BA: 2.5

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If $\{s_k\}_{k=1}^{\infty}$ diverges, we say the series is *divergent*.

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Remark: It is common (but we will not) to write $\displaystyle\sum_{n=1}^{\infty} x_n$ informally as

$$x_1 + x_2 + x_3 + \cdots$$

Example: $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and the limit is 1. That is, $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{2^n} = 1.$

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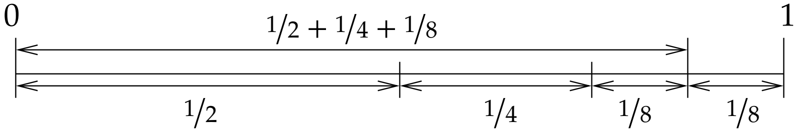

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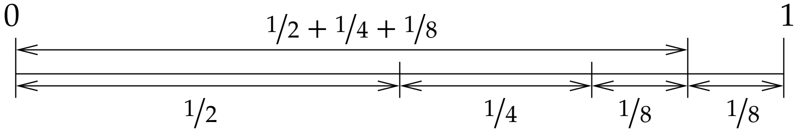

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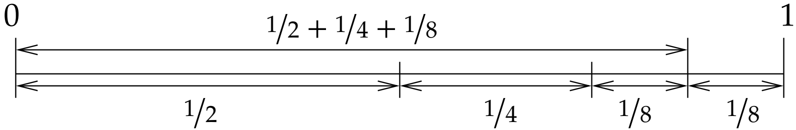

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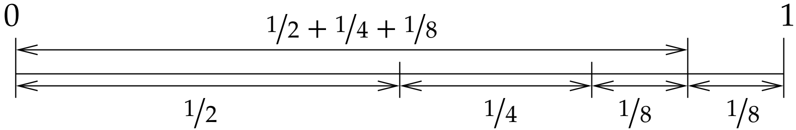

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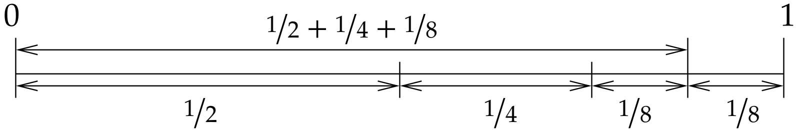

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Suppose $-1 < r < 1$. *Then the geometric series* $\sum_{n=0}^{\infty} r^n$ *converges, and* $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

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and then one takes the limit $k \to \infty$.

Proposition (tail of a series)

Let $\sum_{n=1}^{\infty} x_n$ *be a series. Let* $M \in \mathbb{N}$*. Then*

$$\sum_{n=1}^{\infty} x_n \quad \text{converges if and only if} \quad \sum_{n=M}^{\infty} x_n \quad \text{converges.}$$

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Adding a constant does not change the convergence of a sequence. □

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So $\{s_n\}_{n=1}^{\infty}$ diverges, that is, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $\square$

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Sum of convergent sequences converges to the sum of the limits. $\square$

Example: By the first item, if $|r| < 1$ and $i \in \mathbb{N}$, then

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Obviously, we don't multiply term by term: $(a+b)(c+d) \neq ac + bd$.

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Proof: If $\sum_{n=1}^{\infty} x_n$ converges absolutely, then $\sum_{n=1}^{\infty} |x_n|$ is convergent hence Cauchy.

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$\sum_{n=1}^{\infty} x_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |x_n|$ converges.
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$\Rightarrow$ partial sums of $\sum_{n=1}^{\infty} y_n$ unbounded $\Rightarrow$ $\sum_{n=1}^{\infty} y_n$ diverges. $\square$

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Proposition (p-series or the p-test)

For $p \in \mathbb{R}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.

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Understanding its roots is one of the most famous unsolved problems in mathematics, with many applications.

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$\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} |x_n|$ converges $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} x_n$ converges absolutely. $\square$

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Therefore, the series converges absolutely by the ratio test. $\square$

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Converse still doesn't hold: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges even though $\lim_{n\to\infty} n\left(\frac{1}{n \ln n}\right) = 0$.