BA: 5.2

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If $P = Q \cup \{b\}$, then P is a refinement of P, so $L(Q,f) \le L(P,f)$.

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The upper integral argument is analogous (see book). $\square$

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$\Rightarrow$ f is Riemann integrable on $[a,b]$ and $[b,c]$ and $\int_a^c f = \int_a^b f + \int_b^c f$.

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Corollary

If $f \in \mathcal{R}[a, b]$ and $[c, d] \subset [a, b]$, then the restriction $f|_{[c,d]}$ is in $\mathcal{R}[c, d]$.

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Corollary

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Proof: Exercise.

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(i) *αf is in $\mathcal{R}[a,b]$ and $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.*

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Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions. Then

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If f and g are integrable, $\quad \Rightarrow \quad \int_a^b f \leq \int_a^b g.$ $\square$

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Such functions are Riemann integrable.

Exercise: Prove the *mean value theorem for integrals*: If $f\colon [a,b] \to \mathbb{R}$ is continuous, then there exists a $c \in [a,b]$ such that $\int_a^b f = f(c)(b-a)$.

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Exercise: Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all x.

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Exercise (Challenging)**:** Suppose $f \in \mathcal{R}[a,b]$, then the function that takes x to $|f(x)|$ is also Riemann integrable on $[a,b]$. Then show

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$