BA: 7.5

Jiří Lebl

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$\exists\, \epsilon > 0$, s.t. for every $n \in \mathbb{N}\ \exists\, x_n \in X$, $d_X(x_n, c) < 1/n$ and $d_Y\big(f(x_n), f(c)\big) \geq \epsilon$.

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$\exists\, \epsilon > 0$, s.t. for every $n \in \mathbb{N}\ \exists\, x_n \in X,\ d_X(x_n, c) < 1/n$ and $d_Y\big(f(x_n), f(c)\big) \geq \epsilon$.
$\Rightarrow \quad \{x_n\}_{n=1}^{\infty}$ converges to c, but $\big\{f(x_n)\big\}_{n=1}^{\infty}$ does not converge to $f(c)$. $\square$

Example: Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a polynomial:

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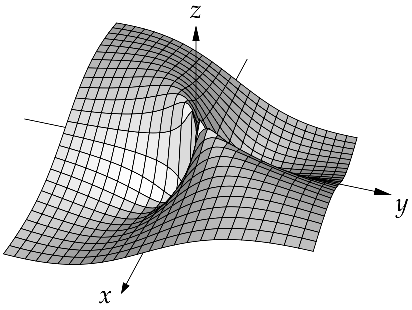

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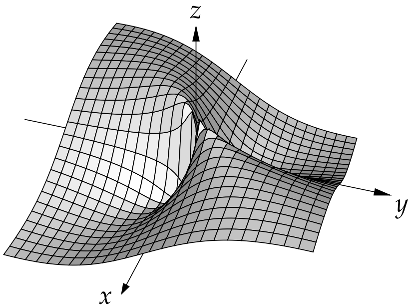

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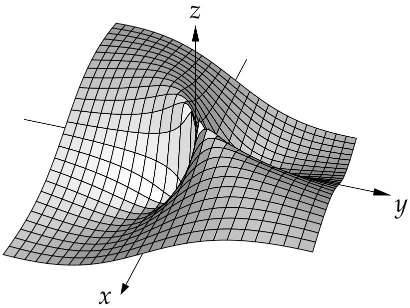

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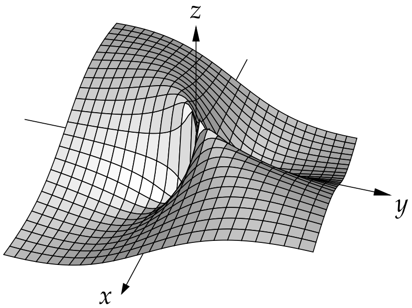

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Claim: f is continuous at $c \in X \quad \Leftrightarrow \quad g$ and h are continuous at c.

Claim follows as $\{f(p_n) = g(p_n) + i\,h(p_n)\}_{n=1}^{\infty}$ converges to $f(p) = g(p) + i\,h(p)$
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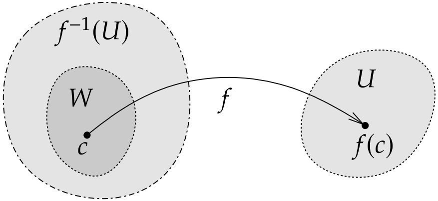

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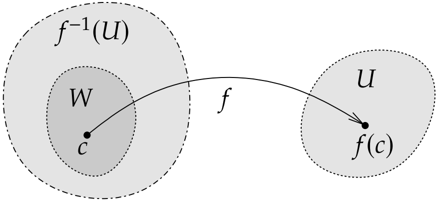

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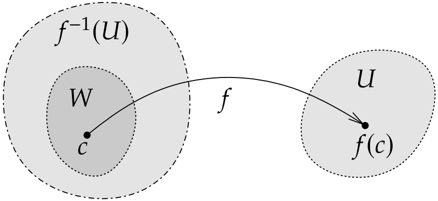

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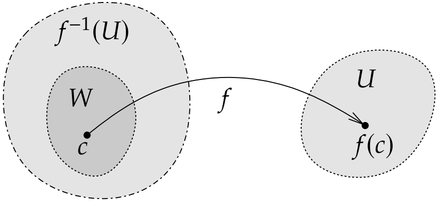

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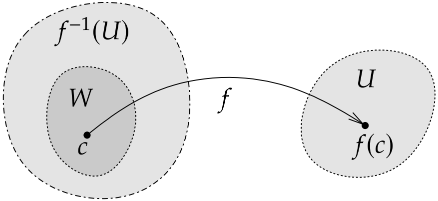

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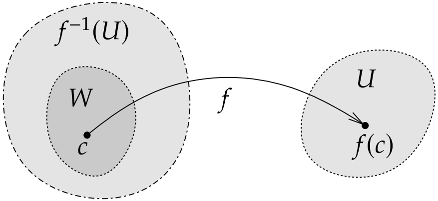

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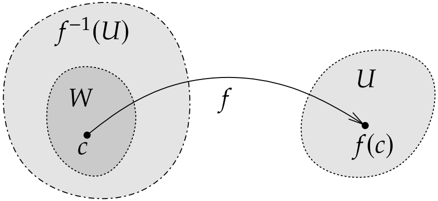

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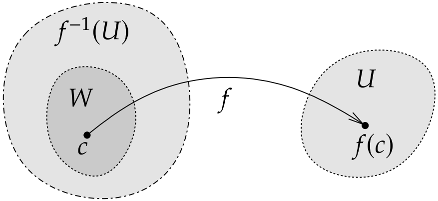

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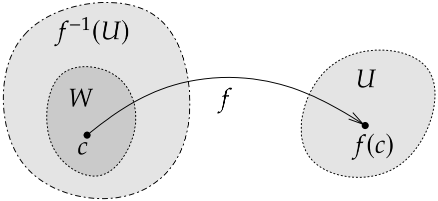

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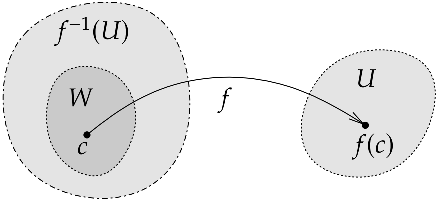

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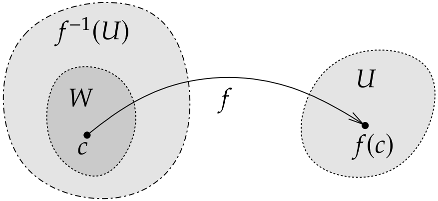

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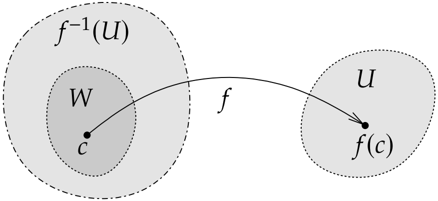

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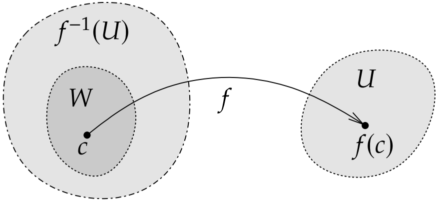

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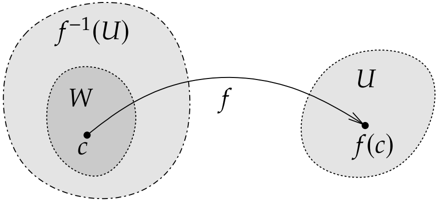

Proof: Suppose f is continuous at c, and
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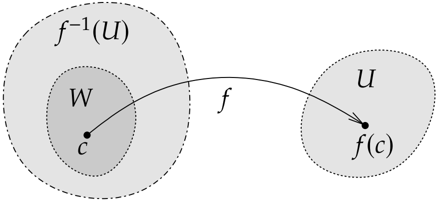

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If $f(x)$ does not converge as x goes to p, then f *diverges* at p.

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Let (X, d_X), (Y, d_Y) *be metric spaces,* $S \subset X$, $p \in X$ *a cluster point of* S*, and* $f : S \longrightarrow Y$ *a function such that* $f(x)$ *converges as* x *goes to* p*.*

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Exercise: As on the real numbers, if p is a cluster point of S, then $f : S \longrightarrow Y$ is continuous at p if and only if

$$\lim_{x \to p} f(x) = f(p).$$

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Exercise: "Intermediate value theorem." Let (X, d) be connected and $f\colon X \to \mathbb{R}$ continuous. If $x_0, x_1 \in X$ and $y \in \mathbb{R}$ such that $f(x_0) < y < f(x_1)$, then $\exists\, z \in X$ such that $f(z) = y$.

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Exercise: Suppose $f: X \to Y$ is one-to-one, onto, and continuous. Suppose X is compact. Then the inverse $f^{-1}: Y \to X$ is continuous.