BA: 1.3

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Case $x \geq 0$: $\quad |x| = x \geq 0$

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Case $x \geq 0$: $\quad |x| = x \geq 0 \quad$ also $|x| = x = 0 \iff x = 0$.

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Case $x \geq 0$: $|x| = x \geq 0$ also $|x| = x = 0 \iff x = 0.$

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$1 = |1 - 2| = |1 + (-2)| \leq |1| + |-2| = 1 + 2 = 3$

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Let $x, y \in \mathbb{R}$.

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Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. *Then*

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That sort of point comes up often in analysis.

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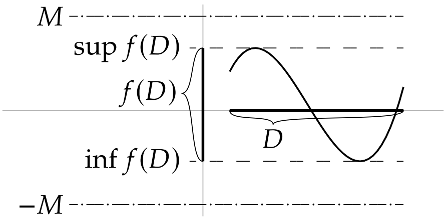

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$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \qquad \textit{and} \qquad \inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

Caution: The x on the LHS is different than the x on the RHS. Perhaps think

$$\sup_{x \in D} f(x) \leq \sup_{y \in D} g(y).$$

Proof: We prove the sup, leave the inf as exercise.
Suppose b is an upper bound for $g(D)$.
$\Rightarrow \quad f(x) \leq g(x) \leq b$ for all $x \in D \quad \Rightarrow \quad b$ is an upper bound for $f(D)$.

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$\implies \quad f(x) \leq g(x) \leq b$ for all $x \in D \quad \implies \quad b$ is an upper bound for $f(D)$.
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$$\Rightarrow \quad \sup_{x\in D} f(x) \leq \sup_{y\in D} g(y) \qquad \square$$

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Proof is an exercise.

Exercise: Let D be nonempty and $f\colon D \to \mathbb{R}$, $g\colon D \to \mathbb{R}$ be bounded.

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a) Show

$$\sup_{x\in D}\big(f(x)+g(x)\big) \le \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$$

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b) Find examples where we obtain strict inequalities.