BA: 0.3 part 3

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Proof: Given $k \in E$, write $k = 2n$ for some $n \in \mathbb{N}$.

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Example bijection: $f(1) := a, f(2) := b, f(3) := c$. The bijection is *not* unique.

Example: The set E of even natural numbers has the same cardinality as $\mathbb{N}$.
Proof: Given $k \in E$, write $k = 2n$ for some $n \in \mathbb{N}$. So $f(n) := 2n$ defines a bijection $f\colon \mathbb{N} \to E$.

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For all $\mathbb{Q}$, also include 0 and the negatives: 0, $^1/_1$, $^{-1}/_1$, $^1/_2$, $^{-1}/_2$, $^2/_1$, $^{-2}/_1$, etc.

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Interestingly, it works for infinite sets too:

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