BA: 7.3

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If $\{n_k\}_{k=1}^{\infty}$ is a sequence of natural numbers such that $n_{k+1} > n_k$ for all k, then $\{x_{n_k}\}_{k=1}^{\infty}$ is a *subsequence* of $\{x_n\}_{n=1}^{\infty}$.

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for every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that $d(x_n, p) < \epsilon$ for all $n \geq M$.

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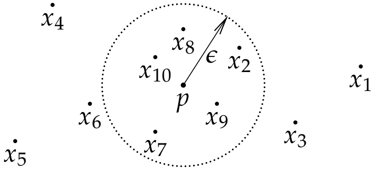

Sequence converging to p. The first 10 points are shown and $M = 7$ for this ϵ.

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Proof: Suppose $\{x_n\}_{n=1}^{\infty}$ has limits x and y.
Take an arbitrary $\epsilon > 0$.
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Hence the limit (if it exists) is unique. $\square$

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A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) converges to $p \in X$ if and only if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers such that

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(i) *If* $\{x_n\}_{n=1}^{\infty}$ *converges to* $p \in X$, *then every subsequence* $\{x_{n_k}\}_{k=1}^{\infty}$ *converges to* p.

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(ii) *If for some* $K \in \mathbb{N}$ *the* K*-tail* $\{x_n\}_{n=K+1}^{\infty}$ *converges to* $p \in X$, *then* $\{x_n\}_{n=1}^{\infty}$ *converges to* p.

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Proofs: Exercises.

Example: Take $C\big([a,b],\mathbb{R}\big)$ be the metric space of continuous functions where $d(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$ (uniform norm).

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That is, $\{f_n\}_{n=1}^{\infty}$ converges uniformly if and only if it converges in the metric space sense.

Remark: No metric on the set of functions $f\colon [a,b] \to \mathbb{R}$ gives pointwise convergence.

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$\Rightarrow \quad \{x_{m,k}\}_{m=1}^{\infty}$ converges to y_k.

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$\{x_n\}_{n=1}^{\infty}$ converges to x and $\{y_n\}_{n=1}^{\infty}$ converges to y.

Proposition

Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Then $\{x_n\}_{n=1}^{\infty}$ converges to $p \in X$ if and only if for every open neighborhood U of p, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n \in U$.

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E^c is open $\quad \Rightarrow \quad \exists\, M$ such that $\forall\, n \geq M,\ x_n \in E^c$.

$\Rightarrow \quad \{x_n\}_{n=1}^{\infty}$ is not a sequence in E. $\square$

Proposition

Let (X, d) be a metric space and $A \subset X$. Then $p \in \overline{A}$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in A such that $\lim_{n\to\infty} x_n = p$.

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The other direction: Exercise. $\square$

Exercise: Let (X, d) be a metric space where d is the discrete metric. Suppose $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in X. Show that there exists a $K \in \mathbb{N}$ such that for all $n \geq K$, we have $x_n = x_K$.

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A set $S \subset X$ is said to be *dense* in X if $X \subset \overline{S}$ or in other words if for every $p \in X$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in S that converges to p.

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Exercise: Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Prove that $\{x_n\}_{n=1}^{\infty}$ converges to $p \in X$ if and only if every subsequence of $\{x_n\}_{n=1}^{\infty}$ has a subsequence that converges to p.