BA: 3.3

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Common technique: Find a sequence with a certain property, then
use Bolzano–Weierstrass to make a convergent sequence.

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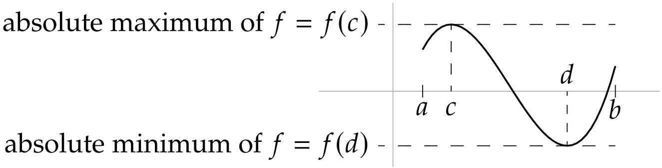

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That f is continuous in the theorem is important.

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Let $f : [a, b] \to \mathbb{R}$ *be continuous. Suppose* $f(a) < 0$ *and* $f(b) > 0$.

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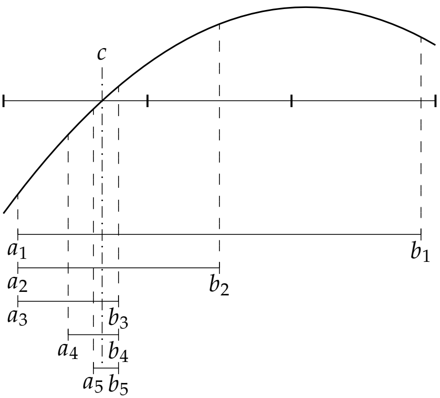

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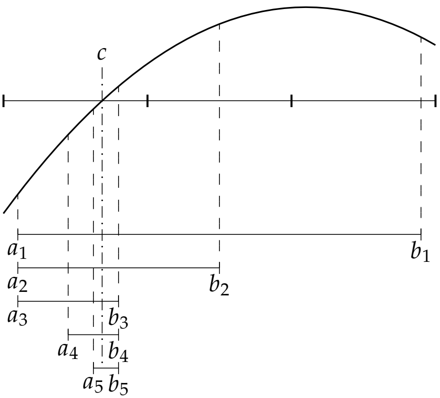

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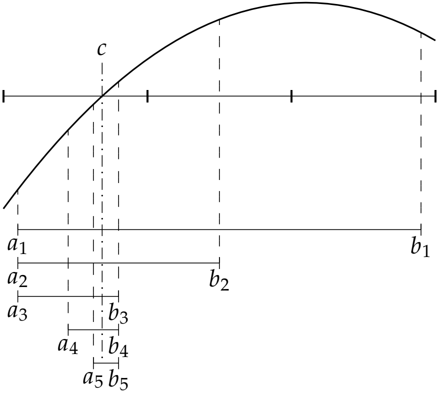

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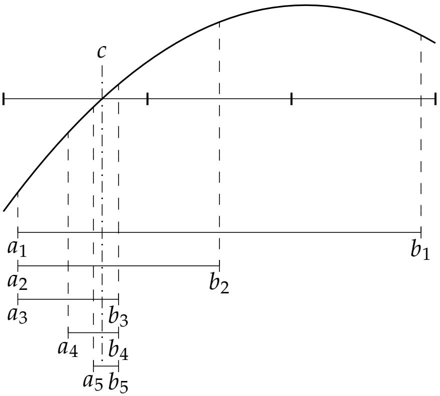

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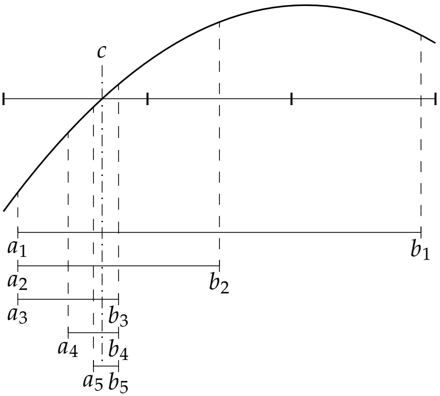

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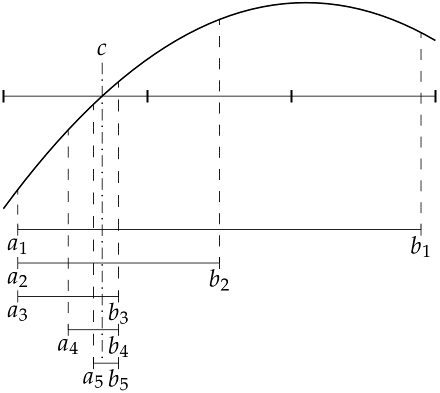

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Note that $c \neq a$ and $c \neq b$ as $f(c) = 0$, $\quad$ so $a < c < b$. $\square$

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If $f\colon S \to \mathbb{R}$ is continuous,
we often apply the theorem to $f|_{[a,b]}$ if $[a,b] \subset S$.

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The theorem/technique gives one root. Must work harder if there are more.

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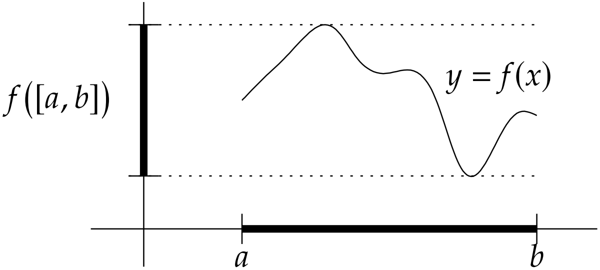

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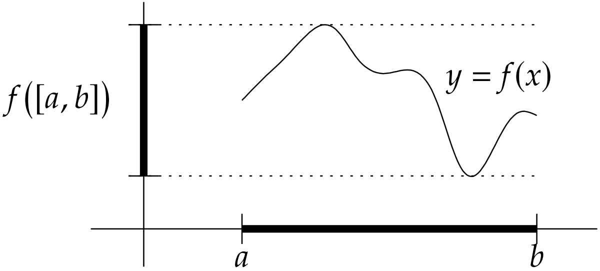

Proof: Exercise.