k=1}^n (x_k - y_k)^2 \sum_{k=1}^n (y_k - z_k)^2}$$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} = \sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}.$$

Note: If $n = 1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$$\begin{aligned}
\big(d(x,z)\big)^2 &= \textstyle\sum_{k=1}^n (x_k-z_k)^2 = \sum_{k=1}^n (x_k-y_k+y_k-z_k)^2 \\
&= \textstyle\sum_{k=1}^n \Big((x_k-y_k)^2+(y_k-z_k)^2+2(x_k-y_k)(y_k-z_k)\Big) \\
&= \textstyle\sum_{k=1}^n (x_k-y_k)^2 + \sum_{k=1}^n (y_k-z_k)^2 + 2\sum_{k=1}^n (x_k-y_k)(y_k-z_k) \\
&\le \textstyle\sum_{k=1}^n (x_k-y_k)^2 + \sum_{k=1}^n (y_k-z_k)^2 + 2\sqrt{\sum_{k=1}^n (x_k-y_k)^2 \sum_{k=1}^n (y_k-z_k)^2} \\
&= \left(\sqrt{\textstyle\sum_{k=1}^n (x_k-y_k)^2} + \sqrt{\textstyle\sum_{k=1}^n (y_k-z_k)^2}\right)^2
\end{aligned}$$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} = \sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}.$$

Note: If $n=1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$$\begin{aligned}
\big(d(x,z)\big)^2 &= \textstyle\sum_{k=1}^n (x_k-z_k)^2 = \sum_{k=1}^n (x_k-y_k+y_k-z_k)^2 \\
&= \textstyle\sum_{k=1}^n \Big((x_k-y_k)^2+(y_k-z_k)^2+2(x_k-y_k)(y_k-z_k)\Big) \\
&= \textstyle\sum_{k=1}^n (x_k-y_k)^2+\sum_{k=1}^n (y_k-z_k)^2+2\sum_{k=1}^n (x_k-y_k)(y_k-z_k) \\
&\le \textstyle\sum_{k=1}^n (x_k-y_k)^2+\sum_{k=1}^n (y_k-z_k)^2+2\sqrt{\sum_{k=1}^n (x_k-y_k)^2\sum_{k=1}^n (y_k-z_k)^2} \\
&= \left(\sqrt{\textstyle\sum_{k=1}^n (x_k-y_k)^2}+\sqrt{\textstyle\sum_{k=1}^n (y_k-z_k)^2}\right)^2 = \big(d(x,y)+d(y,z)\big)^2.
\end{aligned}$$

Example: We define the standard metric for $\mathbb{R}^n$:

$$d(x,y) := \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2} = \sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}.$$

Note: If $n=1$ agrees with above.

The only tricky bit (again) is to check triangle inequality.

$$\begin{aligned}
\big(d(x,z)\big)^2 &= \textstyle\sum_{k=1}^n (x_k-z_k)^2 = \sum_{k=1}^n (x_k-y_k+y_k-z_k)^2 \\
&= \textstyle\sum_{k=1}^n \Big((x_k-y_k)^2+(y_k-z_k)^2+2(x_k-y_k)(y_k-z_k)\Big) \\
&= \textstyle\sum_{k=1}^n (x_k-y_k)^2+\sum_{k=1}^n (y_k-z_k)^2+2\sum_{k=1}^n(x_k-y_k)(y_k-z_k) \\
&\le \textstyle\sum_{k=1}^n (x_k-y_k)^2+\sum_{k=1}^n (y_k-z_k)^2+2\sqrt{\sum_{k=1}^n (x_k-y_k)^2\sum_{k=1}^n (y_k-z_k)^2} \\
&= \Big(\sqrt{\textstyle\sum_{k=1}^n (x_k-y_k)^2}+\sqrt{\textstyle\sum_{k=1}^n (y_k-z_k)^2}\Big)^2 = \big(d(x,y)+d(y,z)\big)^2.
\end{aligned}$$

The triangle inequality follows as $\sqrt{\cdot}$ is an increasing function.

Example: Complex numbers $\mathbb{C}$ is the set of numbers $z = x + iy$, where $x, y \in \mathbb{R}$.

Example: Complex numbers $\mathbb{C}$ is the set of numbers $z = x + iy$, where $x, y \in \mathbb{R}$.

Imposing $i^2 = -1$, makes $\mathbb{C}$ into a field.

Example: Complex numbers $\mathbb{C}$ is the set of numbers $z = x + iy$, where $x, y \in \mathbb{R}$.

Imposing $i^2 = -1$, makes $\mathbb{C}$ into a field.

To make $\mathbb{C}$ into a metric space, identify $\mathbb{C}$ with $\mathbb{R}^2$ by

$$x + iy \in \mathbb{C} \quad \leftrightarrow \quad (x, y) \in \mathbb{R}^2,$$

and use the standard metric.

Example: Complex numbers $\mathbb{C}$ is the set of numbers $z = x + iy$, where $x, y \in \mathbb{R}$.

Imposing $i^2 = -1$, makes $\mathbb{C}$ into a field.

To make $\mathbb{C}$ into a metric space, identify $\mathbb{C}$ with $\mathbb{R}^2$ by

$$x + iy \in \mathbb{C} \quad \leftrightarrow \quad (x, y) \in \mathbb{R}^2,$$

and use the standard metric.

Define the *complex modulus* $|x + iy| := \sqrt{x^2 + y^2}$.

Example: Complex numbers $\mathbb{C}$ is the set of numbers $z = x + iy$, where $x, y \in \mathbb{R}$.

Imposing $i^2 = -1$, makes $\mathbb{C}$ into a field.

To make $\mathbb{C}$ into a metric space, identify $\mathbb{C}$ with $\mathbb{R}^2$ by

$$x + iy \in \mathbb{C} \quad \leftrightarrow \quad (x, y) \in \mathbb{R}^2,$$

and use the standard metric.

Define the *complex modulus* $|x + iy| := \sqrt{x^2 + y^2}$.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|.$$

Example: For any set X, define the *discrete metric*:

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Example: For any set X, define the *discrete metric*:

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

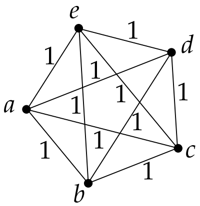
Easy to check the properties.

Example: For any set X, define the *discrete metric*:

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Easy to check the properties.

E.g., if $X = \{a, b, c, d, e\}$, then

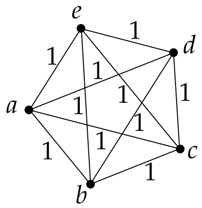


Example: For any set X, define the *discrete metric*:

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Easy to check the properties.

E.g., if $X = \{a, b, c, d, e\}$, then



A very useful "smell test" for statements about metric spaces.

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$.

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b]\to\mathbb{R}$. Define the (standard) metric as $\quad d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as

$$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $\quad d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x)-g(x)| = 0$

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $\quad d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x) - g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $\quad d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$.

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x) - g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x,\, |f(x)-g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x,\, |f(x)-g(x)| \leq d(f,g) = 0$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x) - g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x) - g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x)$

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x) - g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x) - g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \to \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x)-g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x)-g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

(iii) $d(f,g) = d(g,f)$ is trivial.

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $\quad d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x) - g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x) - g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

(iii) $d(f,g) = d(g,f)$ is trivial.

(iv) $d(f,g)$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x)-g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x)-g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

(iii) $d(f,g) = d(g,f)$ is trivial.

(iv) $d(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)|$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as
$$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x)-g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x)-g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

(iii) $d(f,g) = d(g,f)$ is trivial.

(iv) $$d(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| = \sup_{x\in[a,b]} |f(x)-h(x)+h(x)-g(x)|$$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $$d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f=g \quad\Rightarrow\quad \forall x, |f(x)-g(x)| = 0 \quad\Rightarrow\quad d(f,g) = 0.$
$d(f,g) = 0 \quad\Rightarrow\quad \forall x, |f(x)-g(x)| \leq d(f,g) = 0 \quad\Rightarrow\quad \forall x, f(x) = g(x) \quad\Rightarrow\quad f = g.$

(iii) $d(f,g) = d(g,f)$ is trivial.

(iv) $$d(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| = \sup_{x\in[a,b]} |f(x)-h(x)+h(x)-g(x)|$$
$$\leq \sup_{x\in[a,b]} \big(|f(x)-h(x)| + |h(x)-g(x)|\big)$$

Example: Let $C\big([a,b],\mathbb{R}\big)$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $\quad d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|.$

$d(f,g)$ is finite as $|f(x) - g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x) - g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x) - g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

(iii) $d(f,g) = d(g,f)$ is trivial.

(iv)
$$
\begin{aligned}
d(f,g) &= \sup_{x\in[a,b]} |f(x) - g(x)| = \sup_{x\in[a,b]} |f(x) - h(x) + h(x) - g(x)| \\
&\leq \sup_{x\in[a,b]} \big(|f(x) - h(x)| + |h(x) - g(x)|\big) \\
&\leq \sup_{x\in[a,b]} |f(x) - h(x)| + \sup_{x\in[a,b]} |h(x) - g(x)|
\end{aligned}
$$

Example: Let $C([a,b],\mathbb{R})$ be the set of continuous functions $f\colon [a,b] \longrightarrow \mathbb{R}$. Define the (standard) metric as $\displaystyle d(f,g) := \sup_{x\in[a,b]} |f(x)-g(x)|.$

$d(f,g)$ is finite as $|f(x)-g(x)|$ is a continuous function on a closed bounded interval $[a,b]$ and so bounded.

(i) Clearly $d(f,g) \geq 0$.

(ii) $f = g \quad \Rightarrow \quad \forall x, |f(x)-g(x)| = 0 \quad \Rightarrow \quad d(f,g) = 0.$
$d(f,g) = 0 \quad \Rightarrow \quad \forall x, |f(x)-g(x)| \leq d(f,g) = 0 \quad \Rightarrow \quad \forall x, f(x) = g(x) \quad \Rightarrow \quad f = g.$

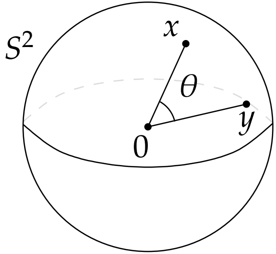
(iii) $d(f,g) = d(g,f)$ is trivial.

(iv)
$$\begin{aligned} d(f,g) &= \sup_{x\in[a,b]} |f(x)-g(x)| = \sup_{x\in[a,b]} |f(x)-h(x)+h(x)-g(x)| \\ &\leq \sup_{x\in[a,b]} \big(|f(x)-h(x)| + |h(x)-g(x)|\big) \\ &\leq \sup_{x\in[a,b]} |f(x)-h(x)| + \sup_{x\in[a,b]} |h(x)-g(x)| = d(f,h)+d(h,g). \end{aligned}$$

Example (Great circle distance)**:** $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is the unit sphere in $\mathbb{R}^3$.

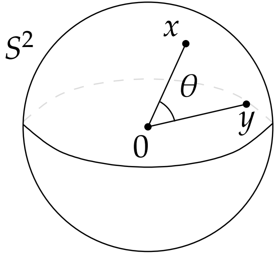
Example (Great circle distance)**:** $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is the unit sphere in $\mathbb{R}^3$.

If x and y are in S^2, the lines they make with the origin meet at angle θ (in radians).



Example (Great circle distance)**:** $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is the unit sphere in $\mathbb{R}^3$.

If x and y are in S^2, the lines they make with the origin meet at angle θ (in radians).

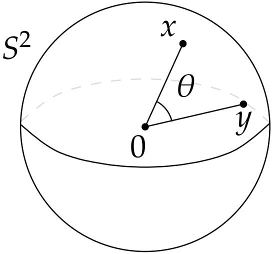


The function defined by

$$d(x, y) := \theta$$

Example (Great circle distance)**:** $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is the unit sphere in $\mathbb{R}^3$.

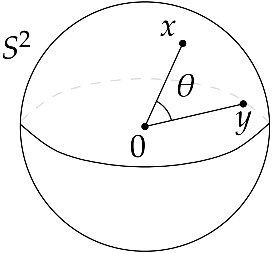
If x and y are in S^2, the lines they make with the origin meet at angle θ (in radians).



The function defined by
$$d(x,y) := \theta = \arccos(x_1y_1 + x_2y_2 + x_3y_3)$$
gives a metric (we skip the proof).

Example (Great circle distance)**:** $S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is the unit sphere in $\mathbb{R}^3$.

If x and y are in S^2, the lines they make with the origin meet at angle θ (in radians).



The function defined by

$$d(x, y) := \theta = \arccos(x_1y_1 + x_2y_2 + x_3y_3)$$

gives a metric (we skip the proof).

It is the shortest path between the points if we travel along the sphere.

Proposition

Let (X, d) be a metric space and $Y \subset X$. Then the restriction $d|_{Y \times Y}$ is a metric on Y.

Proposition

Let (X, d) be a metric space and $Y \subset X$. Then the restriction $d|_{Y \times Y}$ is a metric on Y.

Proof: Obvious.

Proposition

Let (X, d) *be a metric space and* $Y \subset X$*. Then the restriction* $d|_{Y \times Y}$ *is a metric on* Y*.*

Proof: Obvious.

Definition

If (X, d) is a metric space, $Y \subset X$, and $d' := d|_{Y \times Y}$, then (Y, d') is a *subspace* of (X, d), and d' is the *subspace metric* (sometimes we talk about *subspace topology*).

Proposition

Let (X, d) *be a metric space and* $Y \subset X$. *Then the restriction* $d|_{Y \times Y}$ *is a metric on* Y.

Proof: Obvious.

Definition

If (X, d) is a metric space, $Y \subset X$, and $d' := d|_{Y \times Y}$, then (Y, d') is a *subspace* of (X, d), and d' is the *subspace metric* (sometimes we talk about *subspace topology*).

Common to just write d for the metric on Y.

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

(X, d) is bounded if X is a bounded subset.

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

(X, d) is bounded if X is a bounded subset.

Example: $\mathbb{R}$ with standard metric is not bounded.

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

(X, d) is bounded if X is a bounded subset.

Example: $\mathbb{R}$ with standard metric is not bounded.

Example: $\mathbb{R}$ with the discrete metric is bounded.

Definition

Let (X,d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p,x) \leq B \quad \text{for all } x \in S.$$

(X,d) is bounded if X is a bounded subset.

Example: $\mathbb{R}$ with standard metric is not bounded.

Example: $\mathbb{R}$ with the discrete metric is bounded.

Exercise: Suppose $X \neq \emptyset$. Then $S \subset X$ being bounded is equivalent to

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

(X, d) is bounded if X is a bounded subset.

Example: $\mathbb{R}$ with standard metric is not bounded.

Example: $\mathbb{R}$ with the discrete metric is bounded.

Exercise: Suppose $X \neq \emptyset$. Then $S \subset X$ being bounded is equivalent to

(i) $\forall\, p \in X, \exists\, B > 0$ such that $d(p, x) \leq B$ for all $x \in S$.

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

(X, d) is bounded if X is a bounded subset.

Example: $\mathbb{R}$ with standard metric is not bounded.

Example: $\mathbb{R}$ with the discrete metric is bounded.

Exercise: Suppose $X \neq \emptyset$. Then $S \subset X$ being bounded is equivalent to

(i) $\forall\, p \in X, \exists\, B > 0$ such that $d(p, x) \leq B$ for all $x \in S$.

(ii) $\operatorname{diam}(S) := \sup\{d(x, y) : x, y \in S\} < \infty$.

Definition

Let (X, d) be a metric space. $S \subset X$ is *bounded* if $\exists\, p \in X$ and $B \in \mathbb{R}$ such that

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

(X, d) is bounded if X is a bounded subset.

Example: $\mathbb{R}$ with standard metric is not bounded.

Example: $\mathbb{R}$ with the discrete metric is bounded.

Exercise: Suppose $X \neq \emptyset$. Then $S \subset X$ being bounded is equivalent to

(i) $\forall\, p \in X, \exists\, B > 0$ such that $d(p, x) \leq B$ for all $x \in S$.

(ii) $\operatorname{diam}(S) := \sup\{d(x, y) : x, y \in S\} < \infty$.

The quantity $\operatorname{diam}(S)$ is called the *diameter* of a set.

Exercise: Let (X, d_X) and (Y, d_Y) be metric spaces.

a) Show that $(X \times Y, d)$ with $d\big((x_1, y_1), (x_2, y_2)\big) := d_X(x_1, x_2) + d_Y(y_1, y_2)$ is a metric space.

b) Show that $(X \times Y, d)$ with $d\big((x_1, y_1), (x_2, y_2)\big) := \max\big\{d_X(x_1, x_2), d_Y(y_1, y_2)\big\}$ is a metric space.

Exercise: Let (X, d_X) and (Y, d_Y) be metric spaces.

a) Show that $(X \times Y, d)$ with $d\big((x_1, y_1), (x_2, y_2)\big) := d_X(x_1, x_2) + d_Y(y_1, y_2)$ is a metric space.
b) Show that $(X \times Y, d)$ with $d\big((x_1, y_1), (x_2, y_2)\big) := \max\big\{d_X(x_1, x_2), d_Y(y_1, y_2)\big\}$ is a metric space.

Exercise: Let X be the set of continuous functions on $[0, 1]$. Let $\varphi \colon [0, 1] \to (0, \infty)$ be continuous. Define

$$d(f, g) := \int_0^1 |f(x) - g(x)| \, \varphi(x) \, dx.$$

Show that (X, d) is a metric space.