BA: 7.4

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A metric space (X, d) is *complete* or *Cauchy-complete* if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to a $p \in X$.

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Consider $C\big([a,b],\mathbb{R}\big)$ with the uniform norm as the metric

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In chapter 6, we proved a uniform limit of continuous functions is continuous.

The following proposition is then an easy exercise in verifying the definitions:

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The space of continuous functions $C([a,b],\mathbb{R})$ with the uniform norm as metric is a complete metric space.

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Proof: Exercise.

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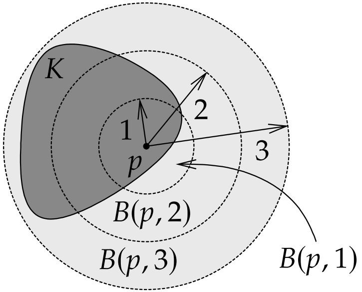

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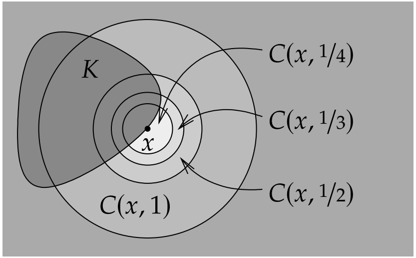

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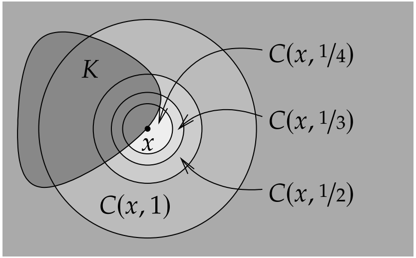

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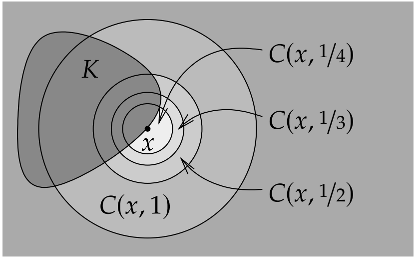

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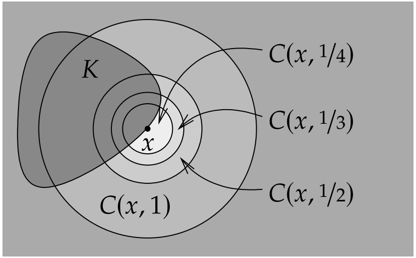

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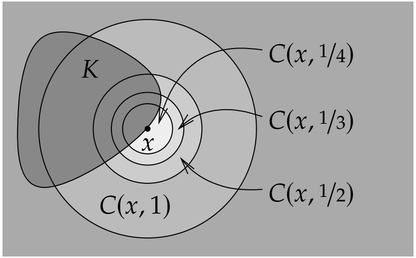

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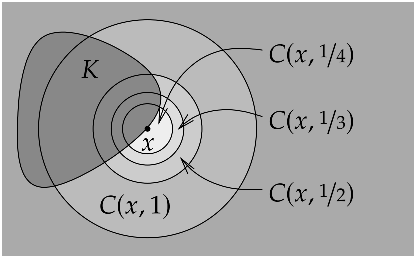

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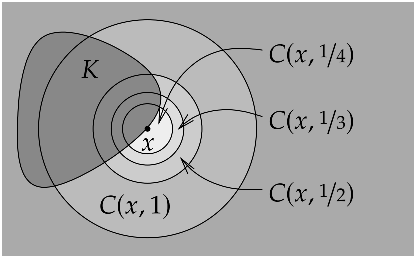

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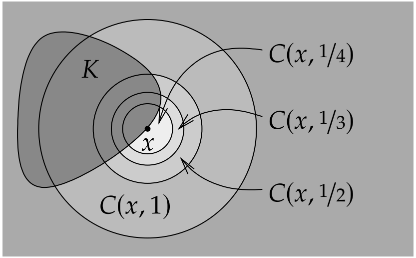

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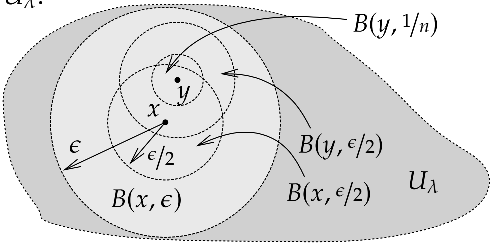

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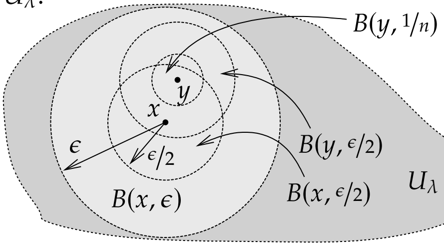

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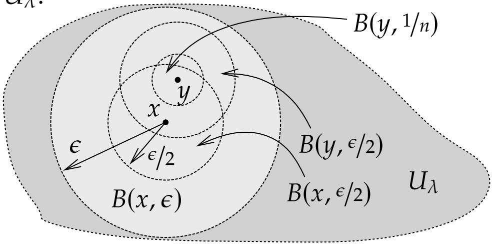

$\Rightarrow \quad y \neq x_n \quad \Rightarrow \quad \{x_n\}_{n=1}^\infty$ has no subsequence converging to x. $\square$

Remark: If K such that every sequence has a convergent subsequence in K is called *sequentially compact*.

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Exercise: Find the cover with no $\delta > 0$.

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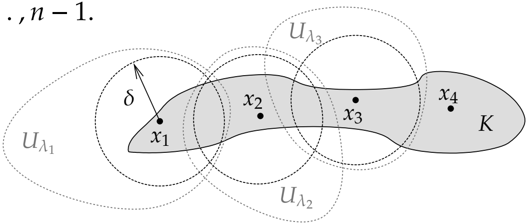

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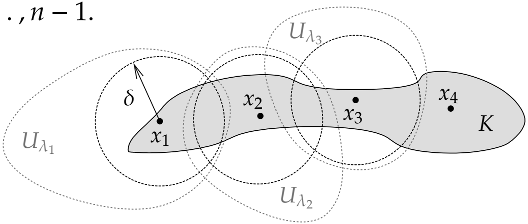

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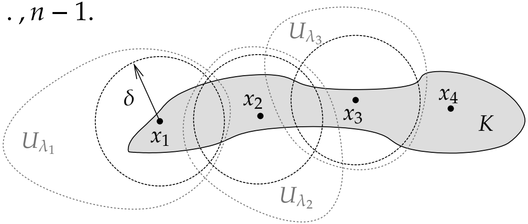

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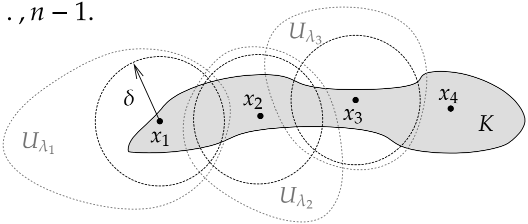

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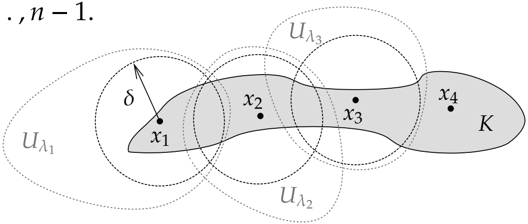

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Exercise: The closed unit ball $C(0, 1)$ in $C([0, 1], \mathbb{R})$ is not compact.

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Let $\{(x_k, y_k)\}_{k=1}^{\infty}$ be a sequence in B.
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Exercise: Let (X, d) be a metric space and $K \subset X$. Prove that K is compact as a subset of (X, d) if and only if K is compact as a subset of itself with the subspace metric.