BA: 1.2

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

We will simply assume that real numbers exist, that is, we won't prove:

Theorem

There exists a unique ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \subset \mathbb{R}$.

We will simply assume that real numbers exist, that is, we won't prove:

Theorem

There exists a unique ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \subset \mathbb{R}$.

We'll simply assume simple properties that follow from the definition.

We will simply assume that real numbers exist, that is, we won't prove:

Theorem

There exists a unique ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \subset \mathbb{R}$.

We'll simply assume simple properties that follow from the definition.

E.g.:

$1 > 0, 2 > 0, \ldots, n > 0$ for all $n \in \mathbb{N} \subset \mathbb{R}$.

We will simply assume that real numbers exist, that is, we won't prove:

Theorem

There exists a unique ordered field $\mathbb{R}$ *with the least-upper-bound property such that* $\mathbb{Q} \subset \mathbb{R}$.

We'll simply assume simple properties that follow from the definition.

E.g.:

$1 > 0, 2 > 0, ..., n > 0$ for all $n \in \mathbb{N} \subset \mathbb{R}$.

$\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

We will simply assume that real numbers exist, that is, we won't prove:

Theorem

There exists a unique ordered field $\mathbb{R}$ with the least-upper-bound property such that $\mathbb{Q} \subset \mathbb{R}$.

We'll simply assume simple properties that follow from the definition.

E.g.:

$1 > 0, 2 > 0, \ldots, n > 0$ for all $n \in \mathbb{N} \subset \mathbb{R}$.

$\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

etc.

Analysis is proving inequalities.

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ *is such that* $x \le \epsilon$ *for all* $\epsilon \in \mathbb{R}$ *where* $\epsilon > 0$,

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ *is such that* $x \leq \epsilon$ *for all* $\epsilon \in \mathbb{R}$ *where* $\epsilon > 0$*, then* $x \leq 0$.

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. □

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. $\square$

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. □

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Or the very common: *If $|x| \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. $\square$

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Or the very common: *If $|x| \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

(To prove $x \geq 0$, you could prove $-x \leq 0$ with the proposition).

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. $\square$

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Or the very common: *If $|x| \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

(To prove $x \geq 0$, you could prove $-x \leq 0$ with the proposition).

We'll see many variations on the above idea.

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. $\square$

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Or the very common: *If $|x| \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

(To prove $x \geq 0$, you could prove $-x \leq 0$ with the proposition).

We'll see many variations on the above idea.

Generalizing the idea of the proof:

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. $\square$

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Or the very common: *If $|x| \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

(To prove $x \geq 0$, you could prove $-x \leq 0$ with the proposition).

We'll see many variations on the above idea.

Generalizing the idea of the proof:
If $a < b$ are real numbers, then there exists $c \in \mathbb{R}$ such that $a < c < b$.

Analysis is proving inequalities.
Here is how an analyst proves a nonstrict inequality:

Proposition

If $x \in \mathbb{R}$ is such that $x \leq \epsilon$ for all $\epsilon \in \mathbb{R}$ where $\epsilon > 0$, then $x \leq 0$.

Proof: Either $x \leq 0$ or $x > 0$.
If $x > 0$, then $0 < x/2 < x$.
$\epsilon = x/2$ gives a contradiction. $\square$

Equivalently: *If $x \geq 0$ is such that $x \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

Or the very common: *If $|x| \leq \epsilon$ for all $\epsilon > 0$, then $x = 0$.*

(To prove $x \geq 0$, you could prove $-x \leq 0$ with the proposition).

We'll see many variations on the above idea.

Generalizing the idea of the proof:
If $a < b$ are real numbers, then there exists $c \in \mathbb{R}$ such that $a < c < b$.
E.g., $c = \frac{a+b}{2}$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2$

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \quad \Rightarrow \quad x < 2$ when $x^2 < 2$ ($x \in A$)

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \quad \Rightarrow \quad x < 2$ when $x^2 < 2$ $(x \in A) \quad \Rightarrow \quad A$ is bounded above.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \;\Rightarrow\; x < 2$ when $x^2 < 2$ $(x \in A) \;\Rightarrow\; A$ is bounded above.

$1 \in A$, so $A \neq \emptyset$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \implies x < 2$ when $x^2 < 2$ $(x \in A) \implies A$ is bounded above.

$1 \in A$, so $A \neq \emptyset$.

So $\sup A$ exists. Let $r := \sup A$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \;\Rightarrow\; x < 2$ when $x^2 < 2$ $(x \in A) \;\Rightarrow\; A$ is bounded above.

$1 \in A$, so $A \neq \emptyset$.

So $\sup A$ exists. Let $r := \sup A$.

Immediate: $r \geq 1 > 0$ as $1 \in A$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \implies x < 2$ when $x^2 < 2$ $(x \in A) \implies A$ is bounded above.

$1 \in A$, so $A \neq \emptyset$.

So $\sup A$ exists. Let $r := \sup A$.

Immediate: $r \geq 1 > 0$ as $1 \in A$.

WTS $r^2 = 2$.

For analysis, the least-upper-bound property is absolutely key (allows limits).

Let's use it to show that $\sqrt{2} \in \mathbb{R}$ exists (it does not exist in $\mathbb{Q}$).

Example: $\exists$ a unique positive $r \in \mathbb{R}$ such that $r^2 = 2$.

Proof: Let $A := \{x \in \mathbb{R} : x^2 < 2\}$.

We claim that A is bounded above and nonempty.

$x^2 \geq 4$ when $x \geq 2 \quad \Rightarrow \quad x < 2$ when $x^2 < 2$ $(x \in A) \quad \Rightarrow \quad A$ is bounded above.

$1 \in A$, so $A \neq \emptyset$.

So $\sup A$ exists. Let $r := \sup A$.

Immediate: $r \geq 1 > 0$ as $1 \in A$.

WTS $r^2 = 2$.

This is analysis, we'll show $r^2 \geq 2$ and $r^2 \leq 2$.

Claim: $r^2 \geq 2$.

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s + h)^2 < 2$.

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0.$

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0$.

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0$.

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$(s+h)^2 - s^2 = h(2s+h)$

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0$.

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$$\begin{aligned}(s+h)^2 - s^2 &= h(2s+h)\\ &< h(2s+1) \qquad \left(\text{since } h < 1\right)\end{aligned}$$

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0$.

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$$\begin{aligned}(s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) \qquad \big(\text{since } h < 1\big) \\ &< 2 - s^2 \qquad \Big(\text{since } h < \tfrac{2-s^2}{2s+1}\Big)\end{aligned}$$

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0.$

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$$\begin{aligned}
(s+h)^2 - s^2 &= h(2s+h) \\
&< h(2s+1) \qquad \big(\text{since } h < 1\big) \\
&< 2 - s^2 \qquad \left(\text{since } h < \tfrac{2-s^2}{2s+1}\right)
\end{aligned}$$

$\Rightarrow \quad (s+h)^2 < 2 \quad \Rightarrow \quad s+h \in A$

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0$.

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$$\begin{aligned}(s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) \qquad \left(\text{since } h < 1\right) \\ &< 2 - s^2 \qquad \left(\text{since } h < \tfrac{2-s^2}{2s+1}\right)\end{aligned}$$

$\Rightarrow \quad (s+h)^2 < 2 \quad \Rightarrow \quad s+h \in A$

but $s+h > s$ as $h > 0 \quad \Rightarrow \quad s < r = \sup A$

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0$.

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$$
\begin{aligned}
(s+h)^2 - s^2 &= h(2s+h) \\
&< h(2s+1) \qquad \big(\text{since } h < 1\big) \\
&< 2 - s^2 \qquad \Big(\text{since } h < \tfrac{2-s^2}{2s+1}\Big)
\end{aligned}
$$

$\Rightarrow \quad (s+h)^2 < 2 \quad \Rightarrow \quad s + h \in A$

but $s + h > s$ as $h > 0 \quad \Rightarrow \quad s < r = \sup A$

s was arbitrary positive s such that $s^2 < 2 \quad \Rightarrow \quad r^2 \geq 2$.

Claim: $r^2 \geq 2$.

Proof of claim: Take $s > 0$ such that $s^2 < 2$. Want $h > 0$ such that $(s+h)^2 < 2$.

$2 - s^2 > 0 \quad \Rightarrow \quad \frac{2-s^2}{2s+1} > 0.$

Take $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^2}{2s+1}$ and $h < 1$.

$$\begin{aligned}(s+h)^2 - s^2 &= h(2s+h) \\ &< h(2s+1) \qquad \left(\text{since } h < 1\right) \\ &< 2 - s^2 \qquad \left(\text{since } h < \tfrac{2-s^2}{2s+1}\right)\end{aligned}$$

$\Rightarrow \quad (s+h)^2 < 2 \quad \Rightarrow \quad s+h \in A$

but $s + h > s$ as $h > 0 \qquad \Rightarrow \qquad s < r = \sup A$

s was arbitrary positive s such that $s^2 < 2 \qquad \Rightarrow \qquad r^2 \geq 2$.

Claim is proved.

Claim: $r^2 \leq 2$.

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$.

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s - h)^2 > 2$ and $s - h > 0$.

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$.

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0$.

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

Claim: $r^2 \leq 2$**.**

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$s^2 - (s-h)^2 = 2sh - h^2$

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \implies h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \end{aligned}$$

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \implies h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$
\begin{aligned}
s^2 - (s-h)^2 &= 2sh - h^2 \\
&< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\
&= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right)
\end{aligned}
$$

Claim: $r^2 \le 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$$\Rightarrow \quad (s-h)^2 > 2$$

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$
\begin{aligned}
s^2 - (s-h)^2 &= 2sh - h^2 \\
&< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\
&= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right)
\end{aligned}
$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \geq s - h \ \Rightarrow \ x^2 \geq (s-h)^2 > 2$

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \Rightarrow h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \geq s - h \Rightarrow x^2 \geq (s-h)^2 > 2 \Rightarrow x \notin A$

Claim: $r^2 \leq 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \geq s - h \Rightarrow x^2 \geq (s-h)^2 > 2 \Rightarrow x \notin A \Rightarrow s - h$ is an upper bound.

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \geq s - h \Rightarrow x^2 \geq (s-h)^2 > 2 \Rightarrow x \notin A \Rightarrow s - h$ is an upper bound.

$s - h < s \quad \Rightarrow \quad s > r = \sup A \quad \Rightarrow \quad r^2 \leq 2.$

Claim: $r^2 \le 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$
\begin{aligned}
s^2 - (s-h)^2 &= 2sh - h^2 \\
&< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \ne 0\right) \\
&= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right)
\end{aligned}
$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \ge s - h \Rightarrow x^2 \ge (s-h)^2 > 2 \Rightarrow x \notin A \Rightarrow s - h$ is an upper bound.

$s - h < s \quad \Rightarrow \quad s > r = \sup A \quad \Rightarrow \quad r^2 \le 2.$ The claim is proved.

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \geq s - h \;\Rightarrow\; x^2 \geq (s-h)^2 > 2 \;\Rightarrow\; x \notin A \;\Rightarrow\; s - h$ is an upper bound.

$s - h < s \quad \Rightarrow \quad s > r = \sup A \quad \Rightarrow \quad r^2 \leq 2.$ The claim is proved.

$r^2 \geq 2$ and $r^2 \leq 2 \quad \Rightarrow \quad r^2 = 2.$

Claim: $r^2 \le 2$.

Proof of claim:
Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \ge s - h \;\Rightarrow\; x^2 \ge (s-h)^2 > 2 \;\Rightarrow\; x \notin A \;\Rightarrow\; s - h$ is an upper bound.

$s - h < s \quad \Rightarrow \quad s > r = \sup A \quad \Rightarrow \quad r^2 \le 2.$ The claim is proved.

$r^2 \ge 2$ and $r^2 \le 2 \quad \Rightarrow \quad r^2 = 2.$ Existence is done.

Claim: $r^2 \le 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \implies h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$
\begin{aligned}
s^2 - (s-h)^2 &= 2sh - h^2 \\
&< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\
&= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right)
\end{aligned}
$$

$\implies (s-h)^2 > 2 \implies s - h \notin A$.

If $x \ge s - h \implies x^2 \ge (s-h)^2 > 2 \implies x \notin A \implies s - h$ is an upper bound.

$s - h < s \implies s > r = \sup A \implies r^2 \le 2$. The claim is proved.

$r^2 \ge 2$ and $r^2 \le 2 \implies r^2 = 2$. Existence is done.

Uniqueness: Suppose $s > 0$ such that $s^2 = 2$

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \implies h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\implies (s-h)^2 > 2 \implies s - h \notin A$.

If $x \geq s - h \implies x^2 \geq (s-h)^2 > 2 \implies x \notin A \implies s - h$ is an upper bound.

$s - h < s \implies s > r = \sup A \implies r^2 \leq 2$. The claim is proved.

$r^2 \geq 2$ and $r^2 \leq 2 \implies r^2 = 2$. Existence is done.

Uniqueness: Suppose $s > 0$ such that $s^2 = 2 \implies s^2 = r^2$.

Claim: $r^2 \le 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$
\begin{aligned}
s^2 - (s-h)^2 &= 2sh - h^2 \\
&< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\
&= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right)
\end{aligned}
$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \ge s - h \Rightarrow x^2 \ge (s-h)^2 > 2 \Rightarrow x \notin A \Rightarrow s - h$ is an upper bound.

$s - h < s \quad \Rightarrow \quad s > r = \sup A \quad \Rightarrow \quad r^2 \le 2.$ The claim is proved.

$r^2 \ge 2$ and $r^2 \le 2 \quad \Rightarrow \quad r^2 = 2.$ Existence is done.

Uniqueness: Suppose $s > 0$ such that $s^2 = 2 \quad \Rightarrow \quad s^2 = r^2.$

$0 < s < r$ would imply $s^2 < r^2$

Claim: $r^2 \le 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $\quad s^2 - 2 > 0 \quad \Rightarrow \quad h = \frac{s^2-2}{2s} > 0. \quad s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0.$

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\Rightarrow \quad (s-h)^2 > 2 \quad \Rightarrow \quad s - h \notin A.$

If $x \ge s - h \Rightarrow x^2 \ge (s-h)^2 > 2 \Rightarrow x \notin A \Rightarrow s - h$ is an upper bound.

$s - h < s \quad \Rightarrow \quad s > r = \sup A \quad \Rightarrow \quad r^2 \le 2.$ The claim is proved.

$r^2 \ge 2$ and $r^2 \le 2 \quad \Rightarrow \quad r^2 = 2.$ Existence is done.

Uniqueness: Suppose $s > 0$ such that $s^2 = 2 \quad \Rightarrow \quad s^2 = r^2.$

$0 < s < r$ would imply $s^2 < r^2 \quad 0 < r < s$ would imply $r^2 < s^2$

Claim: $r^2 \leq 2$.

Proof of claim:

Take $s > 0$ such that $s^2 > 2$. Want $h > 0$ such that $(s-h)^2 > 2$ and $s - h > 0$.

Let $h := \frac{s^2-2}{2s}$. $s^2 - 2 > 0 \implies h = \frac{s^2-2}{2s} > 0$. $s - h = s - \frac{s^2-2}{2s} = \frac{s}{2} + \frac{1}{s} > 0$.

$$\begin{aligned} s^2 - (s-h)^2 &= 2sh - h^2 \\ &< 2sh \qquad \left(\text{since } h^2 > 0 \text{ as } h \neq 0\right) \\ &= s^2 - 2 \quad \left(\text{since } h = \tfrac{s^2-2}{2s}\right) \end{aligned}$$

$\implies (s-h)^2 > 2 \implies s - h \notin A$.

If $x \geq s - h \implies x^2 \geq (s-h)^2 > 2 \implies x \notin A \implies s - h$ is an upper bound.

$s - h < s \implies s > r = \sup A \implies r^2 \leq 2$. The claim is proved.

$r^2 \geq 2$ and $r^2 \leq 2 \implies r^2 = 2$. Existence is done.

Uniqueness: Suppose $s > 0$ such that $s^2 = 2 \implies s^2 = r^2$.

$0 < s < r$ would imply $s^2 < r^2$ $0 < r < s$ would imply $r^2 < s^2 \implies s = r$. $\square$

$x^{1/n}$ exists for any $n \in \mathbb{N}$ and all $x > 0$ (proof harder).

$x^{1/n}$ exists for any $n \in \mathbb{N}$ and all $x > 0$ (proof harder).

$\sqrt{2} \notin \mathbb{Q}$

$x^{1/n}$ exists for any $n \in \mathbb{N}$ and all $x > 0$ (proof harder).

$\sqrt{2} \notin \mathbb{Q}$ so $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$

$x^{1/n}$ exists for any $n \in \mathbb{N}$ and all $x > 0$ (proof harder).

$\sqrt{2} \notin \mathbb{Q} \qquad$ so $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$
(In fact, we'll see $\mathbb{R} \setminus \mathbb{Q}$ is much bigger than $\mathbb{Q}$.)

$x^{1/n}$ exists for any $n \in \mathbb{N}$ and all $x > 0$ (proof harder).

$\sqrt{2} \notin \mathbb{Q}$ so $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$
(In fact, we'll see $\mathbb{R} \setminus \mathbb{Q}$ is much bigger than $\mathbb{Q}$.)

$\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational* numbers.

Theorem

(i) *(Archimedean property) If* $x, y \in \mathbb{R}$ *and* $x > 0$*, then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$*.*

Theorem

(i) (*Archimedean property*) *If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) ($\mathbb{Q}$ *is dense in* $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Theorem

(i) (*Archimedean property*) *If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) ($\mathbb{Q}$ *is dense in* $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Theorem

(i) *(Archimedean property) If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) *(*$\mathbb{Q}$ *is dense in* $\mathbb{R}$*) If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i):

Theorem

(i) *(Archimedean property) If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) *(*$\mathbb{Q}$ *is dense in* $\mathbb{R}$*) If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x.

Theorem

(i) (*Archimedean property*) *If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) ($\mathbb{Q}$ *is dense in* $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}\!/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

Theorem

(i) *(Archimedean property) If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists\, n \in \mathbb{N}$ such that $nx > y$.*

(ii) *($\mathbb{Q}$ is dense in $\mathbb{R}$) If $x, y \in \mathbb{R}$ and $x < y$, then $\exists\, r \in \mathbb{Q}$ such that $x < r < y$.*

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}\!/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Theorem

(i) *(Archimedean property) If* $x, y \in \mathbb{R}$ *and* $x > 0$*, then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$*.*

(ii) *(*$\mathbb{Q}$ *is dense in* $\mathbb{R}$*) If* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$*.*

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}\!/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Suppose $\mathbb{N}$ is bounded (for contradiction).

Theorem

(i) (*Archimedean property*) *If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) ($\mathbb{Q}$ *is dense in* $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Suppose $\mathbb{N}$ is bounded (for contradiction).

Let $b := \sup \mathbb{N}$.

Theorem

(i) (*Archimedean property*) *If* $x, y \in \mathbb{R}$ *and* $x > 0$, *then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) ($\mathbb{Q}$ *is dense in* $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$, *then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Suppose $\mathbb{N}$ is bounded (for contradiction).

Let $b := \sup \mathbb{N}$.

$b - 1$ is not an upper bound as $b - 1 < b$.

Theorem

(i) (*Archimedean property*) *If* $x, y \in \mathbb{R}$ *and* $x > 0$*, then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) ($\mathbb{Q}$ *is dense in* $\mathbb{R}$) *If* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := y/x$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Suppose $\mathbb{N}$ is bounded (for contradiction).

Let $b := \sup \mathbb{N}$.

$b - 1$ is not an upper bound as $b - 1 < b$.

$\Rightarrow \quad \exists m \in \mathbb{N}$ such that $m > b - 1$

Theorem

(i) *(Archimedean property) If* $x, y \in \mathbb{R}$ *and* $x > 0$*, then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$*.*

(ii) *($\mathbb{Q}$ is dense in $\mathbb{R}$) If* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$*.*

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}\!/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Suppose $\mathbb{N}$ is bounded (for contradiction).

Let $b := \sup \mathbb{N}$.

$b - 1$ is not an upper bound as $b - 1 < b$.

$\Rightarrow \quad \exists m \in \mathbb{N}$ such that $m > b - 1 \quad \Rightarrow \quad m + 1 > b$ $(\Rightarrow\Leftarrow)$

Theorem

(i) *(Archimedean property) If* $x, y \in \mathbb{R}$ *and* $x > 0$*, then* $\exists\, n \in \mathbb{N}$ *such that* $nx > y$.

(ii) *(*$\mathbb{Q}$ *is dense in* $\mathbb{R}$*) If* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q}$ *such that* $x < r < y$.

Remark: The two parts are actually equivalent.

Proof: (i): Divide by x. (i) says that $\forall$ real $t := {}^{y}\!/_{x}$ $\exists\, n \in \mathbb{N}$ such that $n > t$.

So (i) says that $\mathbb{N} \subset \mathbb{R}$ is not bounded above.

Suppose $\mathbb{N}$ is bounded (for contradiction).

Let $b := \sup \mathbb{N}$.

$b - 1$ is not an upper bound as $b - 1 < b$.

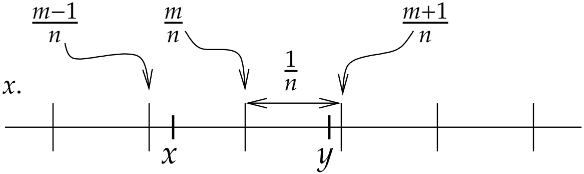
$\Rightarrow \quad \exists m \in \mathbb{N}$ such that $m > b - 1 \quad \Rightarrow \quad m + 1 > b$ $(\Rightarrow\Leftarrow)$

(i) is proved.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.

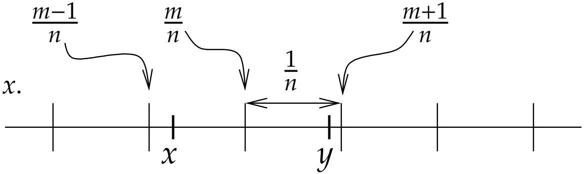


Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.

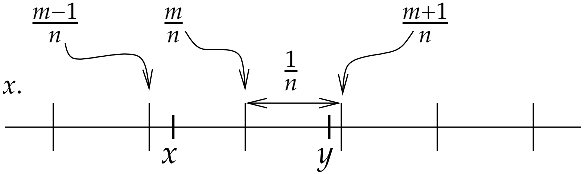
(ii): First assume $x \geq 0$.



Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



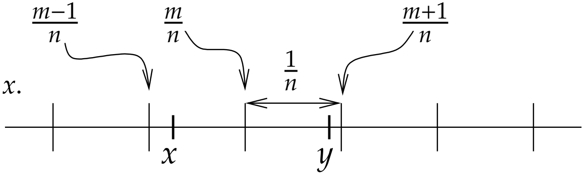
(ii): First assume $x \geq 0$.

$y - x > 0 \quad$ (using (i))$\Rightarrow \quad \exists n \in \mathbb{N}$ such that $\quad n(y - x) > 1 \quad$ or $\quad y - x > 1/n$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

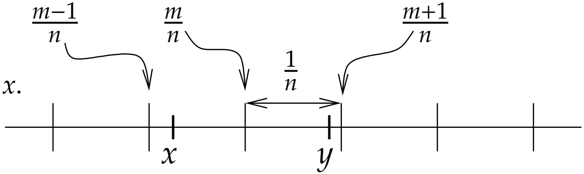
$y - x > 0 \quad$ (using (i))$\Rightarrow \quad \exists n \in \mathbb{N}$ such that $\quad n(y - x) > 1 \quad$ or $\quad y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > 1/n$.

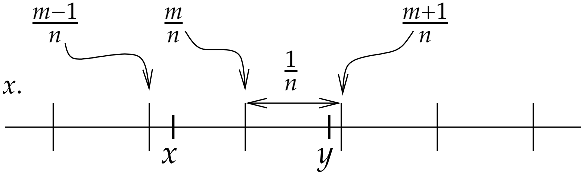
$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

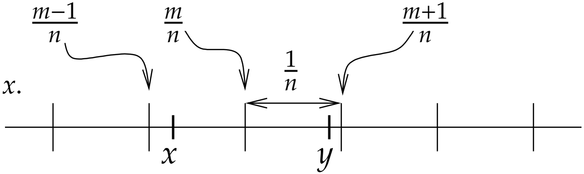
Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < m/n$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > {}^1/_n$,

then find least m such that ${}^m/_n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > {}^1/_n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

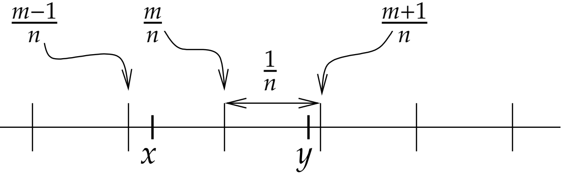
$m \in A \Rightarrow m > nx \Rightarrow x < {}^m/_n$.

m is the least element $\Rightarrow m - 1 \notin A$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \Rightarrow m > nx \Rightarrow x < m/n$.

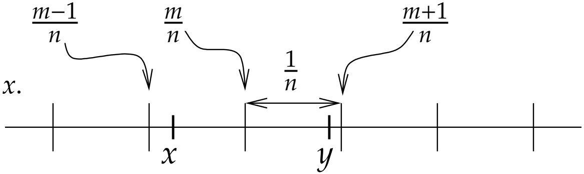
m is the least element $\Rightarrow$ $m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > {}^1/_n$,

then find least m such that ${}^m/_n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0 \quad$ (using (i))$\Rightarrow \quad \exists n \in \mathbb{N}$ such that $\quad n(y - x) > 1 \quad$ or $\quad y - x > {}^1/_n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\quad \Rightarrow \quad \exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < {}^m/_n$.

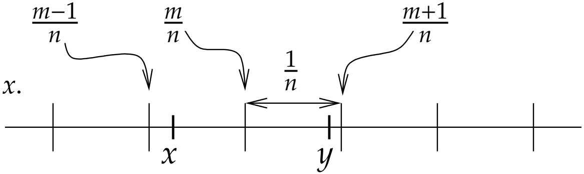
m is the least element $\quad \Rightarrow \quad m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $\quad m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $\quad m - 1 \leq nx \quad$ (as $x \geq 0$).

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > {}^1\!/_n$,

then find least m such that ${}^m\!/_n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0 \quad \text{(using (i))}\Rightarrow \quad \exists n \in \mathbb{N}$ such that $\quad n(y-x) > 1 \quad$ or $\quad y - x > {}^1\!/_n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\quad \Rightarrow \quad \exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < {}^m\!/_n$.

m is the least element $\quad \Rightarrow \quad m - 1 \notin A$.

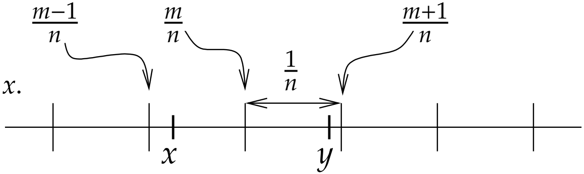
If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $\quad m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $\quad m - 1 \leq nx \quad$ (as $x \geq 0$).

$n(y-x) > 1 \quad \Rightarrow \quad ny > 1 + nx$

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y-x) > 1$ or $y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < m/n$.

m is the least element $\Rightarrow$ $m - 1 \notin A$.

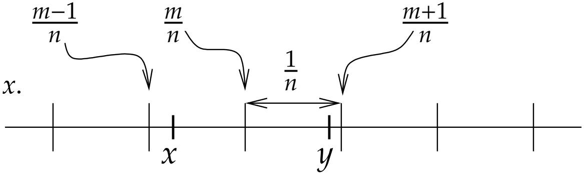
If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $m - 1 \leq nx$ (as $x \geq 0$).

$n(y-x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m$

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0 \quad$ (using (i))$\Rightarrow \quad \exists n \in \mathbb{N}$ such that $\quad n(y-x) > 1 \quad$ or $\quad y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\quad \Rightarrow \quad \exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < m/n$.

m is the least element $\quad \Rightarrow \quad m - 1 \notin A$.

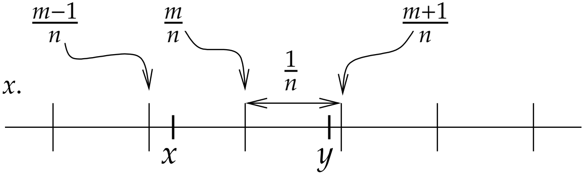
If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $\quad m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $\quad m - 1 \leq nx \quad$ (as $x \geq 0$).

$n(y - x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > m/n$

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < m/n$.

m is the least element $\Rightarrow$ $m - 1 \notin A$.

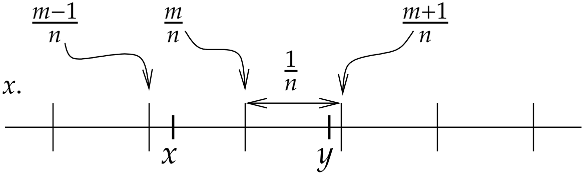
If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $m - 1 \leq nx$ (as $x \geq 0$).

$n(y - x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > m/n$ So $r = m/n$ works.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > {}^1\!/_n$,

then find least m such that ${}^m\!/_n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > {}^1\!/_n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < {}^m\!/_n$.

m is the least element $\Rightarrow$ $m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $m - 1 \leq nx$ (as $x \geq 0$).

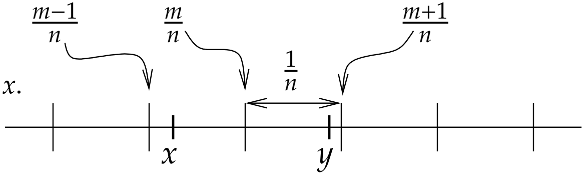
$n(y - x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > {}^m\!/_n$ So $r = {}^m\!/_n$ works.

Now assume $x < 0$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < m/n$.

m is the least element $\Rightarrow$ $m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $m - 1 \leq nx$ (as $x \geq 0$).

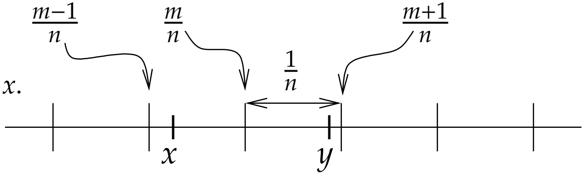
$n(y - x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > m/n$ So $r = m/n$ works.

Now assume $x < 0$. If $y > 0$, then $r = 0$ works.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > 1/n$,

then find least m such that $m/n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$ or $y - x > 1/n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < m/n$.

m is the least element $\Rightarrow$ $m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $m - 1 \leq nx$ (as $x \geq 0$).

$n(y - x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > m/n$ So $r = m/n$ works.

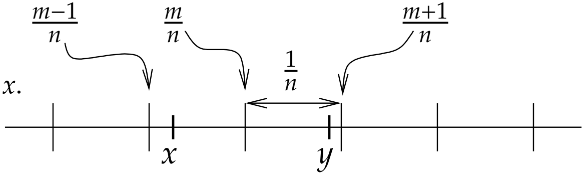
Now assume $x < 0$. If $y > 0$, then $r = 0$ works.

If $y \leq 0$, then $0 \leq -y < -x$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > {}^1/_n$,

then find least m such that ${}^m/_n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0$ (using (i))$\Rightarrow$ $\exists n \in \mathbb{N}$ such that $n(y-x) > 1$ or $y - x > {}^1/_n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\Rightarrow$ $\exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < {}^m/_n$.

m is the least element $\Rightarrow$ $m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $m - 1 \leq nx$ (as $x \geq 0$).

$n(y-x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > {}^m/_n$ So $r = {}^m/_n$ works.

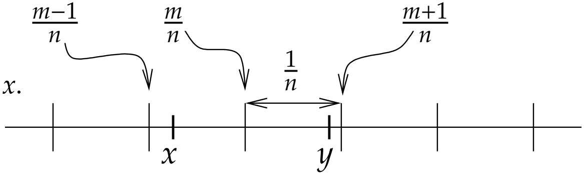
Now assume $x < 0$. If $y > 0$, then $r = 0$ works.

If $y \leq 0$, then $0 \leq -y < -x$. Find $q \in \mathbb{Q}$ s.t. $-y < q < -x$.

Idea for (ii) (density of $\mathbb{Q}$):

Find n such that $y - x > {}^1/_n$,

then find least m such that ${}^m/_n > x$.



(ii): First assume $x \geq 0$.

$y - x > 0 \quad$ (using (i))$\Longrightarrow \quad \exists n \in \mathbb{N}$ such that $\quad n(y-x) > 1 \quad$ or $\quad y - x > {}^1/_n$.

$A := \{k \in \mathbb{N} : k > nx\}$ is nonempty by (i).

Well ordering property $\quad \Rightarrow \quad \exists$ a least element m of A.

$m \in A \quad \Rightarrow \quad m > nx \quad \Rightarrow \quad x < {}^m/_n$.

m is the least element $\quad \Rightarrow \quad m - 1 \notin A$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \notin A$ so $\quad m - 1 \leq nx$.
If $m = 1$, then $m - 1 = 0$ and so also $\quad m - 1 \leq nx \quad$ (as $x \geq 0$).

$n(y-x) > 1 \quad \Rightarrow \quad ny > 1 + nx \quad \Rightarrow \quad ny > 1 + nx \geq m \quad \Rightarrow \quad y > {}^m/_n \qquad$ So $r = {}^m/_n$ works.

Now assume $x < 0$. $\qquad$ If $y > 0$, then $r = 0$ works.

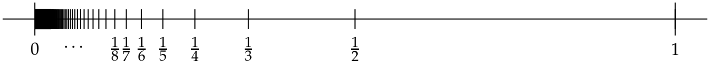
If $y \leq 0$, then $0 \leq -y < -x$. $\quad$ Find $q \in \mathbb{Q}$ s.t. $-y < q < -x$. $\quad r = -q$ works. $\square$

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$

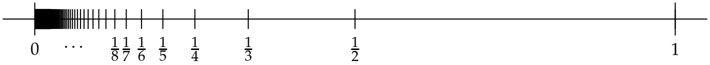
Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$



Corollary

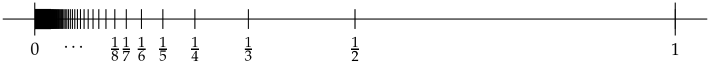
$\inf\{1/n : n \in \mathbb{N}\} = 0.$



Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$

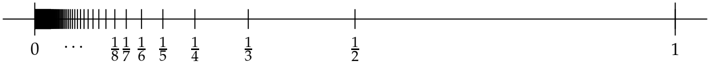


Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

$1/n > 0$ for all $n \in \mathbb{N} \quad \Rightarrow \quad$ 0 is a lower bound

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$



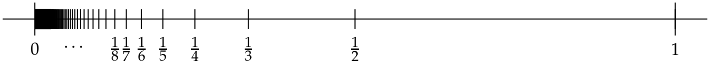
Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

$1/n > 0$ for all $n \in \mathbb{N} \quad \Rightarrow \quad 0$ is a lower bound

$\Rightarrow \quad b := \inf A$ exists and $b \geq 0$.

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$



Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

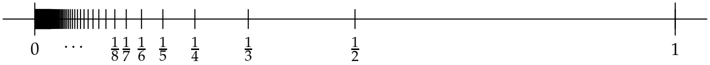
$1/n > 0$ for all $n \in \mathbb{N} \quad \Longrightarrow \quad 0$ is a lower bound

$\Longrightarrow \quad b := \inf A$ exists and $b \geq 0$.

Let $a > 0$ be arbitrary.

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$



Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

$1/n > 0$ for all $n \in \mathbb{N} \quad \Rightarrow \quad 0$ is a lower bound

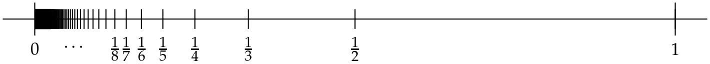
$\Rightarrow \quad b := \inf A$ exists and $b \geq 0$.

Let $a > 0$ be arbitrary.

Archimedean property $\quad \Rightarrow \quad \exists n \in \mathbb{N}$ such that $na > 1$

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$



Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

$1/n > 0$ for all $n \in \mathbb{N} \quad \Rightarrow \quad 0$ is a lower bound

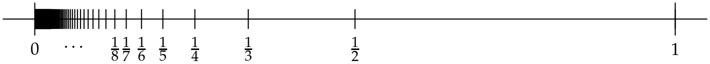
$\Rightarrow \quad b := \inf A$ exists and $b \geq 0$.

Let $a > 0$ be arbitrary.

Archimedean property $\quad \Rightarrow \quad \exists n \in \mathbb{N}$ such that $na > 1 \quad \Rightarrow \quad a > 1/n \in A$

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0.$



Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

$1/n > 0$ for all $n \in \mathbb{N} \quad \Rightarrow \quad$ 0 is a lower bound

$\Rightarrow \quad b := \inf A$ exists and $b \geq 0$.

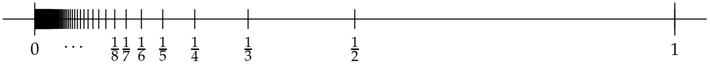
Let $a > 0$ be arbitrary.

Archimedean property $\quad \Rightarrow \quad \exists n \in \mathbb{N}$ such that $na > 1 \quad \Rightarrow \quad a > 1/n \in A$

$\Rightarrow \quad a$ is not a lower bound

Corollary

$\inf\{1/n : n \in \mathbb{N}\} = 0$.



Proof: $A := \{1/n : n \in \mathbb{N}\}$ is nonempty.

$1/n > 0$ for all $n \in \mathbb{N} \quad \Rightarrow \quad 0$ is a lower bound

$\Rightarrow \quad b := \inf A$ exists and $b \geq 0$.

Let $a > 0$ be arbitrary.

Archimedean property $\quad \Rightarrow \quad \exists n \in \mathbb{N}$ such that $na > 1 \quad \Rightarrow \quad a > 1/n \in A$

$\Rightarrow \quad a$ is not a lower bound $\quad \Rightarrow \quad b = 0$. $\square$

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

(ii) *If $x \in \mathbb{R}$ and A is bounded below, then* $\inf(x + A) = x + \inf A$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

(ii) *If $x \in \mathbb{R}$ and A is bounded below, then* $\inf(x + A) = x + \inf A$.

(iii) *If $x > 0$ and A is bounded above, then* $\sup(xA) = x(\sup A)$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

(ii) *If $x \in \mathbb{R}$ and A is bounded below, then* $\inf(x + A) = x + \inf A$.

(iii) *If $x > 0$ and A is bounded above, then* $\sup(xA) = x(\sup A)$.

(iv) *If $x > 0$ and A is bounded below, then* $\inf(xA) = x(\inf A)$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

(ii) *If $x \in \mathbb{R}$ and A is bounded below, then* $\inf(x + A) = x + \inf A$.

(iii) *If $x > 0$ and A is bounded above, then* $\sup(xA) = x(\sup A)$.

(iv) *If $x > 0$ and A is bounded below, then* $\inf(xA) = x(\inf A)$.

(v) *If $x < 0$ and A is bounded below, then* $\sup(xA) = x(\inf A)$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

(ii) *If $x \in \mathbb{R}$ and A is bounded below, then* $\inf(x + A) = x + \inf A$.

(iii) *If $x > 0$ and A is bounded above, then* $\sup(xA) = x(\sup A)$.

(iv) *If $x > 0$ and A is bounded below, then* $\inf(xA) = x(\inf A)$.

(v) *If $x < 0$ and A is bounded below, then* $\sup(xA) = x(\inf A)$.

(vi) *If $x < 0$ and A is bounded above, then* $\inf(xA) = x(\sup A)$.

For $A \subset \mathbb{R}$, $x \in \mathbb{R}$ define $\quad x + A := \{x + y \in \mathbb{R} : y \in A\} \quad xA := \{xy \in \mathbb{R} : y \in A\}$.

E.g., if $A = \{1, 2, 3\}$, then $5 + A = \{6, 7, 8\}$ and $3A = \{3, 6, 9\}$.

Proposition

Let $A \subset \mathbb{R}$ be nonempty.

(i) *If $x \in \mathbb{R}$ and A is bounded above, then* $\sup(x + A) = x + \sup A$.

(ii) *If $x \in \mathbb{R}$ and A is bounded below, then* $\inf(x + A) = x + \inf A$.

(iii) *If $x > 0$ and A is bounded above, then* $\sup(xA) = x(\sup A)$.

(iv) *If $x > 0$ and A is bounded below, then* $\inf(xA) = x(\inf A)$.

(v) *If $x < 0$ and A is bounded below, then* $\sup(xA) = x(\inf A)$.

(vi) *If $x < 0$ and A is bounded above, then* $\inf(xA) = x(\sup A)$.

Note that if $x < 0$, then for multiplication supremum and infimum switch.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$",
rest are exercises.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup\ A$",
rest are exercises.

Suppose b is an upper bound for A.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup\ A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup\ A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup\ A$, then $\quad \sup(x + A) \leq x + b = x + \sup\ A$.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup\ A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup\ A$, then $\quad \sup(x + A) \leq x + b = x + \sup\ A$.

Opposite inequality is similar:

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup A$, then $\quad \sup(x + A) \leq x + b = x + \sup A$.

Opposite inequality is similar: Suppose c is an upper bound for $x + A$.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup\ A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup\ A$, then $\quad \sup(x + A) \leq x + b = x + \sup\ A$.

Opposite inequality is similar: Suppose c is an upper bound for $x + A$.

$\Rightarrow \quad x + y \leq c$ for all $y \in A$

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup A$, then $\quad \sup(x + A) \leq x + b = x + \sup A$.

Opposite inequality is similar: Suppose c is an upper bound for $x + A$.

$\Rightarrow \quad x + y \leq c$ for all $y \in A \quad \Rightarrow \quad y \leq c - x$ for all $y \in A$

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \le b$ for all $y \in A \quad \Rightarrow \quad x + y \le x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup A$, then $\quad \sup(x + A) \le x + b = x + \sup A$.

Opposite inequality is similar: Suppose c is an upper bound for $x + A$.

$\Rightarrow \quad x + y \le c$ for all $y \in A \quad \Rightarrow \quad y \le c - x$ for all $y \in A$

$\Rightarrow \quad c - x$ is an upper bound for A.

Proof: Let's prove (i):
"If $x \in \mathbb{R}$ and A is bounded above, then $\sup(x + A) = x + \sup A$",
rest are exercises.

Suppose b is an upper bound for A.

$\Rightarrow \quad y \leq b$ for all $y \in A \quad \Rightarrow \quad x + y \leq x + b$ for all $y \in A$

$\Rightarrow \quad x + b$ is an upper bound for $x + A$.

If $b = \sup A$, then $\quad \sup(x + A) \leq x + b = x + \sup A$.

Opposite inequality is similar: $\quad$ Suppose c is an upper bound for $x + A$.

$\Rightarrow \quad x + y \leq c$ for all $y \in A \quad \Rightarrow \quad y \leq c - x$ for all $y \in A$

$\Rightarrow \quad c - x$ is an upper bound for A.

If $c = \sup(x + A)$, then $\quad \sup A \leq c - x = \sup(x + A) - x$. $\square$

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$.

Proposition

Let $A, B \subset \mathbb{R}$ *be nonempty sets such that* $x \le y$ *whenever* $x \in A$ *and* $y \in B$.
Then A *is bounded above,* B *is bounded below, and* $\sup A \le \inf B$.

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$.
Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Longrightarrow \quad x \leq \inf B$ for all $x \in A$

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \le y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \le \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\implies$ $x \le \inf B$ for all $x \in A$

$\implies$ $\inf B$ is an upper bound for A

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Longrightarrow \quad x \leq \inf B$ for all $x \in A$

$\Longrightarrow \quad \inf B$ is an upper bound for A

$\Longrightarrow \quad \sup A \leq \inf B$. $\square$

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Rightarrow \quad x \leq \inf B$ for all $x \in A$

$\Rightarrow \quad \inf B$ is an upper bound for A

$\Rightarrow \quad \sup A \leq \inf B$. $\square$

Care must be taken with suprema and infima and strict inequalities.

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Rightarrow$ $x \leq \inf B$ for all $x \in A$

$\Rightarrow$ $\inf B$ is an upper bound for A

$\Rightarrow$ $\sup A \leq \inf B$. $\square$

Care must be taken with suprema and infima and strict inequalities.

"$x < y$ for all $x \in A$ and $y \in B$" still only implies $\sup A \leq \inf B$ (nonstrict).

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Rightarrow \quad x \leq \inf B$ for all $x \in A$

$\Rightarrow \quad \inf B$ is an upper bound for A

$\Rightarrow \quad \sup A \leq \inf B$. $\square$

Care must be taken with suprema and infima and strict inequalities.

"$x < y$ for all $x \in A$ and $y \in B$" still only implies $\sup A \leq \inf B$ (nonstrict).

E.g., $A := \{0\}$ and $B := \{1/n : n \in \mathbb{N}\}$

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$.
Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Rightarrow \quad x \leq \inf B$ for all $x \in A$

$\Rightarrow \quad \inf B$ is an upper bound for A

$\Rightarrow \quad \sup A \leq \inf B$. $\square$

Care must be taken with suprema and infima and strict inequalities.

"$x < y$ for all $x \in A$ and $y \in B$" still only implies $\sup A \leq \inf B$ (nonstrict).

E.g., $A := \{0\}$ and $B := \{1/n : n \in \mathbb{N}\} \quad \Rightarrow \quad 0 < 1/n$ for all $n \in \mathbb{N}$.

Proposition

Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \leq \inf B$.

Proof: Any $x \in A$ is a lower bound for B.

$\Rightarrow \quad x \leq \inf B$ for all $x \in A$

$\Rightarrow \quad \inf B$ is an upper bound for A

$\Rightarrow \quad \sup A \leq \inf B$. $\square$

Care must be taken with suprema and infima and strict inequalities.

"$x < y$ for all $x \in A$ and $y \in B$" still only implies $\sup A \leq \inf B$ (nonstrict).

E.g., $A := \{0\}$ and $B := \{1/n : n \in \mathbb{N}\} \quad \Rightarrow \quad 0 < 1/n$ for all $n \in \mathbb{N}$.

However, $\sup A = 0$ and $\inf B = 0$.

Proposition

If $S \subset \mathbb{R}$ is nonempty and bounded above, then for every $\epsilon > 0$ there exists an $x \in S$ such that $(\sup S) - \epsilon < x \leq \sup S$.

Proposition

If $S \subset \mathbb{R}$ *is nonempty and bounded above, then for every* $\epsilon > 0$ *there exists an* $x \in S$ *such that* $(\sup S) - \epsilon < x \leq \sup S$.

Proof is an exercise.

Definition

Let $A \subset \mathbb{R}$ be a set.

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty.$

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\sup A := -\infty$.

(ii) If A is not bounded above, then $\sup A := \infty$.

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty.$

(ii) If A is not bounded above, then $\quad \sup A := \infty.$

(iii) If A is empty, then $\quad \inf A := \infty.$

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty$.

(ii) If A is not bounded above, then $\quad \sup A := \infty$.

(iii) If A is empty, then $\quad \inf A := \infty$.

(iv) If A is not bounded below, then $\quad \inf A := -\infty$.

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty$.

(ii) If A is not bounded above, then $\quad \sup A := \infty$.

(iii) If A is empty, then $\quad \inf A := \infty$.

(iv) If A is not bounded below, then $\quad \inf A := -\infty$.

Remark: ∞ and $-\infty$ are sometimes treated as numbers.

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty$.

(ii) If A is not bounded above, then $\quad \sup A := \infty$.

(iii) If A is empty, then $\quad \inf A := \infty$.

(iv) If A is not bounded below, then $\quad \inf A := -\infty$.

Remark: ∞ and $-\infty$ are sometimes treated as numbers.

$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ (extended real numbers) is an ordered set
$(-\infty < \infty \quad$ and $\quad -\infty < x$ and $x < \infty$ for all $x \in \mathbb{R})$.

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty$.

(ii) If A is not bounded above, then $\quad \sup A := \infty$.

(iii) If A is empty, then $\quad \inf A := \infty$.

(iv) If A is not bounded below, then $\quad \inf A := -\infty$.

Remark: ∞ and $-\infty$ are sometimes treated as numbers.

$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ (extended real numbers) is an ordered set
($-\infty < \infty \quad$ and $\quad -\infty < x$ and $x < \infty$ for all $x \in \mathbb{R}$).

Some (but not all) arithmetic can be done in the obvious way,

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty.$

(ii) If A is not bounded above, then $\quad \sup A := \infty.$

(iii) If A is empty, then $\quad \inf A := \infty.$

(iv) If A is not bounded below, then $\quad \inf A := -\infty.$

Remark: ∞ and $-\infty$ are sometimes treated as numbers.

$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ (extended real numbers) is an ordered set $(-\infty < \infty \quad$ and $\quad -\infty < x$ and $x < \infty$ for all $x \in \mathbb{R})$.

Some (but not all) arithmetic can be done in the obvious way,

leave $\infty - \infty$, $0 \cdot (\pm\infty)$, and $\frac{\pm\infty}{\pm\infty}$ undefined.

Definition

Let $A \subset \mathbb{R}$ be a set.

(i) If A is empty, then $\quad \sup A := -\infty.$

(ii) If A is not bounded above, then $\quad \sup A := \infty.$

(iii) If A is empty, then $\quad \inf A := \infty.$

(iv) If A is not bounded below, then $\quad \inf A := -\infty.$

Remark: ∞ and $-\infty$ are sometimes treated as numbers.

$\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ (extended real numbers) is an ordered set
$(-\infty < \infty \quad \text{and} \quad -\infty < x$ and $x < \infty$ for all $x \in \mathbb{R})$.

Some (but not all) arithmetic can be done in the obvious way,

leave $\infty - \infty$, $0 \cdot (\pm\infty)$, and $\frac{\pm\infty}{\pm\infty}$ undefined.

We avoid using this arithmetic; $\mathbb{R}^*$ is not a field!

If $A \neq \emptyset$ is finite, then $\inf A \in A$ and $\sup A \in A$.

If $A \neq \emptyset$ is finite, then $\inf A \in A$ and $\sup A \in A$.

We then usually write

$\min A := \inf A$ (minimum)

$\max A := \sup A$ (maximum)

If $A \neq \emptyset$ is finite, then $\inf A \in A$ and $\sup A \in A$.

We then usually write

$\min A := \inf A$ (minimum)

$\max A := \sup A$ (maximum)

E.g., $\max\{1, 2.4, \pi, 100\} = 100 \qquad \min\{1, 2.4, \pi, 100\} = 1.$

Some exercises with useful conclusions.

Some exercises with useful conclusions.

Exercise: Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.

Some exercises with useful conclusions.

Exercise: Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.

Exercise: Let $x, y \in \mathbb{R}$. Suppose $x^2 + y^2 = 0$. Prove that $x = 0$ and $y = 0$.

Some exercises with useful conclusions.

Exercise: Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.

Exercise: Let $x, y \in \mathbb{R}$. Suppose $x^2 + y^2 = 0$. Prove that $x = 0$ and $y = 0$.

Exercise: Let A and B be two nonempty bounded sets of real numbers. Let $C := \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that

$$\sup C = \sup A + \sup B \qquad \text{and} \qquad \inf C = \inf A + \inf B.$$

Some exercises with useful conclusions.

Exercise: Prove that if $t \geq 0$ ($t \in \mathbb{R}$), then there exists an $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.

Exercise: Let $x, y \in \mathbb{R}$. Suppose $x^2 + y^2 = 0$. Prove that $x = 0$ and $y = 0$.

Exercise: Let A and B be two nonempty bounded sets of real numbers. Let $C := \{a + b : a \in A, b \in B\}$. Show that C is a bounded set and that

$$\sup C = \sup A + \sup B \qquad \text{and} \qquad \inf C = \inf A + \inf B.$$

Exercise: Prove the so-called *Bernoulli's inequality*: If $1 + x > 0$, then for all $n \in \mathbb{N}$, we have $(1 + x)^n \geq 1 + nx$.