BA: 5.1

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We will follow the Darboux approach rather than Riemann approach,
they are equivalent.

Definition

A *partition* P of the interval $[a, b]$ is a finite set $\{x_0, x_1, x_2, \ldots, x_n\}$ such that

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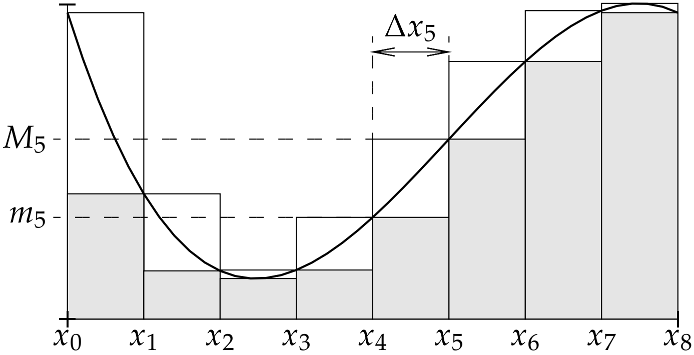
We call $L(P,f)$ the *lower Darboux sum* and $U(P,f)$ the *upper Darboux sum*.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Remark: If $f \colon S \to \mathbb{R}$ and $[a,b] \subset S$, use $f|_{[a,b]}$. But we'll just write $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$.

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There are integers $k_0 < k_1 < \cdots < k_n$ such that $x_i = \widetilde{x}_{k_i}$ for $i = 0, 1, 2, \ldots, n$.

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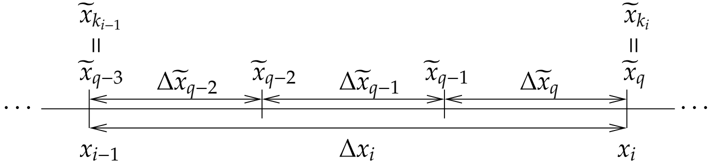

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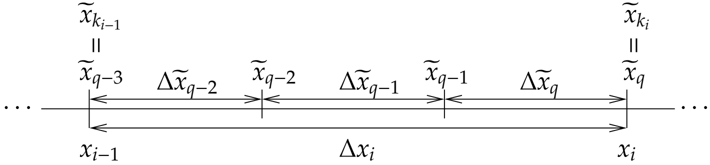

Δx_i

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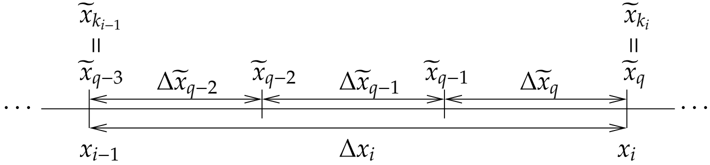

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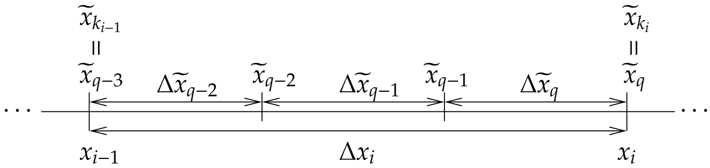

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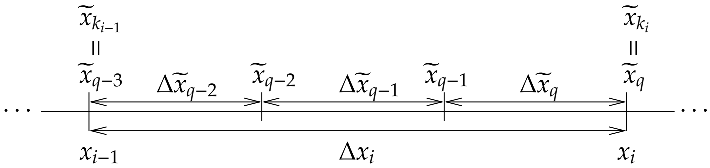

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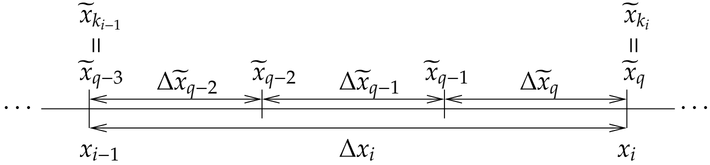

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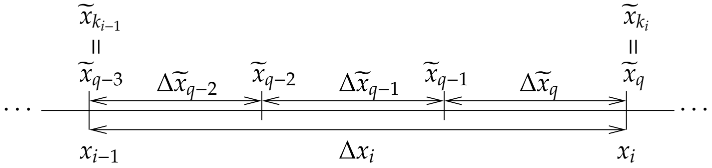
$\widetilde{x}_{k_{i-1}}$
$\parallel$
$\widetilde{x}_{q-3}$
$\Delta\widetilde{x}_{q-2}$
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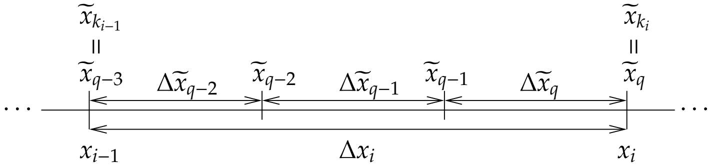
$\cdots$
$\cdots$
x_{i-1}
Δx_i
x_i



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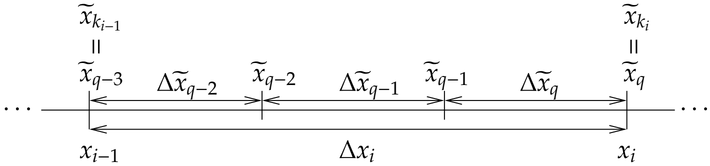

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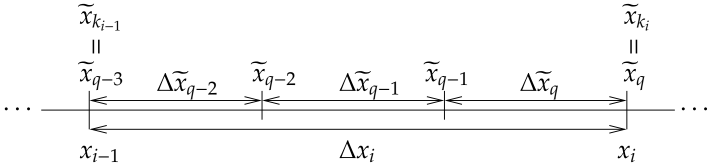

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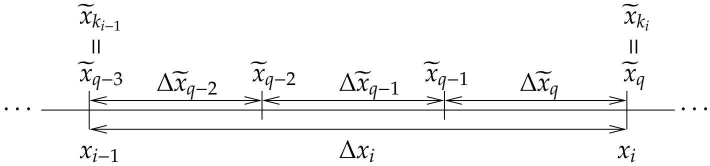

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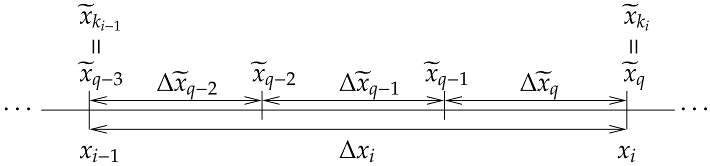

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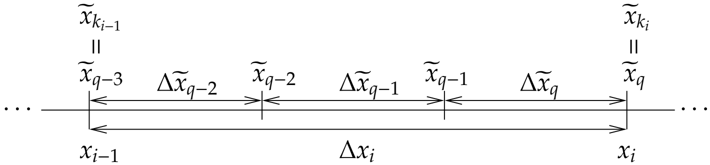

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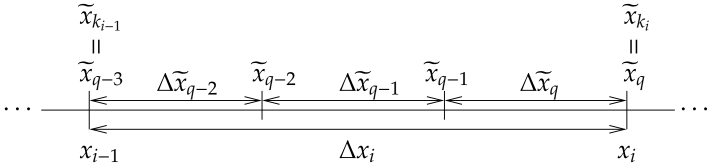

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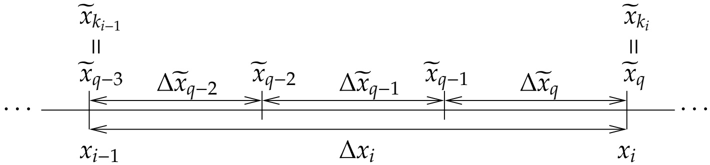

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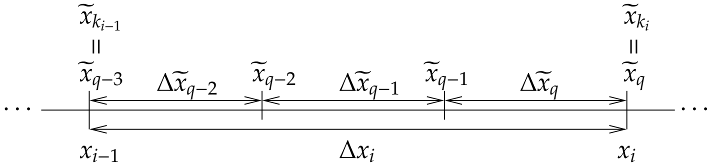

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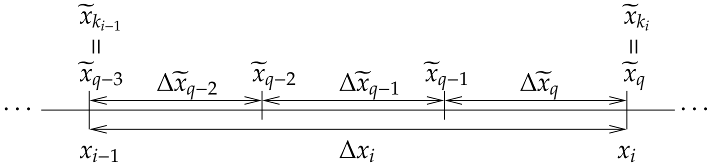

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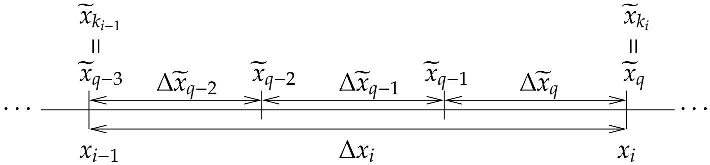

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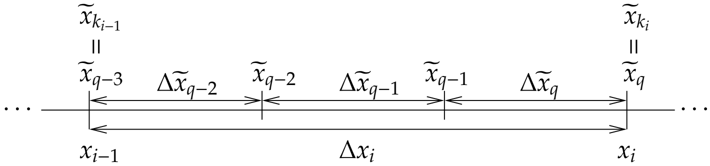

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$U(\widetilde{P},f) \le U(P,f)$ is left as an exercise. $\square$

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Let P_1, P_2 be partitions of $[a, b]$. Define $\widetilde{P} := P_1 \cup P_2$.

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$\int_a^b f$ is called the *Riemann integral* of f, or simply the *integral* of f.

We immediately get:

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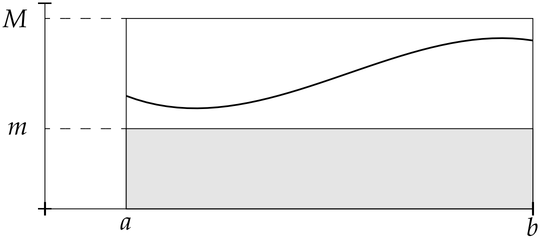

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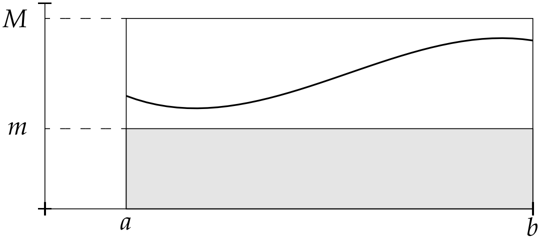

Useful version: If $|f(x)| \leq M$ for all $x \in [a, b]$, then $\left| \int_a^b f \right| \leq M(b-a)$.

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So f is integrable on any interval $[a, b]$ and

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Example: Let $f\colon [0,2] \to \mathbb{R}$ be defined by

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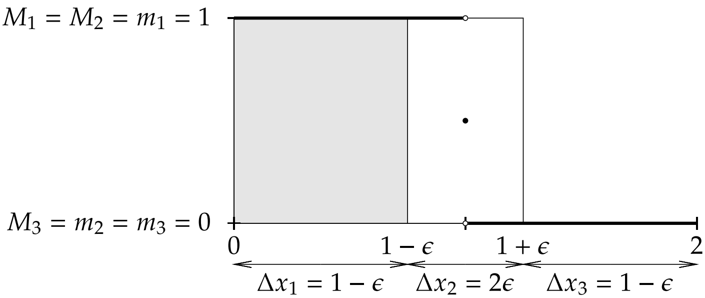
Furthermore, $\Delta x_1 = 1 - \epsilon$, $\Delta x_2 = 2\epsilon$, and $\Delta x_3 = 1 - \epsilon$.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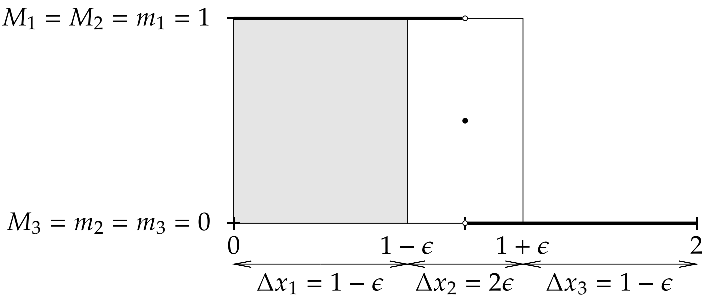
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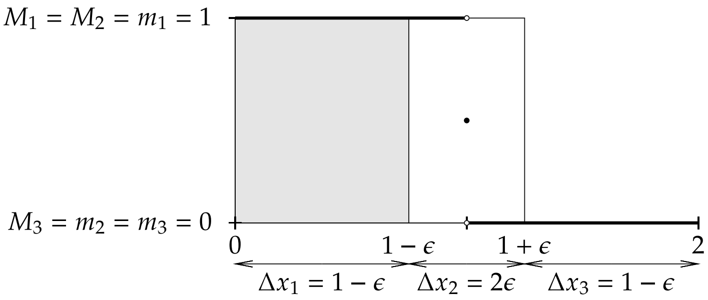
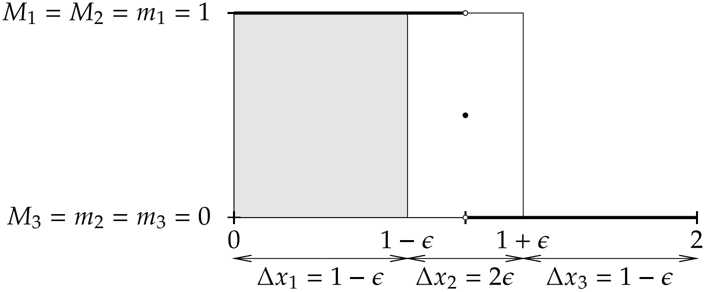
$L(P, f)$



$$L(P,f) = \sum_{i=1}^{3} m_i \Delta x_i$$



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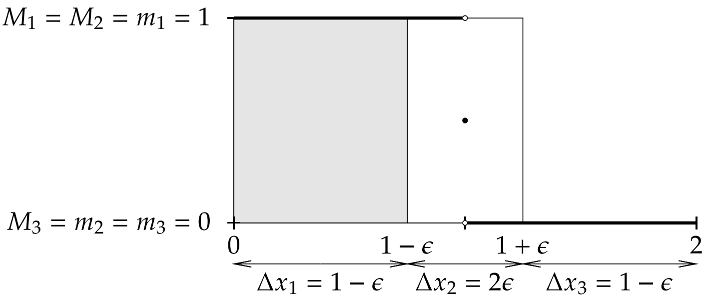

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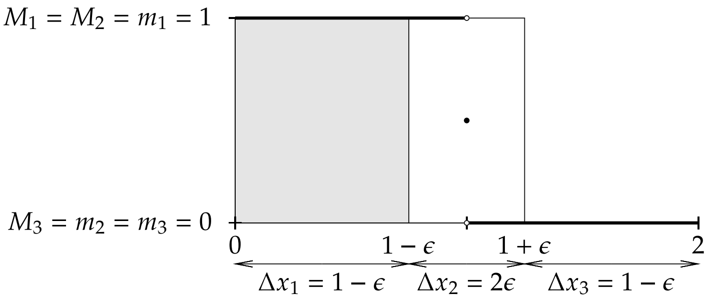

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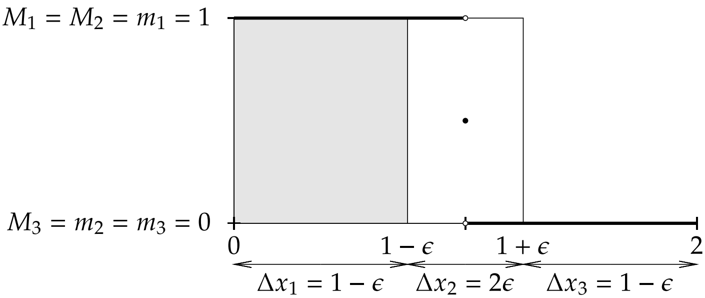

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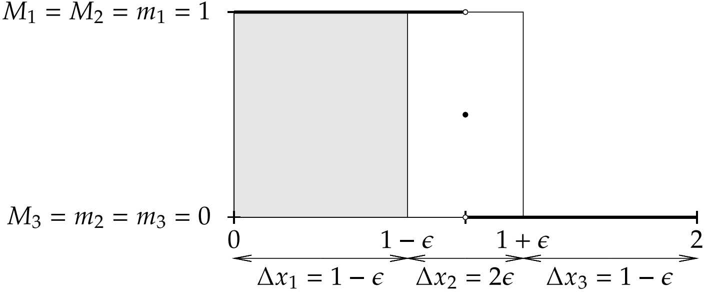

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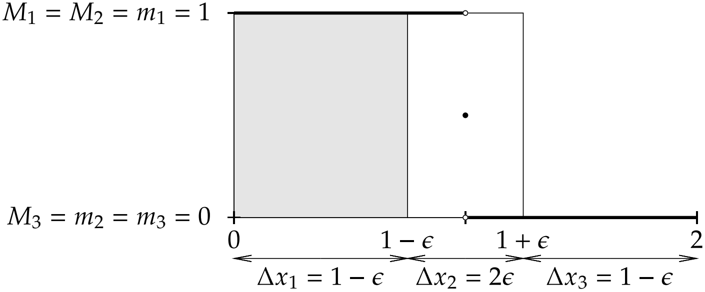

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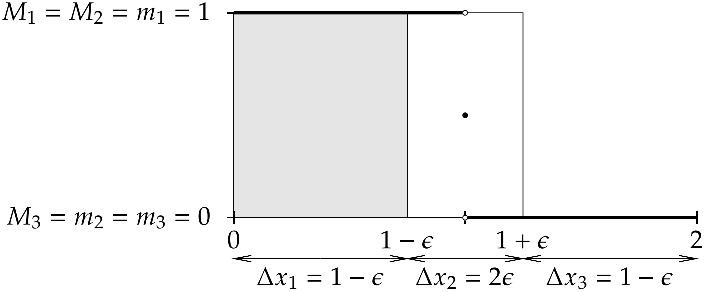

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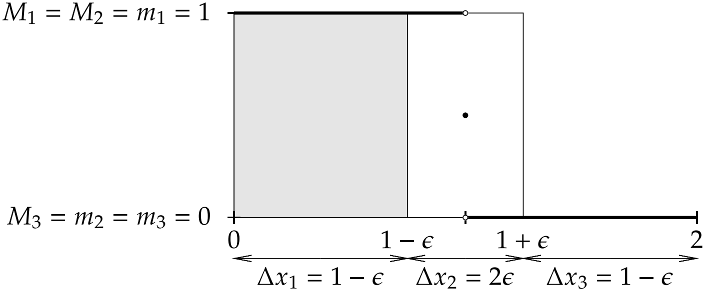

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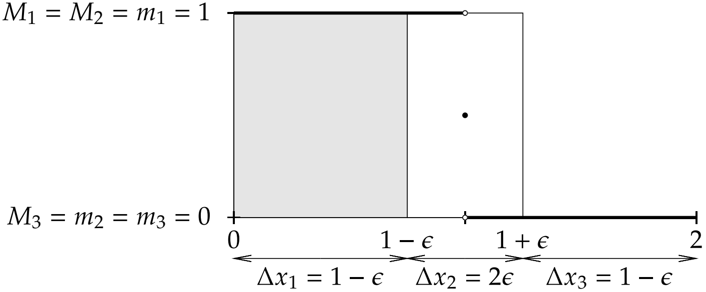

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The x in dx is just a dummy variable, so, e.g., $\displaystyle\int_a^b f(s)\,ds := \int_a^b f(x)\,dx.$

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Exercise: Prove that if f is Riemann integrable, then for any $\epsilon > 0$ there exists a Riemann sum that is within ϵ of the integral.