BA: 4.1

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Definition

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ be a function, and let $c \in I$.

Definition

Let I be an interval, let $f : I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

Definition

Let I be an interval, let $f\colon I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

then f is *differentiable* at c, L is the *derivative* of f at c, and write $f'(c) := L$.

Definition

Let I be an interval, let $f\colon I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

then f is *differentiable* at c, L is the *derivative* of f at c, and write $f'(c) := L$.

If f is differentiable at all $c \in I$, say f is *differentiable*, and obtain a function $f'\colon I \to \mathbb{R}$.

Definition

Let I be an interval, let $f : I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

then f is *differentiable* at c, L is the *derivative* of f at c, and write $f'(c) := L$.

If f is differentiable at all $c \in I$, say f is *differentiable*, and obtain a function $f' : I \to \mathbb{R}$.

Other notation: $\dfrac{df}{dx}$ or $\dfrac{d}{dx}\big(f(x)\big)$.

Definition

Let I be an interval, let $f\colon I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

then f is *differentiable* at c, L is the *derivative* of f at c, and write $f'(c) := L$.

If f is differentiable at all $c \in I$, say f is *differentiable*, and obtain a function $f'\colon I \to \mathbb{R}$.

Other notation: $\dfrac{df}{dx}$ or $\dfrac{d}{dx}\big(f(x)\big)$.

The expression $\dfrac{f(x) - f(c)}{x - c}$ is called the *difference quotient*.

Definition

Let I be an interval, let $f : I \to \mathbb{R}$ be a function, and let $c \in I$. If the limit

$$L := \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{exists,}$$

then f is *differentiable* at c, L is the *derivative* of f at c, and write $f'(c) := L$.

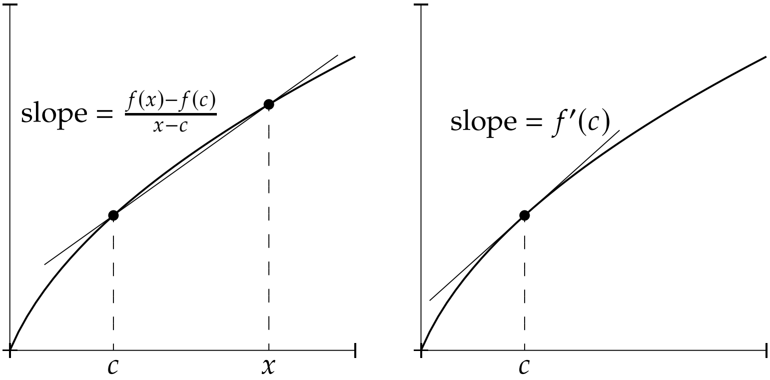
If f is differentiable at all $c \in I$, say f is *differentiable*, and obtain a function $f' : I \to \mathbb{R}$.

Other notation: $\dfrac{df}{dx}$ or $\dfrac{d}{dx}\big(f(x)\big)$.

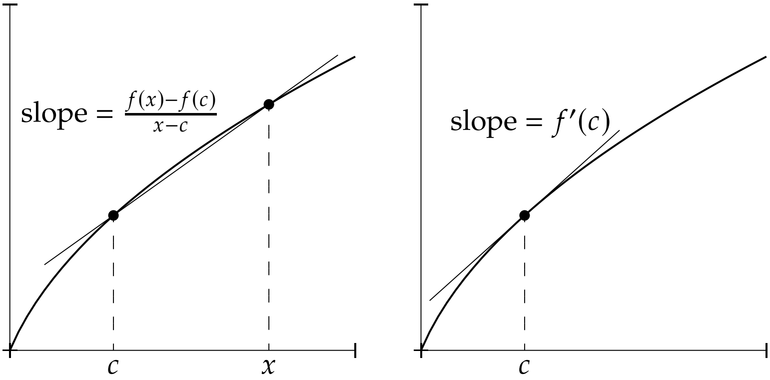
The expression $\dfrac{f(x) - f(c)}{x - c}$ is called the *difference quotient*.

Remark: We allow I to be a closed interval, and we allow c to be an endpoint of I.

The graphical interpretation of the derivative:

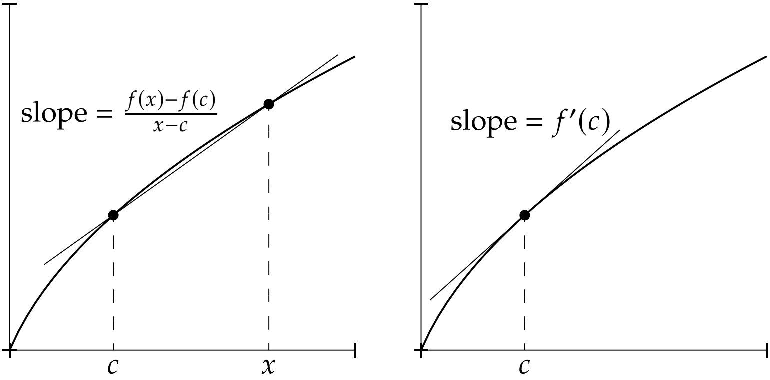


The graphical interpretation of the derivative:



$$\frac{f(x)-f(c)}{x-c}$$ is the slope of the *secant line* through $(c, f(c))$ and $(x, f(x))$.

The graphical interpretation of the derivative:



$\dfrac{f(x)-f(c)}{x-c}$ is the slope of the *secant line* through $\big(c, f(c)\big)$ and $\big(x, f(x)\big)$.

$f'(c)$ is the limit of these slopes as $x \to c$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c}$$

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c}$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c} = (x + c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c}$$

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c}(x+c)$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c} = (x + c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c}(x + c) = 2c.$$

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c}(x+c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$.

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x+c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$.

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c}$$

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c} = (x + c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = \frac{a(x - c)}{x - c}$$

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = \frac{a(x-c)}{x-c} = a.$$

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c}(x+c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = \frac{a(x-c)}{x-c} = a.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x - c} = (x + c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = \frac{a(x - c)}{x - c} = a.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} a = a.$$

Example: Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) := x^2$ and $c \in \mathbb{R}$. If $x \neq c$, then

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = (x+c).$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c}(x+c) = 2c.$$

Example: Let $f(x) := ax + b$ for $a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. For $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = \frac{a(x-c)}{x-c} = a.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} a = a.$$

Remark: Every differentiable f “infinitesimally” behaves like the affine function $ax + b$.

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$.

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

Example: $f(x) := \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x} - \sqrt{c}}{x - c}$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})}$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c}$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}}$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}. \qquad \square$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}. \qquad \square$$

Example: $f(x) \coloneqq |x|$ is not differentiable at 0.

Example: $f(x) := \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

□

Example: $f(x) := |x|$ is not differentiable at 0.
Proof: When $x > 0$,

$$\frac{|x|-|0|}{x-0} = \frac{x-0}{x-0} = 1.$$

Example: $f(x) \coloneqq \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

□

Example: $f(x) \coloneqq |x|$ is not differentiable at 0.
Proof: When $x > 0$,

$$\frac{|x|-|0|}{x-0} = \frac{x-0}{x-0} = 1.$$

When $x < 0$,

$$\frac{|x|-|0|}{x-0} = \frac{-x-0}{x-0} = -1.$$

Example: $f(x) := \sqrt{x}$ is differentiable for $x > 0$. **Proof:** Fix $c > 0$ and take $x \neq c$, $x > 0$.

$$\frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{(\sqrt{x}-\sqrt{c})(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

□

Example: $f(x) := |x|$ is not differentiable at 0.
Proof: When $x > 0$,

$$\frac{|x|-|0|}{x-0} = \frac{x-0}{x-0} = 1.$$

When $x < 0$,

$$\frac{|x|-|0|}{x-0} = \frac{-x-0}{x-0} = -1.$$

So the limit $x \to 0$ does not exist.

□

A continuous function exists which is not differentiable at any point (Weierstrass).

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \to c} (x - c) = 0.$$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \to c} (x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c).$$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \to c} (x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c).$$

$\Rightarrow$ limit of $f(x) - f(c)$ as $x \to c$ exists

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \longrightarrow c} (x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c).$$

$\Rightarrow$ limit of $f(x) - f(c)$ as $x \longrightarrow c$ exists and

$$\lim_{x \longrightarrow c} \big(f(x) - f(c) \big)$$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \longrightarrow c} (x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c).$$

$\Longrightarrow$ limit of $f(x) - f(c)$ as $x \longrightarrow c$ exists and

$$\lim_{x \longrightarrow c} \big(f(x) - f(c) \big) = \left(\lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c} \right) \left(\lim_{x \longrightarrow c} (x - c) \right)$$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \to c}(x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c).$$

$\Rightarrow$ limit of $f(x) - f(c)$ as $x \to c$ exists and

$$\lim_{x \to c}\big(f(x) - f(c)\big) = \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) \left(\lim_{x \to c}(x - c) \right) = f'(c) \cdot 0 = 0.$$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \to c}(x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right)(x - c).$$

$\Rightarrow$ limit of $f(x) - f(c)$ as $x \longrightarrow c$ exists and

$$\lim_{x \to c}\big(f(x) - f(c)\big) = \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) \left(\lim_{x \to c}(x - c) \right) = f'(c) \cdot 0 = 0.$$

$\Rightarrow \quad \lim_{x \to c} f(x) = f(c)$

A continuous function exists which is not differentiable at any point (Weierstrass). However,

Proposition

Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.

Proof:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad \text{and} \qquad \lim_{x \to c}(x - c) = 0.$$

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right)(x - c).$$

$\Longrightarrow$ limit of $f(x) - f(c)$ as $x \longrightarrow c$ exists and

$$\lim_{x \to c}\big(f(x) - f(c)\big) = \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \right) \left(\lim_{x \to c}(x - c) \right) = f'(c) \cdot 0 = 0.$$

$\Longrightarrow$ $\lim_{x \to c} f(x) = f(c)$ $\Longrightarrow$ f is continuous at c. $\square$

Proposition (Linearity)

Let I be an interval, let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

Proposition (Linearity)

Let I be an interval, let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h: I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

Proposition (Linearity)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g \colon I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h \colon I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h \colon I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proposition (Linearity)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$,

Proposition (Linearity)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\quad \dfrac{h(x) - h(c)}{x - c}$

Proposition (Linearity)

Let I be an interval, let $f: I \longrightarrow \mathbb{R}$ and $g: I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\quad \dfrac{h(x) - h(c)}{x - c} = \dfrac{\alpha f(x) - \alpha f(c)}{x - c}$

Proposition (Linearity)

Let I be an interval, let $f: I \longrightarrow \mathbb{R}$ and $g: I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $$\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \frac{f(x) - f(c)}{x - c}.$$

Proposition (Linearity)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\quad \dfrac{h(x) - h(c)}{x - c} = \dfrac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \dfrac{f(x) - f(c)}{x - c}.$

The limit as $x \longrightarrow c$ of the RHS exists

Proposition (Linearity)

Let I be an interval, let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h: I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h: I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $$\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \frac{f(x) - f(c)}{x - c}.$$

The limit as $x \to c$ of the RHS exists and $$\lim_{x \to c} \frac{h(x) - h(c)}{x - c}$$

Proposition (Linearity)

Let I be an interval, let $f: I \longrightarrow \mathbb{R}$ and $g: I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $$\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \frac{f(x) - f(c)}{x - c}.$$

The limit as $x \to c$ of the RHS exists and $$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Proposition (Linearity)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h\colon I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h\colon I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $$\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \frac{f(x) - f(c)}{x - c}.$$

The limit as $x \to c$ of the RHS exists and $$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

(ii) For $x \in I$, $x \neq c$,

Proposition (Linearity)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h\colon I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h\colon I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\dfrac{h(x) - h(c)}{x - c} = \dfrac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \dfrac{f(x) - f(c)}{x - c}$.

The limit as $x \to c$ of the RHS exists and $\displaystyle\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

(ii) For $x \in I$, $x \neq c$,

$$\frac{h(x) - h(c)}{x - c}$$

Proposition (Linearity)

Let I be an interval, let $f: I \longrightarrow \mathbb{R}$ and $g: I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h: I \longrightarrow \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\quad \dfrac{h(x) - h(c)}{x - c} = \dfrac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \dfrac{f(x) - f(c)}{x - c}$.

The limit as $x \to c$ of the RHS exists and $\quad \displaystyle\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

(ii) For $x \in I$, $x \neq c$,

$$\frac{h(x) - h(c)}{x - c} = \frac{\big(f(x) + g(x)\big) - \big(f(c) + g(c)\big)}{x - c}$$

Proposition (Linearity)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h\colon I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h\colon I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $$\frac{h(x)-h(c)}{x-c} = \frac{\alpha f(x) - \alpha f(c)}{x-c} = \alpha \frac{f(x)-f(c)}{x-c}.$$

The limit as $x \to c$ of the RHS exists and $$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c}.$$

(ii) For $x \in I$, $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{\big(f(x)+g(x)\big) - \big(f(c)+g(c)\big)}{x-c} = \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}.$$

Proposition (Linearity)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h\colon I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h\colon I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\dfrac{h(x)-h(c)}{x-c} = \dfrac{\alpha f(x) - \alpha f(c)}{x-c} = \alpha \dfrac{f(x)-f(c)}{x-c}$.

The limit as $x \to c$ of the RHS exists and $\displaystyle \lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \alpha \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.

(ii) For $x \in I$, $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{\big(f(x)+g(x)\big) - \big(f(c)+g(c)\big)}{x-c} = \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}.$$

The limit as $x \to c$ of RHS exists

Proposition (Linearity)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h\colon I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h\colon I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$, $\dfrac{h(x)-h(c)}{x-c} = \dfrac{\alpha f(x) - \alpha f(c)}{x-c} = \alpha \dfrac{f(x)-f(c)}{x-c}$.

The limit as $x \to c$ of the RHS exists and $\displaystyle\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \alpha \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

(ii) For $x \in I$, $x \neq c$,

$$\frac{h(x)-h(c)}{x-c} = \frac{\big(f(x)+g(x)\big) - \big(f(c)+g(c)\big)}{x-c} = \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}.$$

The limit as $x \to c$ of RHS exists and

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c}$$

Proposition (Linearity)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be differentiable at $c \in I$, and $\alpha \in \mathbb{R}$.

(i) *Define $h\colon I \to \mathbb{R}$ by $h(x) := \alpha f(x)$. Then h is differentiable at c and $h'(c) = \alpha f'(c)$.*

(ii) *Define $h\colon I \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*

Proof: (i) For $x \in I$, $x \neq c$,
$$\frac{h(x) - h(c)}{x - c} = \frac{\alpha f(x) - \alpha f(c)}{x - c} = \alpha \frac{f(x) - f(c)}{x - c}.$$

The limit as $x \to c$ of the RHS exists and
$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \alpha \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

(ii) For $x \in I$, $x \neq c$,
$$\frac{h(x) - h(c)}{x - c} = \frac{\big(f(x) + g(x)\big) - \big(f(c) + g(c)\big)}{x - c} = \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}.$$

The limit as $x \to c$ of RHS exists and
$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$ $\square$

Proposition (Product rule / Leibniz rule)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be functions differentiable at c. If $h\colon I \to \mathbb{R}$ is defined by $h(x) := f(x)g(x)$,

Proposition (Product rule / Leibniz rule)

Let I be an interval, let $f\colon I \longrightarrow \mathbb{R}$ and $g\colon I \longrightarrow \mathbb{R}$ be functions differentiable at c. If $h\colon I \longrightarrow \mathbb{R}$ is defined by $h(x) := f(x)g(x)$, then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

Proposition (Product rule / Leibniz rule)

Let I be an interval, let $f\colon I \longrightarrow \mathbb{R}$ and $g\colon I \longrightarrow \mathbb{R}$ be functions differentiable at c. If $h\colon I \longrightarrow \mathbb{R}$ is defined by $h(x) := f(x)g(x)$, then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

Proof: Exercise.

Proposition (Product rule / Leibniz rule)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be functions differentiable at c. If $h\colon I \to \mathbb{R}$ is defined by $h(x) := f(x)g(x)$, then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

Proof: Exercise. Hint: $f(x)g(x) - f(c)g(c) = f(x)\big(g(x) - g(c)\big) + \big(f(x) - f(c)\big)g(c)$

Proposition (Product rule / Leibniz rule)

Let I be an interval, let $f\colon I \longrightarrow \mathbb{R}$ and $g\colon I \longrightarrow \mathbb{R}$ be functions differentiable at c. If $h\colon I \longrightarrow \mathbb{R}$ is defined by $h(x) := f(x)g(x)$, then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

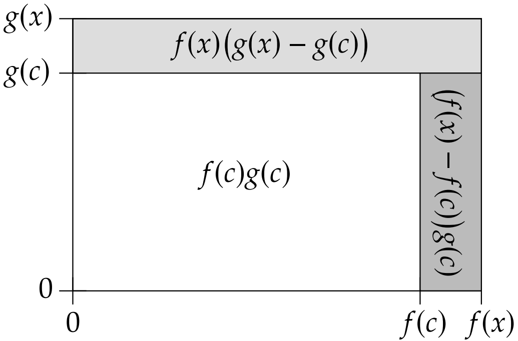
Proof: Exercise. Hint: $f(x)g(x) - f(c)g(c) = f(x)\big(g(x) - g(c)\big) + \big(f(x) - f(c)\big)g(c)$
or $\quad \Delta(f \cdot g) = f \cdot \Delta g + \Delta f \cdot g.$

Proposition (Product rule / Leibniz rule)

Let I be an interval, let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be functions differentiable at c. If $h\colon I \to \mathbb{R}$ is defined by $h(x) := f(x)g(x)$, then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

Proof: Exercise. Hint: $f(x)g(x) - f(c)g(c) = f(x)\big(g(x) - g(c)\big) + \big(f(x) - f(c)\big)g(c)$
or $\quad \Delta(f \cdot g) = f \cdot \Delta g + \Delta f \cdot g$.



Proposition (Quotient rule)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g\colon I \longrightarrow \mathbb{R}$ be differentiable at c and $g(x) \neq 0$ for all $x \in I$. If $h\colon I \longrightarrow \mathbb{R}$ is defined by $h(x) := \frac{f(x)}{g(x)}$,

Proposition (Quotient rule)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g \colon I \longrightarrow \mathbb{R}$ be differentiable at c and $g(x) \neq 0$ for all $x \in I$. If $h \colon I \longrightarrow \mathbb{R}$ is defined by $h(x) := \frac{f(x)}{g(x)}$, then h is differentiable at c and

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\big(g(c)\big)^2}.$$

Proposition (Quotient rule)

Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be differentiable at c and $g(x) \neq 0$ for all $x \in I$. If $h : I \longrightarrow \mathbb{R}$ is defined by $h(x) := \frac{f(x)}{g(x)}$, then h is differentiable at c and

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{\big(g(c)\big)^2}.$$

Proof: Exercise.

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$.

Proposition (Chain rule)

Let I_1, I_2 *be intervals, let* $g\colon I_1 \to I_2$ *be differentiable at* $c \in I_1$*, and* $f\colon I_2 \to \mathbb{R}$ *be differentiable at* $g(c)$*. If* $h\colon I_1 \to \mathbb{R}$ *is defined by* $h(x) \coloneqq (f \circ g)(x) = f\bigl(g(x)\bigr)$*,*

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases}$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c) \quad \Rightarrow \quad u$ is continuous at d.

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c)$ $\quad\Rightarrow\quad$ u is continuous at d.
g differentiable at c $\quad\Rightarrow\quad$ v is continuous at c.

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\bigl(g(x)\bigr)$, then h is differentiable at c and

$$h'(c) = f'\bigl(g(c)\bigr)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c)$ $\quad\Rightarrow\quad$ u is continuous at d.
g differentiable at c $\quad\Rightarrow\quad$ v is continuous at c.

$f(y) - f(d) = u(y)(y - d)$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c)$ $\quad\Rightarrow\quad$ u is continuous at d.
g differentiable at c $\quad\Rightarrow\quad$ v is continuous at c.

$$f(y) - f(d) = u(y)(y-d) \quad \text{and} \quad g(x) - g(c) = v(x)(x-c) \qquad \forall x, y.$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c) \quad \Longrightarrow \quad u$ is continuous at d.
g differentiable at $c \quad \Longrightarrow \quad v$ is continuous at c.

$$f(y) - f(d) = u(y)(y-d) \quad \text{and} \quad g(x) - g(c) = v(x)(x-c) \qquad \forall x, y.$$

$$\Longrightarrow \quad h(x) - h(c)$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c)$ $\quad\Rightarrow\quad$ u is continuous at d.
g differentiable at c $\quad\Rightarrow\quad$ v is continuous at c.

$$f(y) - f(d) = u(y)(y-d) \quad \text{and} \quad g(x) - g(c) = v(x)(x-c) \qquad \forall x, y.$$

$$\Rightarrow \quad h(x) - h(c) = f\big(g(x)\big) - f\big(g(c)\big)$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c) \quad \Rightarrow \quad u$ is continuous at d.
g differentiable at $c \quad \Rightarrow \quad v$ is continuous at c.

$$f(y) - f(d) = u(y)(y-d) \quad \text{and} \quad g(x) - g(c) = v(x)(x-c) \qquad \forall x, y.$$

$$\Rightarrow \quad h(x) - h(c) = f\big(g(x)\big) - f\big(g(c)\big) = u\big(g(x)\big)\big(g(x) - g(c)\big)$$

Proposition (Chain rule)

Let I_1, I_2 be intervals, let $g\colon I_1 \to I_2$ be differentiable at $c \in I_1$, and $f\colon I_2 \to \mathbb{R}$ be differentiable at $g(c)$. If $h\colon I_1 \to \mathbb{R}$ is defined by $h(x) := (f \circ g)(x) = f\big(g(x)\big)$, then h is differentiable at c and

$$h'(c) = f'\big(g(c)\big)g'(c).$$

Proof: Let $d := g(c)$.
Define $u\colon I_2 \to \mathbb{R}$ and $v\colon I_1 \to \mathbb{R}$ by

$$u(y) := \begin{cases} \frac{f(y)-f(d)}{y-d} & \text{if } y \neq d, \\ f'(d) & \text{if } y = d, \end{cases} \qquad v(x) := \begin{cases} \frac{g(x)-g(c)}{x-c} & \text{if } x \neq c, \\ g'(c) & \text{if } x = c. \end{cases}$$

f differentiable at $d = g(c)$ $\quad\Longrightarrow\quad$ u is continuous at d.
g differentiable at c $\quad\Longrightarrow\quad$ v is continuous at c.

$$f(y) - f(d) = u(y)(y-d) \quad \text{and} \quad g(x) - g(c) = v(x)(x-c) \qquad \forall x, y.$$

$$\Longrightarrow \quad h(x) - h(c) = f\big(g(x)\big) - f\big(g(c)\big) = u\big(g(x)\big)\big(g(x) - g(c)\big) = u\big(g(x)\big)\big(v(x)(x-c)\big).$$

If $x \neq c$, then $\qquad \dfrac{h(x) - h(c)}{x - c}$

If $x \neq c$, then $$\frac{h(x) - h(c)}{x - c} = u\big(g(x)\big)v(x).$$

If $x \neq c$, then $$\frac{h(x) - h(c)}{x - c} = u\big(g(x)\big)v(x).$$

u is continuous at d $\Rightarrow$ $\lim\limits_{y \to d} u(y) = f'(d) = f'\big(g(c)\big).$

If $x \neq c$, then $$\frac{h(x) - h(c)}{x - c} = u\big(g(x)\big)v(x).$$

u is continuous at d $\quad \Rightarrow \quad \lim_{y \to d} u(y) = f'(d) = f'\big(g(c)\big).$

v is continuous at c $\quad \Rightarrow \quad \lim_{x \to c} v(x) = g'(c).$

If $x \neq c$, then $$\frac{h(x) - h(c)}{x - c} = u\big(g(x)\big)v(x).$$

u is continuous at $d \quad \Rightarrow \quad \lim_{y \to d} u(y) = f'(d) = f'\big(g(c)\big).$

v is continuous at $c \quad \Rightarrow \quad \lim_{x \to c} v(x) = g'(c).$

g is continuous at $c \quad \Rightarrow \quad \lim_{x \to c} g(x) = g(c).$

If $x \neq c$, then $\qquad \dfrac{h(x) - h(c)}{x - c} = u\big(g(x)\big)v(x).$

u is continuous at $d \quad \Rightarrow \quad \lim_{y \to d} u(y) = f'(d) = f'\big(g(c)\big).$

v is continuous at $c \quad \Rightarrow \quad \lim_{x \to c} v(x) = g'(c).$

g is continuous at $c \quad \Rightarrow \quad \lim_{x \to c} g(x) = g(c).$

$\Rightarrow \quad \lim_{x \to c} u\big(g(x)\big)v(x) \quad$ exists

If $x \neq c$, then $$\frac{h(x)-h(c)}{x-c} = u\big(g(x)\big)v(x).$$

u is continuous at d $\Rightarrow$ $\lim\limits_{y \to d} u(y) = f'(d) = f'\big(g(c)\big)$.

v is continuous at c $\Rightarrow$ $\lim\limits_{x \to c} v(x) = g'(c)$.

g is continuous at c $\Rightarrow$ $\lim\limits_{x \to c} g(x) = g(c)$.

$\Rightarrow$ $\lim\limits_{x \to c} u\big(g(x)\big)v(x)$ exists and equals $f'\big(g(c)\big)g'(c)$.

If $x \neq c$, then $$\frac{h(x) - h(c)}{x - c} = u\big(g(x)\big)v(x).$$

u is continuous at d $\quad \Rightarrow \quad \lim_{y \to d} u(y) = f'(d) = f'\big(g(c)\big).$

v is continuous at c $\quad \Rightarrow \quad \lim_{x \to c} v(x) = g'(c).$

g is continuous at c $\quad \Rightarrow \quad \lim_{x \to c} g(x) = g(c).$

$\Rightarrow \quad \lim_{x \to c} u\big(g(x)\big)v(x)$ exists and equals $f'\big(g(c)\big)g'(c).$

$\Rightarrow \quad h$ is differentiable at c

If $x \neq c$, then $\quad \dfrac{h(x)-h(c)}{x-c} = u\big(g(x)\big)v(x).$

u is continuous at $d \quad \Rightarrow \quad \lim_{y \to d} u(y) = f'(d) = f'\big(g(c)\big).$

v is continuous at $c \quad \Rightarrow \quad \lim_{x \to c} v(x) = g'(c).$

g is continuous at $c \quad \Rightarrow \quad \lim_{x \to c} g(x) = g(c).$

$\Rightarrow \quad \lim_{x \to c} u\big(g(x)\big)v(x)$ exists and equals $f'\big(g(c)\big)g'(c).$

$\Rightarrow \quad h$ is differentiable at c and $h'(c) = f'\big(g(c)\big)g'(c).$ $\square$

Exercise: Prove the following simple version of L'Hôpital's rule:

Exercise: Prove the following simple version of L'Hôpital's rule:
Suppose $f\colon (a, b) \longrightarrow \mathbb{R}$ and $g\colon (a, b) \longrightarrow \mathbb{R}$ are differentiable functions whose derivatives f' and g' are continuous functions.

Exercise: Prove the following simple version of L'Hôpital's rule:
Suppose $f\colon (a, b) \longrightarrow \mathbb{R}$ and $g\colon (a, b) \longrightarrow \mathbb{R}$ are differentiable functions whose derivatives f' and g' are continuous functions. Suppose that at $c \in (a, b)$, $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ for all $x \in (a, b)$, and $g(x) \neq 0$ whenever $x \neq c$.

Exercise: Prove the following simple version of L'Hôpital's rule:
Suppose $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are differentiable functions whose derivatives f' and g' are continuous functions. Suppose that at $c \in (a,b)$, $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ for all $x \in (a,b)$, and $g(x) \neq 0$ whenever $x \neq c$.

Note that the limit of $f'(x)/g'(x)$ as x goes to c exists.

Exercise: Prove the following simple version of L'Hôpital's rule:
Suppose $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are differentiable functions whose derivatives f' and g' are continuous functions. Suppose that at $c \in (a,b)$, $f(c) = 0$, $g(c) = 0$, $g'(x) \neq 0$ for all $x \in (a,b)$, and $g(x) \neq 0$ whenever $x \neq c$.

Note that the limit of ${}^{f'(x)}\!/_{g'(x)}$ as x goes to c exists.

Show that
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$