Tasty Bits of Several Complex Variables

A whirlwind tour of the subject

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Introduction

This book is a polished version of my course notes for Math 6283, Several Complex Variables, given in Spring 2014 and then Spring 2016 semesters at Oklahoma State University. Overall it should be enough material for a semester-long course, though that of course depends on the speed of the lecturer and the number of detours one takes. There are quite a few exercises sprinkled throughout the text, and I am assuming that a reader is at least attempting all of them. Many of them are required later in the text. The reader should attempt exercises in sequence as earlier exercises can help or even be required to solve later exercises.

The prerequisites are a decent knowledge of vector calculus, basic real-analysis and a working knowledge of complex analysis in one variable. It should be accessible to beginning graduate students after a complex analysis course, although the background actually required is quite minimal.

This book is not meant as an exhaustive reference. It is simply a whirlwind tour of several complex variables. To find the list of the books useful for reference and further reading, see the end of the book.

0.1 Motivation, single variable, and Cauchy's formula

Let us start with some standard notation. We use $\mathbb{C}$ for the complex numbers, $\mathbb{R}$ for real numbers, $\mathbb{Z}$ for integers, $\mathbb{N} = \{1,2,3,\ldots\}$ for natural numbers, $i = \sqrt{-1}$, etc… Throughout this book, we will use the standard terminology of *domain* to mean connected open set.

As complex analysis deals with the complex numbers perhaps we should start with $\sqrt{-1}$. We start with the real numbers, $\mathbb{R}$, and wish to add $\sqrt{-1}$ into our field. We call this square root i, and write the complex numbers, $\mathbb{C}$, by identifying $\mathbb{C}$ with $\mathbb{R}^2$ using

$$z = x + iy,$$

where $z \in \mathbb{C}$, and $(x,y) \in \mathbb{R}^2$. A subtle philosophical issue is that there are two square roots of -1. There are two chickens running around in our yard, and because we like to know which is which, we catch one and write "i" on it. If we happened to catch the other chicken, we would have gotten an exactly equivalent theory, which we could not tell apart from the original.

Given a complex number z, its “opposite” is the *complex conjugate* of z and is defined as

$$\bar{z} \stackrel{\text{def}}{=} x - iy.$$

The size of z is measured by the so-called *modulus*, which is really just the *Euclidean distance*:

$$|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2+y^2}.$$

Complex analysis is the study of holomorphic (or complex analytic) functions. There is an awful lot one can do with polynomials, but sometimes they are just not enough. For example, there is no polynomial function that solves the simplest of differential equations $f' = f$. We need the exponential function, which is holomorphic. Holomorphic functions are a generalization of polynomials, and to get there one leaves the land of algebra to arrive in the realm of analysis.

Let us start with polynomials. In one variable a polynomial in z is an expression of the form

$$P(z) = \sum_{j=0}^{d} c_j z^j,$$

where $c_j \in \mathbb{C}$. The number d is called the *degree* of the polynomial P. We can plug in some number z and simply compute $P(z)$, so we have a function $P \colon \mathbb{C} \to \mathbb{C}$.

We try to write

$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$

and all is very fine, until we wish to know what $f(z)$ is for some number $z \in \mathbb{C}$. What we usually mean is

$$\sum_{j=0}^{\infty} c_j z^j = \lim_{d\to\infty} \sum_{j=0}^{d} c_j z^j.$$

As long as the limit exists, we have a function. You know all this; it is your one variable complex analysis. We usually start with the functions and prove that we can expand into series.

A function $f \colon U \subset \mathbb{C} \to \mathbb{C}$ can be written as $f(z) = u(x,y) + iv(x,y)$, where u and v are real-valued. Recall that f is *holomorphic* (or *complex analytic*) if it satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

An easier way to understand these equations is to define the following formal differential operators (the so-called *Wirtinger operators*):

$$\frac{\partial}{\partial z} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The form of these operators is determined by insisting

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0, \quad \frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1.$$

The function f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Let us check:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

This expression is zero if and only if the real parts and the imaginary parts are zero. In other words if and only if

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \qquad \text{and} \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

That is, the Cauchy-Riemann equations are satisfied.

If f is holomorphic, the derivative in z is the standard complex derivative you know and love:

$$\frac{\partial f}{\partial z}(z_0) = f'(z_0) = \lim_{w \to z_0} \frac{f(w) - f(z_0)}{w - z_0}.$$

We can compute it as

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

A function on $\mathbb{C}$ is really a function defined on $\mathbb{R}^2$ as identified above and hence it is a function of x and y. Writing $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, we think of it as a function of two complex variables z and $\bar{z}$. Pretend for a moment as if $\bar{z}$ did not depend on z. The Wirtinger operators work as if z and $\bar{z}$ really were independent variables. For example:

$$\frac{\partial}{\partial z}\left[z^2\bar{z}^3 + z^{10}\right] = 2z\bar{z}^3 + 10z^9 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}\left[z^2\bar{z}^3 + z^{10}\right] = z^2(3\bar{z}^2) + 0.$$

So a holomorphic function is a function not depending on $\bar{z}$.

One of the most important theorems in one variable is the *Cauchy integral formula*.

Theorem 0.1.1 (Cauchy integral formula)**.** *Let $U \subset \mathbb{C}$ be a domain where ∂U is a piecewise smooth simple closed curve (a Jordan curve). Let $f\colon \overline{U} \to \mathbb{C}$ be a continuous function, holomorphic in U. Orient ∂U positively (going around counter clockwise). Then for $z \in U$:*

$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The Cauchy formula is the essential ingredient we need from the study of one complex variable and follows for example from Green's theorem (Stokes' theorem in two dimensions). You can look forward to Theorem 4.1.1 for a proof of a more general formula, the Cauchy-Pompeiu integral formula.

As a differential form $dz = dx + i\,dy$. If you are uneasy about differential forms you probably defined this integral using the Riemann-Stieltjes integral in your one variable class. Let us write down the formula in terms of the standard Riemann integral in a special case. Let

$$\mathbb{D} \overset{\text{def}}{=} \{z : |z| < 1\},$$

be the *unit disc*. The boundary is the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$ oriented positively, that is, counterclockwise. We parametrize $\partial\mathbb{D}$ by e^{it}, where t goes from 0 to 2π. If $\zeta = e^{it}$, then $d\zeta = ie^{it}\,dt$, and

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{it})e^{it}}{e^{it} - z}\,dt.$$

If you are not completely comfortable with path or surface integrals try to think about how you would parametrize the path and write the integral as an integral any calculus student would recognize.

I venture a guess that 90% of what you learned in a complex analysis course (depending on who taught it) is more or less a straightforward consequence of the Cauchy integral formula. An important theorem from one variable that follows from the Cauchy formula is the *maximum principle*, which has several versions, let us give the simplest one.

Theorem 0.1.2 (Maximum principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f \colon U \to \mathbb{C}$ holomorphic function. If*

$$\sup_{z\in U} |f(z)| = f(z_0)$$

for some $z_0 \in U$, then f is constant ($f \equiv f(z_0)$).

That is, if the supremum is attained in the interior of the domain, then the function must be constant. Another way to state the maximum principle is to say: if f extends continuously to the boundary of a domain, then the supremum of $|f(z)|$ is attained on the boundary. In one variable you learned that the maximum principle is really a property of harmonic functions.

Theorem 0.1.3 (Maximum principle)**.** *Let $U \subset \mathbb{C}$ be a domain and $h \colon U \to \mathbb{R}$ harmonic, that is,*

$$\nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

If

$$\sup_{z\in U} h(z) = h(z_0)$$

for some $z_0 \in U$, then h is constant ($h \equiv h(z_0)$).

In one variable if $f = u + iv$ is holomorphic then u and v are harmonic. In fact, locally, any harmonic function is the real (or imaginary) part of a holomorphic function, so studying harmonic functions is almost equivalent to studying holomorphic functions in one complex variable. Things are decidedly different two or more variables.

Holomorphic functions admit a power series representation in z at each point a:

$$f(z) = \sum_{j=0}^{\infty} c_j (z-a)^j.$$

There is no $\bar{z}$ necessary there since $\frac{\partial f}{\partial \bar{z}} = 0$.

Let us see the proof using the Cauchy integral formula as we will require this computation in several variables as well. Given $a \in \mathbb{C}$ and $\rho > 0$ define the disc of radius ρ around a

$$\Delta_\rho(a) \overset{\text{def}}{=} \{z : |z-a| < \rho\}.$$

Suppose $U \subset \mathbb{C}$ is a domain, $f\colon U \to \mathbb{C}$ is holomorphic, $a \in U$, and $\overline{\Delta_\rho(a)} \subset U$ (that is, the closure of the disc is in U, and so its boundary is also in U).

For $z \in \Delta_\rho(a)$ and $\zeta \in \partial\Delta_\rho(a)$,

$$\left| \frac{z-a}{\zeta - a} \right| = \frac{|z-a|}{\rho} < 1.$$

In fact, if $|z-a| \le \rho' < \rho$, then $\left| \frac{z-a}{\zeta-a} \right| \le \frac{\rho'}{\rho} < 1$. Therefore the geometric series

$$\sum_{j=0}^{\infty} \left(\frac{z-a}{\zeta-a} \right)^j = \frac{1}{1-\frac{z-a}{\zeta-a}} = \frac{\zeta-a}{\zeta-z}$$

converges absolutely uniformly for $(z,\zeta) \in \overline{\Delta_{\rho'}(a)} \times \partial\Delta_\rho(a)$. In particular the series converges uniformly in z on compact subsets of $\Delta_\rho(a)$.

Let γ be the curve going around $\partial\Delta_\rho(a)$ once in the positive direction. Compute

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \frac{\zeta - a}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} \sum_{j=0}^{\infty} \left(\frac{z-a}{\zeta-a} \right)^j d\zeta \\
&= \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{j+1}} d\zeta \right) (z-a)^j.
\end{aligned}$$

In the last equality, we are allowed to interchange the limit on the sum by Fubini theorem because for some fixed M

$$\left| \frac{f(\zeta)}{\zeta - a} \left(\frac{z-a}{\zeta - a} \right)^j \right| \leq M \left(\frac{|z-a|}{\rho} \right)^j, \qquad \text{and} \qquad \frac{|z-a|}{\rho} < 1.$$

The key point is writing the *Cauchy kernel* $\frac{1}{\zeta - z}$ as

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \frac{\zeta - a}{\zeta - z},$$

and then using the geometric series.

Not only have we proved that f has a power series, but we computed that the radius of convergence is at least R, where R is the maximum R such that $\Delta_R(a) \subset U$. We also obtained a formula for the coefficients

$$c_j = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{j+1}} d\zeta.$$

For a set K denote the *supremum norm*

$$\|f\|_K \overset{\text{def}}{=} \sup_{z \in K} |f(z)|.$$

By a brute force estimation we obtain the very useful *Cauchy estimates*

$$|c_j| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - a)^{j+1}} d\zeta \right| \leq \frac{1}{2\pi} \int_\gamma \frac{\|f\|_\gamma}{\rho^{j+1}} |d\zeta| = \frac{\|f\|_\gamma}{\rho^j}.$$

We differentiate Cauchy's formula j times,

$$\frac{\partial^j f}{\partial z^j}(z) = \frac{1}{2\pi i} \int_\gamma \frac{j! f(\zeta)}{(\zeta - z)^{j+1}} d\zeta,$$

and therefore

$$j! c_j = \frac{\partial^j f}{\partial z^j}(a).$$

Consequently,

$$\left| \frac{\partial^j f}{\partial z^j}(a) \right| \leq \frac{j! \|f\|_\gamma}{\rho^j}.$$

Chapter 1

Holomorphic functions in several variables

1.1 Onto several variables

Let $\mathbb{C}^n$ denote the *complex Euclidean space*. We denote by $z = (z_1, z_2, \dots, z_n)$ the coordinates of $\mathbb{C}^n$. Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ denote the coordinates in $\mathbb{R}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ by letting $z = x + iy$. Just as in one complex variable we write $\bar{z} = x - iy$. We call z the *holomorphic coordinates* and $\bar{z}$ the *antiholomorphic coordinates*.

Definition 1.1.1. For $\rho = (\rho_1, \dots, \rho_n)$ where $\rho_j > 0$ and $a \in \mathbb{C}^n$, define a *polydisc*

$$\Delta_\rho(a) \overset{\text{def}}{=} \{z \in \mathbb{C}^n : |z_j - a_j| < \rho_j\}.$$

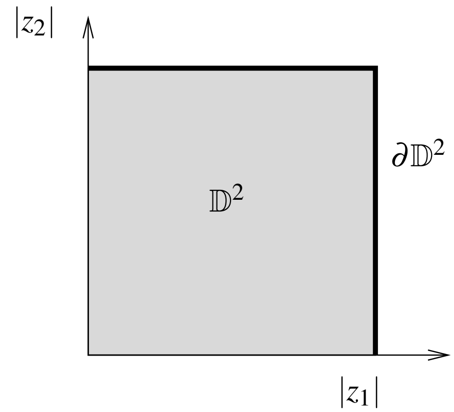
We call a the *center* and ρ the *polyradius* or simply the *radius* of the polydisc $\Delta_\rho(a)$. If $\rho > 0$ is a number then

$$\Delta_\rho(a) \overset{\text{def}}{=} \{z \in \mathbb{C}^n : |z_j - a_j| < \rho\}.$$

As there is the unit disc $\mathbb{D}$ in one variable, so is there the *unit polydisc* in several variables:

$$\mathbb{D}^n = \mathbb{D} \times \mathbb{D} \times \cdots \times \mathbb{D} = \Delta_1(0) = \{z \in \mathbb{C}^n : |z_j| < 1\}.$$

In more than one complex dimension, it is difficult to draw exact pictures for lack of real dimensions on our paper. We can visualize a polydisc in two variables (a *bidisc*) by drawing the following picture by plotting just against the modulus of the variables:



Recall the *Euclidean inner product* on $\mathbb{C}^n$

$$\langle z, w \rangle \overset{\text{def}}{=} z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n.$$

Using the inner product we obtain the standard *Euclidean norm* on $\mathbb{C}^n$

$$\|z\| \overset{\text{def}}{=} \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

This norm agrees with the standard Euclidean norm on $\mathbb{R}^{2n}$. We define balls as in $\mathbb{R}^{2n}$:

$$B_\rho(a) \overset{\text{def}}{=} \{z \in \mathbb{C}^n : \|z - a\| < \rho\},$$

And we define the unit ball as

$$\mathbb{B}_n \overset{\text{def}}{=} B_1(0) = \{z \in \mathbb{C}^n : \|z\| < 1\}.$$

To define holomorphic functions, as in one variable we define the Wirtinger operators

$$\frac{\partial}{\partial z_j} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$
$$\frac{\partial}{\partial \bar{z}_j} \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Definition 1.1.2. Let $U \subset \mathbb{C}^n$ be an open set, and let $f \colon U \to \mathbb{C}$ be a locally bounded function*. Suppose the first partial derivatives exist and f satisfies the *Cauchy-Riemann equations*

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

We then say f is *holomorphic*.

*For every $p \in U$, there is a neighborhood N of p such that $f|_N$ is bounded.

In other words, f is holomorphic if it is holomorphic in each variable separately as a function of one variable. Let us first prove that we may as well have assumed differentiability in the definition.

Proposition 1.1.3. *Let $U \subset \mathbb{C}^n$ be a domain and suppose $f : U \to \mathbb{C}$ is holomorphic. Then f is infinitely differentiable.*

Proof. Suppose $\Delta = \Delta_\rho(a) = \Delta_1 \times \cdots \times \Delta_n$ is a polydisc centered at a, where each Δ_j is a disc and suppose $\overline{\Delta} \subset U$, that is, f is holomorphic on a neighborhood of the closure of Δ. Orient $\partial \Delta_1$ positively and apply the Cauchy formula (after all f is holomorphic in z_1):

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta_1} \frac{f(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1} d\zeta_1.$$

Apply it again on the second variable, again orienting $\partial \Delta_2$ positively:

$$f(z) = \frac{1}{(2\pi i)^2} \int_{\partial \Delta_1} \int_{\partial \Delta_2} \frac{f(\zeta_1, \zeta_2, z_3, \ldots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 \, d\zeta_1.$$

Applying the formula n times we obtain

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial \Delta_1} \int_{\partial \Delta_2} \cdots \int_{\partial \Delta_n} \frac{f(\zeta_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_2 \, d\zeta_1.$$

At this point we notice that we can simply differentiate underneath the integral. We can do so because f is bounded on the compact set where we are integrating and so are the derivatives with respect to z as long as z is a positive distance away from the boundary. We are really differentiating only in the real and imaginary parts of the z_j variables, and the function underneath the integral is infinitely differentiable in those variables. □

In the definition of holomorphicity, we may have assumed f was smooth and satisfies the Cauchy-Riemann equations. However, the way we stated the definition makes it easier to apply.

Above, we really derived the Cauchy integral formula in several variables. To write the formula more concisely we apply the Fubini theorem to write it as a single integral. We will write it down using differential forms. If you are unfamiliar with differential forms, think of the integral as the iterated integral above. It is enough to understand real differential forms; we simply allow complex coefficients. For a good reference on differential forms see Rudin [R2].

Recall that a one-form dx_j is a linear functional on tangent vectors such that when $dx_j\left(\frac{\partial}{\partial x_j}\right) = 1$ and $dx_j\left(\frac{\partial}{\partial x_k}\right) = 0$ if $j \neq k$. Therefore because $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$,

$$dz_j = dx_j + i dy_j, \qquad d\bar{z}_j = dx_j - i dy_j.$$

One can check that as expected,

$$\begin{aligned} dz_j\left(\frac{\partial}{\partial z_k}\right) &= \delta_j^k, & dz_j\left(\frac{\partial}{\partial \bar{z}_k}\right) &= 0, \\ d\bar{z}_j\left(\frac{\partial}{\partial z_k}\right) &= 0, & d\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_k}\right) &= \delta_j^k, \end{aligned}$$

where δ_j^k is the Kronecker delta, that is, $\delta_j^j = 1$, and $\delta_j^k = 0$ if $j \neq k$. One-forms are objects of the form
$$\sum_{j=1}^{n} \alpha_j dz_j + \beta_j d\bar{z}_j,$$
where α_j and β_j are functions (of z). Also recall the wedge product $\omega \wedge \eta$ is anti-commutative on the one-forms, that is, for one-forms ω and η, $\omega \wedge \eta = -\eta \wedge \omega$. A wedge product of two one-forms is a two-form, and so on. A k-form is an object that can then be integrated on a so-called k-chain, for example a k-dimensional surface. The wedge product takes care of the orientation.

At this point we need to talk about orientation in $\mathbb{C}^n$. There are really two natural real-linear isomorphisms of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. That is we identify $z = x + iy$ as
$$(x,y) = (x_1,\ldots,x_n,y_1,\ldots,y_n) \qquad \text{or} \qquad (x_1,y_1,x_2,y_2,\ldots,x_n,y_n).$$
If we took the natural orientation of $\mathbb{R}^{2n}$, we can possibly obtain two opposite orientations (if n is even). The one we take as the natural orientation of $\mathbb{C}^n$ corresponds to the second ordering above, that is, $(x_1,y_1,\ldots,x_n,y_n)$. Both isomorphisms can be used in computation as long as they are used consistently, and the underlying orientation is kept in mind.

Theorem 1.1.4 (Cauchy integral formula)**.** *Let Δ be a polydisc centered at $a \in \mathbb{C}^n$. Suppose $f : \overline{\Delta} \to \mathbb{C}$ is a continuous function holomorphic in Δ. Write $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ oriented appropriately (each $\partial\Delta_j$ has positive orientation). Then for $z \in \Delta$*
$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)\cdots(\zeta_n - z_n)} \, d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n.$$

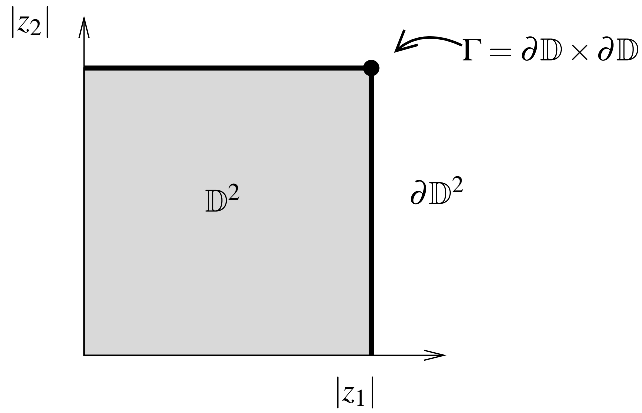
We stated a more general result where f is only continuous on $\overline{\Delta}$ and holomorphic in Δ. The proof of this slight generalization is contained within the next two exercises.

Exercise* 1.1.1*: *Suppose $f : \overline{\mathbb{D}^2} \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{D}^2$. For any $\theta \in \mathbb{R}$, prove*
$$g_1(\xi) = f(\xi, e^{i\theta}) \qquad \text{and} \qquad g_2(\xi) = f(e^{i\theta}, \xi)$$
are holomorphic in $\mathbb{D}$.

Exercise* 1.1.2*: *Prove the theorem above, that is, the slightly more general Cauchy integral formula given f is only continuous on $\overline{\Delta}$ and holomorphic in Δ.*

The Cauchy integral formula shows an important and subtle point about holomorphic functions in several variables: the value of the function f on Δ is completely determined by the values of f on the set Γ, which is much smaller than the boundary of the polydisc $\partial\Delta$. In fact, the Γ is of real dimension n, while the boundary of the polydisc has dimension $2n-1$.

The set $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ is called the *distinguished boundary*. For the unit bidisc we have:



The set Γ is a 2-dimensional torus, like the surface of a donut. Whereas the set $\partial\mathbb{D}^2 = (\partial\mathbb{D} \times \overline{\mathbb{D}}) \cup (\overline{\mathbb{D}} \times \partial\mathbb{D})$ is the union of two filled donuts, or more precisely it is both the inside and the outside of the donut put together, and these two things meet on the surface of the donut. So you can see the set Γ is quite small in comparison to the entire boundary.

Exercise* 1.1.3*: *Suppose Δ is a polydisc, Γ its distinguished boundary, and $f \colon \overline{\Delta} \to \mathbb{C}$ is continuous and holomorphic on Δ. Prove $|f(z)|$ achieves its maximum on Γ.*

Exercise* 1.1.4*: *Show that differentiable in each variable separately does not imply differentiable even in the case where the function is locally bounded. Show that $\frac{xy}{x^2+y^2}$ is a locally bounded function in $\mathbb{R}^2$, that is differentiable in each variable separately (all partial derivatives exist), but the function is not even continuous. There is something very special about the holomorphic category.*

1.2 Power series representation

As you noticed, writing out all the components can be a pain. It would become even more painful later on. Just as we write vectors as z instead of $(z_1, z_2, \ldots, z_n)$ we similarly define notation to deal with the formulas as above.

We will often use the so-called *multi-index notation*. Let $\alpha \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a vector of nonnegative integers. We write

$$z^\alpha \overset{\text{def}}{=} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \qquad |\alpha| \overset{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

$$\frac{1}{z} \overset{\text{def}}{=} \frac{1}{z_1 z_2 \cdots z_n} \qquad \alpha! \overset{\text{def}}{=} \alpha_1! \alpha_2! \cdots \alpha_n!$$

$$dz \overset{\text{def}}{=} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \qquad \frac{\partial^{|\alpha|}}{\partial z^\alpha} \overset{\text{def}}{=} \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

We can also make sense of this notation, especially the notation z^α, if $\alpha \in \mathbb{Z}^n$, although usually α is assumed to be $\mathbb{N}_0^n$. Furthermore, when we use 1 as a vector it will mean $(1,1,\ldots,1)$. For example if $z \in \mathbb{C}^n$ then,

$$1 - z = (1 - z_1, 1 - z_2, \ldots, 1 - z_n).$$

In this notation, the Cauchy formula becomes the perhaps deceptively simple

$$f(z) = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)}\, d\zeta.$$

It goes without saying that when using this notation it is important to be careful to always realize which symbol lives where.

Let us move to power series. For simplicity let us first start with power series at the origin. Using the multinomial notation we write such a series as

$$\sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha.$$

It is important to note what this means. The sum does not have some natural ordering. We are summing over $\alpha \in \mathbb{N}_0$ and there just is not any natural ordering. So it does not make sense to talk about conditional convergence. When we say the series converges, we mean absolutely. Fortunately power series converge absolutely, and so the ordering does not matter. You have to admit that the above is far nicer to write than for example for 3 variables writing

$$\sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{\ell=0}^\infty c_{jk\ell} z_1^j z_2^k z_3^\ell.$$

We will often write just

$$\sum_\alpha c_\alpha z^\alpha,$$

when it is clear from context that we are talking about a power series and therefore all the powers are nonnegative.

To begin, we need the *geometric series in several variables*. If $z \in \mathbb{D}^n$ (unit polydisc) then

$$\begin{aligned}\frac{1}{1-z} = \frac{1}{(1-z_1)(1-z_2)\cdots(1-z_n)} &= \left(\sum_{j=0}^\infty {z_1}^j\right)\left(\sum_{j=0}^\infty {z_2}^j\right)\cdots\left(\sum_{j=0}^\infty {z_n}^j\right) \\ &= \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \cdots \sum_{j_n=0}^\infty \left({z_1}^{j_1} {z_n}^{j_2} \cdots {z_n}^{j_n}\right) = \sum_\alpha z^\alpha.\end{aligned}$$

The convergence is uniform and absolute on compact subsets of the unit polydisc. In fact any compact set in the unit polydisc is contained in a polydisc Δ centered at 0 of radius $1 - \varepsilon$ for some $\varepsilon > 0$. Then the convergence is uniform on Δ (or in fact on the closure of Δ). This claim follows by simply noting the same fact for each factor is true in one dimension.

We now prove that holomorphic functions are precisely those having a power series expansion.

Theorem 1.2.1. *Let $\Delta = \Delta_\rho(a) \subset \mathbb{C}^n$ be a polydisc. Suppose $f : \overline{\Delta} \to \mathbb{C}$ is continuous and holomorphic on Δ. Then on Δ, f is equal to a power series converging uniformly on compact subsets of Δ:*

$$f(z) = \sum_\alpha c_\alpha (z-a)^\alpha. \tag{1.1}$$

Conversely, if f is defined by (1.1) *converging uniformly on compact subsets of Δ, then f is holomorphic on Δ.*

Proof. First assume f is holomorphic. We write $\Gamma = \partial\Delta_1 \times \cdots \times \partial\Delta_n$ and orient it positively. Take $z \in \Delta$ and $\zeta \in \Gamma$. As in one variable we write the kernel of the Cauchy formula as

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} \frac{1}{\left(1 - \frac{z-a}{\zeta-a}\right)} = \frac{1}{\zeta - a} \sum_\alpha \left(\frac{z-a}{\zeta-a}\right)^\alpha.$$

Notice that the geometric series is just a product of geometric series in one variable, and geometric series in one variable is uniformly absolutely convergent on compact subsets of the unit disc. Therefore the series above converges absolutely uniformly for z in compact subsets of Δ and $\zeta \in \Gamma$.

Compute

$$\begin{aligned}
f(z) &= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - a} \frac{\zeta - a}{\zeta - z} d\zeta \\
&= \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{\zeta - a} \sum_\alpha \left(\frac{z-a}{\zeta-a}\right)^\alpha d\zeta \\
&= \sum_\alpha \left(\frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - a)^{\alpha+1}} d\zeta \right) (z-a)^\alpha.
\end{aligned}$$

The last equality follows by Fubini just as it does in one variable.

Uniform convergence (as z moves) on compact subsets of the final series follows from the uniform convergence of the geometric series. It is also a direct consequence of the Cauchy estimates below.

We have shown that

$$f(z) = \sum_\alpha c_\alpha (z-a)^\alpha,$$

where

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}} d\zeta.$$

Notice how strikingly similar the computation is to one variable.

The converse follows by applying the Cauchy-Riemann equations to the series term-wise. To do this you have to show that the term-by-term derivative series also converges uniformly on compact subsets. It is left as an exercise. Then you apply the well-known theorem from real analysis. The proof of this fact is very similar to one variable series that you know.

The conclusion also follows by restricting to one variable for each variable in turn, and then using the corresponding one-variable result. □

> ***Exercise* 1.2.1:** *Prove the claim above that if a power series converges uniformly on compact subsets of a polydisc Δ, then the term by term derivative converges. Do the proof without using the analogous result for single variable series.*

Using Leibniz rule, as long as $z \in \Delta$ and not on the boundary, we can differentiate under the integral. Let us do a single derivative to get the idea:

$$\begin{aligned} \frac{\partial f}{\partial z_1}(z) &= \frac{\partial}{\partial z_1}\left[\frac{1}{(2\pi i)^n}\int_\Gamma \frac{f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{(\zeta_1-z_1)(\zeta_2-z_2)\cdots(\zeta_n-z_n)}\,d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n\right] \\ &= \frac{1}{(2\pi i)^n}\int_\Gamma \frac{f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{(\zeta_1-z_1)^2(\zeta_2-z_2)\cdots(\zeta_n-z_n)}\,d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n. \end{aligned}$$

How about we do it a second time:

$$\frac{\partial^2 f}{\partial z_1^2}(z) = \frac{1}{(2\pi i)^n}\int_\Gamma \frac{2f(\zeta_1,\zeta_2,\ldots,\zeta_n)}{(\zeta_1-z_1)^3(\zeta_2-z_2)\cdots(\zeta_n-z_n)}\,d\zeta_1\wedge d\zeta_2\wedge\cdots\wedge d\zeta_n.$$

Notice the 2 before the f. Next time 3 is coming out, so after j derivatives in z_1 you will get the constant $j!$. It is exactly the same thing that is happening in one variable. A moment's thought will convince you that the following formula is correct for $\alpha \in \mathbb{N}_0^n$:

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) = \frac{1}{(2\pi i)^n}\int_\Gamma \frac{\alpha!\,f(\zeta)}{(\zeta-z)^{\alpha+1}}\,d\zeta.$$

Therefore

$$\alpha!\,c_\alpha = \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a).$$

And as before, we obtain the *Cauchy estimates*:

$$\left|\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a)\right| = \left|\frac{1}{(2\pi i)^n}\int_\Gamma \frac{\alpha!\,f(\zeta)}{(\zeta-a)^{\alpha+1}}\,d\zeta\right| \le \frac{1}{(2\pi)^n}\int_\Gamma \frac{\alpha!\,|f(\zeta)|}{\rho^{\alpha+1}}\,|d\zeta| \le \frac{\alpha!}{\rho^\alpha}\|f\|_\Gamma.$$

Or

$$|c_\alpha| \le \frac{\|f\|_\Gamma}{\rho^\alpha}.$$

As in one variable theory the Cauchy estimates prove the following proposition.

Proposition 1.2.2. *Let $U \subset \mathbb{C}^n$ be a domain. Suppose $f_j \colon U \to \mathbb{C}$ converge uniformly on compact subsets to $f \colon U \to \mathbb{C}$. If every f_j is holomorphic, then f is holomorphic and $\frac{\partial^{|\alpha|} f_j}{\partial z^\alpha}$ converge to $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}$ uniformly on compact subsets.*

> ***Exercise* 1.2.2:** *Prove the above proposition.*

Let $W \subset \mathbb{C}^n$ be the set where a power series converges such that it diverges on the complement. The interior of W is called the *domain of convergence*. In one variable, every domain of convergence is a disc, and hence it can be described with a single number (the radius). In several variables, the domain where a series converges is not as easy to describe. For the geometric series it is easy to see that the domain of convergence is the unit polydisc, but more complicated examples are easy to find.

Example 1.2.3: The power series

$$\sum_{k=0}^{\infty} z_1 z_2^k$$

converges absolutely on the set

$$\{z : |z_2| < 1\} \cup \{z : z_1 = 0\},$$

and nowhere else. This set is not quite a polydisc. It is not even an open set, and is not a closure of the domain of convergence.

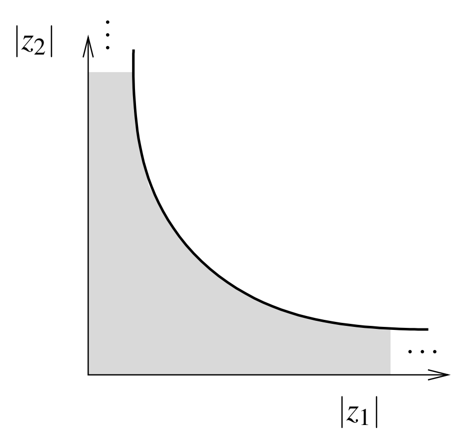
Example 1.2.4: The power series

$$\sum_{k=0}^{\infty} z_1^k z_2^k$$

converges absolutely exactly on the set

$$\{z : |z_1 z_2| < 1\}.$$

Here the picture is definitely more complicated than a polydisc:



Exercise* 1.2.3*: *Find the domain of convergence of $\sum_{j,k} \frac{1}{k!} z_1^j z_2^k$ and draw the corresponding picture.*

Exercise* 1.2.4*: *Find the domain of convergence of $\sum_{j,k} c_{j,k} z_1^j z_2^k$ and draw the corresponding picture if $c_{k,k} = 2^k$, $c_{0,k} = c_{k,0} = 1$ and $c_{j,k} = 0$ otherwise.*

Exercise* 1.2.5*: *Suppose a power series in two variables can be written as a sum of a power series in z_1 and a power series in z_2. Show that the domain of convergence is a polydisc.*

Suppose a domain $U \subset \mathbb{C}^n$ is such that if $z \in U$, then $w \in U$ whenever $|z_j| = |w_j|$ for all j. Such a U is called a *Reinhardt domain*. The domains we were drawing so far have been Reinhardt domains, they are exactly the domains that you can draw by plotting what happens for the moduli of the variables. A domain is called a *complete Reinhardt domain* if whenever $z \in U$ then for $r = (r_1, \ldots, r_n)$ where $r_j = |z_j|$ for all j, we have that the whole polydisc $\Delta_r(0) \subset U$. So a complete Reinhardt domain is a union (possibly infinite) of polydiscs centered at the origin.

Exercise* 1.2.6*: *Let $W \subset \mathbb{C}^n$ be the set where a certain power series at the origin converges. Show that the interior of W is a complete Reinhardt domain.*

Theorem 1.2.5 (Identity theorem)**.** *Let $U \subset \mathbb{C}^n$ be a domain (connected open set) and let $f : U \to \mathbb{C}$ be holomorphic. Suppose $f|_N \equiv 0$ for an open subset $N \subset U$. Then $f \equiv 0$.*

Proof. Let Z be set where all derivatives of f are zero; then $N \subset Z$. The set Z is closed in U as all derivatives are continuous. Take an arbitrary $a \in Z$. We find $\Delta_\rho(a) \subset U$. If we expand f in a power series around a. As the coefficients are given by derivatives of f, we see that the power series is identically zero and hence f is identically zero in $\Delta_\rho(a)$. Therefore Z is open in U and $Z = U$. □

The theorem is often used to show that if two holomorphic functions f and g are equal on a small open set, then $f \equiv g$.

Theorem 1.2.6 (Maximum principle)**.** *Let $U \subset \mathbb{C}^n$ be a domain (connected open set). Let $f : U \to \mathbb{C}$ be holomorphic and suppose $|f(z)|$ attains a maximum at some $a \in U$. Then $f \equiv f(a)$.*

Proof. Suppose $|f(z)|$ attains its maximum at $a \in U$. Consider a polydisc $\Delta = \Delta_1 \times \cdots \times \Delta_n \subset U$ centered at a. The function

$$z_1 \mapsto f(z_1, a_2, \ldots, a_n)$$

is holomorphic on Δ_1 and its modulus attains the maximum at the center. Therefore it is constant by maximum principle in one variable, that is, $f(z_1, a_2, \ldots, a_n) = f(a)$ for all $z_1 \in \Delta_1$. For any fixed $z_1 \in \Delta_1$ consider the function

$$z_2 \mapsto f(z_1, z_2, a_3, \ldots, a_n).$$

This function again attains its maximum modulus at the center of Δ_2 and hence is constant on Δ_2. Iterating this procedure we obtain that $f(z) = f(a)$ for all $z \in \Delta$. By the identity theorem we have that $f(z) = f(a)$ for all $z \in U$. □

Exercise* 1.2.7*: *Let V be the volume measure on $\mathbb{R}^{2n}$ and hence on $\mathbb{C}^n$. Suppose Δ centered at $a \in \mathbb{C}^n$, and f is a function holomorphic on a neighborhood of $\overline{\Delta}$. Prove*

$$f(a) = \frac{1}{V(\Delta)} \int_{\Delta} f(\zeta)\, dV(\zeta).$$

That is, $f(a)$ is an average of the values on a polydisc centered at a.

Exercise* 1.2.8*: *Prove the maximum principle by using the Cauchy formula instead. (Hint: use previous exercise)*

Exercise* 1.2.9*: *Prove a several variables analogue of the* Schwarz's lemma*: Suppose f is holomorphic in a neighborhood of $\overline{\mathbb{D}^n}$, $f(0) = 0$, and for some $k \in \mathbb{N}$ we have $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = 0$ whenever $|\alpha| < k$. Further suppose for all $z \in \mathbb{D}^n$, $|f(z)| \leq M$ for some M. Show that*

$$|f(z)| \leq M \|z\|^k$$

for all $z \in \overline{\mathbb{D}^n}$.

Exercise* 1.2.10*: *Apply the one variable Liouville's theorem to prove it for several variables. That is, suppose $f \colon \mathbb{C}^n \to \mathbb{C}$ is holomorphic and bounded. Prove f is constant.*

Exercise* 1.2.11*: *Improve Liouville's theorem slightly. A complex line though the origin is the image of a linear map $L \colon \mathbb{C} \to \mathbb{C}^n$. a) Prove that for any collection of finitely many complex lines through the origin, there exists an entire nonconstant holomorphic function ($n \geq 2$) bounded on these complex lines. b) Prove that if an entire holomorphic function is bounded on countable many complex lines through the origin then it is constant.*

Exercise* 1.2.12*: *Prove the several variables version of* Montel's theorem*: Suppose $\{f_k\}$ is a sequence of holomorphic functions on $U \subset \mathbb{C}^n$ that is uniformly bounded. Show that there exists a subsequence $\{f_{k_j}\}$ that converges uniformly on compact subsets to some holomorphic function f. Hint: Mimic the one-variable proof.*

Exercise* 1.2.13*: *Prove a several variables version of* Hurwitz's theorem*: Suppose $\{f_k\}$ is a sequence of nowhere zero holomorphic functions on a domain $U \subset \mathbb{C}^n$ converging uniformly on compact subsets to a function f. Show that either f is identically zero, or that f is nowhere zero. Hint: Feel free to use the one variable result.*

Let us define some notation for the set of holomorphic functions. At the same time, we notice that the set of holomorphic functions is naturally a ring under pointwise addition and multiplication.

Definition 1.2.7. Let $U \subset \mathbb{C}^n$ be an open set. Define $\mathscr{O}(U)$ to be the *ring of holomorphic functions*. The letter $\mathscr{O}$ is used to recognize the fundamental contribution to several complex variables by Kiyoshi Oka*.

***Exercise* 1.2.14:** *Prove that $\mathscr{O}(U)$ is actually a ring with the operations*

$$(f+g)(z) = f(z)+g(z), \qquad (fg)(z) = f(z)g(z).$$

***Exercise* 1.2.15:** *Show that $\mathscr{O}(U)$ is an integral domain (has no zero divisors) if and only if U is connected. That is, show that U being connected is equivalent to showing that if $h(z) = f(z)g(z)$ is identically zero for $f, g \in \mathscr{O}(U)$, then either $f(z)$ or $g(z)$ are identically zero.*

A function F defined on a dense open subset of U is *meromorphic* if locally near every $p \in U$, $F = f/g$ for f and g holomorphic in some neighbourhood of p. We remark that it follows from a deep result of Oka that for domains $U \subset \mathbb{C}^n$, every meromorphic function can be represented as f/g globally. That is, the ring of meromorphic functions is the ring of fractions of $\mathscr{O}(U)$. This problem is the so-called *Poincarè problem*, and its solution is no longer positive once we generalize U to complex manifolds. The points where $g = 0$ are called the *poles* of F, just as in one variable.

***Exercise* 1.2.16:** *In two variables one can no longer think of a meromorphic function F simply taking on the value of ∞, when the denominator vanishes. Show that $F(z,w) = z/w$ achieves all values of $\mathbb{C}$ in every neighbourhood of the origin. The origin is called a* point of indeterminacy.

1.3 Derivatives

When you apply a conjugate to a holomorphic function you get a so-called *antiholomorphic function*. An antiholomorphic function is a function that does not depend on z, but only on $\bar{z}$. Given a holomorphic function f, we define the function $\bar{f}$ by $\overline{f(z)}$, that we write as $\bar{f}(\bar{z})$. Then for all j

$$\frac{\partial \bar{f}}{\partial z_j} = 0, \qquad \frac{\partial \bar{f}}{\partial \bar{z}_j} = \overline{\left(\frac{\partial f}{\partial z_j}\right)}.$$

Let us figure out how chain rule works for the holomorphic and antiholomorphic derivatives.

*See `http://en.wikipedia.org/wiki/Kiyoshi_Oka`

Proposition 1.3.1 (Complex chain rule)**.** *Suppose $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and suppose $f\colon U \to V$, and $g\colon V \to \mathbb{C}$ are differentiable functions (mappings). Write the variables as $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$. Then for any $j = 1, \ldots, n$ we have*

$$\frac{\partial}{\partial z_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial \bar{f}_\ell}{\partial z_j} \right), \qquad \frac{\partial}{\partial \bar{z}_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial \bar{f}_\ell}{\partial \bar{z}_j} \right). \tag{1.2}$$

Proof. Write $f = u + iv$, $z = x + iy$, $w = s + it$. The composition replaces w with f, and so it replaces s with u, and t with v. Compute

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) [g \circ f] \\
&= \frac{1}{2} \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial s_\ell} \frac{\partial u_\ell}{\partial x_j} + \frac{\partial g}{\partial t_\ell} \frac{\partial v_\ell}{\partial x_j} - i \left(\frac{\partial g}{\partial s_\ell} \frac{\partial u_\ell}{\partial y_j} + \frac{\partial g}{\partial t_\ell} \frac{\partial v_\ell}{\partial y_j} \right) \right) \\
&= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial s_\ell} \frac{1}{2} \left(\frac{\partial u_\ell}{\partial x_j} - i \frac{\partial u_\ell}{\partial y_j} \right) + \frac{\partial g}{\partial t_\ell} \frac{1}{2} \left(\frac{\partial v_\ell}{\partial x_j} - i \frac{\partial v_\ell}{\partial y_j} \right) \right) \\
&= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial s_\ell} \frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell} \frac{\partial v_\ell}{\partial z_j} \right).
\end{aligned}$$

For $\ell = 1, \ldots, m$,

$$\frac{\partial}{\partial s_\ell} = \frac{\partial}{\partial w_\ell} + \frac{\partial}{\partial \bar{w}_\ell}, \qquad \frac{\partial}{\partial t_\ell} = i \left(\frac{\partial}{\partial w_\ell} - \frac{\partial}{\partial \bar{w}_\ell} \right).$$

Continuing:

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial s_\ell} \frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell} \frac{\partial v_\ell}{\partial z_j} \right) \\
&= \sum_{\ell=1}^{m} \left(\left(\frac{\partial g}{\partial w_\ell} \frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial u_\ell}{\partial z_j} \right) + i \left(\frac{\partial g}{\partial w_\ell} \frac{\partial v_\ell}{\partial z_j} - \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial v_\ell}{\partial z_j} \right) \right) \\
&= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \left(\frac{\partial u_\ell}{\partial z_j} + i \frac{\partial v_\ell}{\partial z_j} \right) + \frac{\partial g}{\partial \bar{w}_\ell} \left(\frac{\partial u_\ell}{\partial z_j} - i \frac{\partial v_\ell}{\partial z_j} \right) \right) \\
&= \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell} \frac{\partial \bar{f}_\ell}{\partial z_j} \right).
\end{aligned}$$

The $\bar{z}$ derivative works similarly. □

Because of the proposition, when we deal with arbitrary possibly nonholomorphic functions we often write $f(z, \bar{z})$ and treat them as functions of z and $\bar{z}$.

Remark 1.3.2. It is good to notice the subtlety of what we just said. Formally it seems as if we are treating z and $\bar{z}$ as independent variables when taking derivatives, but in reality they are not independent if we actually wish to evaluate the function. Underneath, a smooth function that is not necessarily holomorphic is really a function of real variables x and y, where $z = x + iy$.

Remark 1.3.3. Another remark to make is that we could have swapped z and $\bar{z}$, by just flipping the bars everywhere. There is no difference between the two, they are twins in effect. We just need to know which one is which. After all, it all starts with taking the two square roots of -1 and deciding which one is i. There is no "natural choice" for that, but once we make that choice we must be consistent. And once we picked which root is i, we have also picked what is holomorphic and what is antiholomorphic. This is a subtle philosophical as much as a mathematical point.

Definition 1.3.4. Let $U \subset \mathbb{C}^n$ be an open set. A mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic if each component is holomorphic. That is, if $f = (f_1, \ldots, f_m)$ then each f_j is a holomorphic function.

As in one variable the composition of holomorphic functions (mappings) is holomorphic.

Theorem 1.3.5. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets and suppose $f : U \to V$, and $g : V \to \mathbb{C}^k$ are both holomorphic. Then the composition $g \circ f$ is holomorphic.*

Proof. The proof is almost trivial by chain rule. Again let g be a function of $w \in V$ and f be a function of $z \in U$. For any $j = 1, \ldots, n$ and any $p = 1, \ldots, k$ we compute

$$\frac{\partial}{\partial \bar{z}_j} \left[g_p \circ f \right] = \sum_{\ell=1}^{m} \left(\frac{\partial g_p}{\partial w_\ell} \cancelto{0}{\frac{\partial f_\ell}{\partial \bar{z}_j}} + \cancelto{0}{\frac{\partial g_p}{\partial \bar{w}_\ell}} \frac{\partial \bar{f}_\ell}{\partial \bar{z}_j} \right) = 0. \qquad \square$$

Let us also state the chain rule for holomorphic functions then. Again suppose $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and $f : U \to V$, and $g : V \to \mathbb{C}$ are holomorphic. And again let the variables be named $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$. In formula (1.2) for the z_j derivative we notice that the $\bar{w}_j$ derivative of g is zero and the z_j derivative of $\bar{f}_\ell$ is also zero because f and g are holomorphic. Therefore for any $j = 1, \ldots, n$,

$$\frac{\partial}{\partial z_j} \left[g \circ f \right] = \sum_{\ell=1}^{m} \frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j}.$$

Exercise* 1.3.1*: *Prove using only the Wirtinger derivatives that a holomorphic function that is real-valued must be constant.*

Exercise* 1.3.2*: *Let f be a holomorphic function on $\mathbb{C}^n$. When we write $\bar{f}$ we mean the function $z \mapsto \overline{f(z)}$ and we usually write $\bar{f}(\bar{z})$ as the function is antiholomorphic. However if we write $\bar{f}(z)$ we really mean $z \mapsto \overline{f(\bar{z})}$, that is composing both the function and the argument with conjugation. Prove $z \mapsto \bar{f}(z)$ is holomorphic and prove f is real-valued on $\mathbb{R}^n$ (when $y = 0$) if and only if $f(z) = \bar{f}(z)$ for all z.*

The regular implicit function theorem and the chain rule give that the implicit function theorem holds in the holomorphic setting. The main thing to check is to check that the solution given by the standard implicit function theorem is holomorphic, which follows by the chain rule.

Theorem 1.3.6 (Implicit function theorem)**.** *Let $U \subset \mathbb{C}^n \times \mathbb{C}^m$ be a domain, let $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$ be our coordinates, and let $f\colon U \to \mathbb{R}^m$ be a holomorphic mapping. Let $(z^0,w^0) \in U$ be a point such that $f(z^0,w^0) = 0$ and such that the $m \times m$ matrix*

$$\left[\frac{\partial f_j}{\partial w_k}(z^0,w^0) \right]_{jk}$$

is invertible. Then there exists an open set $V \subset \mathbb{C}^n$ with $z^0 \in V$, open set $W \subset \mathbb{C}^m$ with $w^0 \in W$, $V \times W \subset U$, and a holomorphic mapping $g\colon V \to W$, with $g(z^0) = w^0$ such that for every $z \in V$, the point $g(z)$ is the unique point in W such that

$$f\big(z, g(z)\big) = 0.$$

Exercise* 1.3.3*: *Prove the holomorphic implicit function theorem above. Hint: Check that you can use the normal implicit function theorem for C^1 functions, and then show that the g you obtain is holomorphic.*

For a $U \subset \mathbb{C}^n$, a holomorphic mapping $f\colon U \to \mathbb{C}^m$, and a point $a \in U$, define the holomorphic derivative, sometimes called the *Jacobian matrix*:

$$Df(a) \overset{\text{def}}{=} \left[\frac{\partial f_j}{\partial z_k}(a) \right]_{jk} .$$

Sometimes the notation $f'(a) = Df(a)$ is used.

Using the holomorphic chain rule, as in the theory of real functions we get that the derivative of the composition is the composition of derivatives (multiplied as matrices).

Proposition 1.3.7 (Chain rule for holomorphic mappings)**.** *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets. Suppose $f\colon U \to V$ and $g\colon V \to \mathbb{C}^k$ are both holomorphic, and $a \in U$. Then*

$$D(g \circ f)(a) = Dg\big(f(a)\big)\, Df(a).$$

In short-hand we often simply write $D(g \circ f) = DgDf$.

Exercise* 1.3.4*: *Prove the proposition.*

Proposition 1.3.8. *Let $U \subset \mathbb{C}^n$ be open sets and $f\colon U \to \mathbb{C}^n$ be holomorphic. If we let $D_{\mathbb{R}}f(a)$ be the real Jacobian matrix of f (a $2n \times 2n$ real matrix), then*

$$\lvert \det Df(a) \rvert^2 = \det D_{\mathbb{R}} f(a).$$

The expression $\det Df(a)$ is called the *Jacobian determinant* and clearly it is important to know if we are talking about the holomorphic Jacobian determinant or the standard real Jacobian determinant $\det D_{\mathbb{R}} f(a)$. Recall from vector calculus that if the real Jacobian determinant $\det D_{\mathbb{R}} f(a)$ of a smooth function is positive, then the function preserves orientation. In particular, holomorphic maps preserve orientation.

Proof. The statement is simply about matrices. We have a complex $n \times n$ matrix A, that we rewrite as a real $2n \times 2n$ matrix B by using the identity $z = x + iy$. If we change basis from (x,y) to $(z,\bar{z})$ that is $(x+iy, x-iy)$, we are really just changing a basis via a matrix M as $M^{-1}BM$. Then we notice

$$M^{-1}BM = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix},$$

where $\overline{A}$ is the complex conjugate of A. Thus

$$\det(B) = \det(M^{-1}MB) = \det(M^{-1}BM) = \det(A)\det(\overline{A}) = \det(A)\overline{\det(A)} = \left|\det(A)\right|^2. \qquad \square$$

1.4 Inequivalence of ball and polydisc

Definition 1.4.1. Two domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^n$ are said to be *biholomorphic* if there exists a one-to-one and onto holomorphic map $f \colon U \to V$ such that f^{-1} is holomorphic. The mapping f is said to be a *biholomorphic map* or a *biholomorphism*.

One of the main questions in complex analysis is to classify domains up to biholomorphic transformations. In one variable, there is the rather striking theorem due to Riemann:

Theorem 1.4.2 (Riemann mapping theorem)**.** *If $U \subset \mathbb{C}$ is a simply connected domain such that $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$.*

In one variable, a topological property on U is enough to classify a whole class of domains. It is one of the reasons why studying the disc is so important in one variable, and why many theorems are stated for the disc only. There is simply no such theorem in several variables. We will show momentarily that the unit ball and the polydisc,

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| < 1\} \qquad \text{and} \qquad \mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| < 1\},$$

are *not* equivalent. Both are simply connected (have no holes), and they are the two most obvious generalizations of the disc to several dimensions. Let us stick with $n = 2$. Instead of proving that $\mathbb{B}_2$ and $\mathbb{D}^2$ are inequivalent we will prove a stronger theorem. First a definition.

Definition 1.4.3. Suppose $f \colon X \to Y$ is a continuous map between two topological spaces. Then f is a *proper map* if for every compact $K \subset\subset Y$, the set $f^{-1}(K)$ is compact.

The notation "$\subset\subset$" is a common notation for compact, or relatively compact subset. Often the distinction between compact and relatively compact is not important, for example in the above definition we can replace compact with relatively compact.

Vaguely, "proper" means that "boundary goes to the boundary." As a continuous map, f pushes compacts to compacts; a proper map is one where the inverse does so too. If the inverse is a continuous function, then clearly f is proper, but not every proper map is invertible. For example, the map $f\colon \mathbb{D} \to \mathbb{D}$ given by $f(z) = z^2$ is proper, but not invertible. The codomain of f is important. If we replace f by $g\colon \mathbb{D} \to \mathbb{C}$, still given by $g(z) = z^2$, the map is no longer proper. Let us state the main result of this section.

Theorem 1.4.4 (Rothstein 1935)**.** *There exists no proper holomorphic mapping of the unit bidisc $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ to the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$.*

As a biholomorphic mapping is proper, the unit bidisc is not biholomorphically equivalent to the unit ball in $\mathbb{C}^2$. This fact was first proved by Poincaré by computing the automorphism groups of $\mathbb{D}^2$ and $\mathbb{B}_2$, although his proof assumed the maps extended past the boundary. The first complete proof was by Henri Cartan in 1931, though popularly the theorem is attributed to Poincaré. It seems standard that any general audience talk about several complex variables contains a mention of Poincaré, and often the reference is to this exact theorem.

We need some lemmas before we get to the proof of the result. First, a certain one-dimensional object plays a very important role in the geometry of several complex variables. It allows us to apply one-dimensional results in several dimensions. It is especially important in understanding the boundary behavior of holomorphic functions. It also prominently appears in complex geometry.

Definition 1.4.5. A non-constant holomorphic mapping $\varphi\colon \mathbb{D} \to \mathbb{C}^n$ is called an *analytic disc*. If the mapping φ extends continuously to the closed unit disc $\overline{\mathbb{D}}$, then the mapping $\varphi\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ is called a *closed analytic disc*.

Often we call the image $\Delta = \varphi(\mathbb{D})$ the analytic disc rather than the mapping. For a closed analytic disc we write $\partial\Delta = \varphi(\partial\mathbb{D})$ and call it the boundary of the analytic disc.

In some sense, analytic discs play the role of line segments in $\mathbb{C}^n$. It is important to always have in mind that there is a mapping defining the disc, even if we are more interested in the set. Obviously for a given image, the mapping φ is not unique.

Let us consider the boundaries of the unit bidisc $\mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$ and the unit ball $\mathbb{B}_2 \subset \mathbb{C}^2$. We notice that the boundary of the unit bidisc contains analytic discs $\{p\} \times \mathbb{D}$ and $\mathbb{D} \times \{p\}$ for $p \in \partial\mathbb{D}$. That is, through every point in the boundary, with the exception of the distinguished boundary $\partial\mathbb{D} \times \partial\mathbb{D}$ there exists an analytic disc lying entirely inside the boundary. On the other hand for the ball we have the following proposition.

Proposition 1.4.6. *The unit sphere $S^{2n-1} = \partial\mathbb{B}_n \subset \mathbb{C}^n$ contains no analytic discs.*

Proof. Suppose we have a holomorphic function $g\colon \mathbb{D} \to \mathbb{C}^n$ such that the image of g is inside the unit sphere. In other words

$$\|g(z)\|^2 = |g_1(z)|^2 + |g_2(z)|^2 + \cdots + |g_n(z)|^2 = 1$$

for all $z \in \mathbb{D}$. Without loss of generality (after composing with a unitary matrix) we assume that $g(0) = (1,0,0,\ldots,0)$. We look at the first component and notice that $g_1(0) = 1$. If a sum of positive numbers is less than or equal to 1, they all are, and hence $|g_1(z)| \leq 1$. By maximum principle we have that $g_1(z) = 1$ for all $z \in \mathbb{D}$. But then $g_j(z) = 0$ for all $j = 2,\ldots,n$ and all $z \in \mathbb{D}$. Therefore g is constant and thus not an analytic disc. ☐

The fact that the sphere contains no analytic discs is the most important geometric distinction between the boundary of the polydisc and the sphere.

> ***Exercise* 1.4.1:** *Modify the proof to show some stronger results.*
> *a) Let Δ be an analytic disc and $\Delta \cap \partial\mathbb{B}_n \neq \emptyset$. Prove Δ contains points not in $\overline{\mathbb{B}_n}$.*
> *b) Let Δ be an analytic disc such that $\mathbb{B}_n \cap \Delta = \emptyset$, but there is a point $p \in \Delta \cap \partial\mathbb{B}_n$. Prove p is isolated in $\Delta \cap \overline{\mathbb{B}_n}$.*

Before we prove the theorem let us prove a lemma making the statement about proper maps taking boundary to boundary precise.

Lemma 1.4.7. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be bounded domains and let $f \colon U \to V$ be continuous. Then f is proper if and only if for every sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$, the set of limit points of $\{f(p_k)\}$ lies in ∂V.*

Proof. First suppose f is proper. Take a sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$. Then take any convergent subsequence $\{f(p_{k_j})\}$ of $\{f(p_k)\}$ converging to some $q \in \overline{V}$. Take $E = \{f(p_{k_j})\}$ as a set. Let $\overline{E}$ be the closure of E in V (relative topology). If $q \in V$, then $\overline{E} = E \cup \{q\}$, otherwise $\overline{E} = E$. The inverse image $f^{-1}(\overline{E})$ is not compact (it contains a sequence going to $p \in \partial U$) and hence $\overline{E}$ is not compact either as f is proper. Thus $q \notin V$, and hence $q \in \partial V$. As we took an arbitrary subsequence of $\{f(p_k)\}$, q was an arbitrary limit point. Therefore, all limit points are in ∂V.

Let us prove the converse. Suppose that for every sequence $\{p_k\}$ in U such that $p_k \to p \in \partial U$, the set of limit points of $\{f(p_k)\}$ lies in ∂V. Take a closed set $E \subset V$ (relative topology) and look at $f^{-1}(E)$. If $f^{-1}(E)$ is not compact, then there exists a sequence $\{p_k\}$ in $f^{-1}(E)$ such that $p_k \to p \in \partial U$. That is because $f^{-1}(E)$ is closed (in U) but not compact. The hypothesis then says that the limit points of $\{f(p_k)\}$ are in ∂V, and hence E has limit points in ∂V and thus is not compact. ☐

A more general version of the above characterization of proper maps is the following exercise:

> ***Exercise* 1.4.2:** *Let $f \colon X \to Y$ be a continuous function of locally compact Hausdorff topological spaces. Let X_∞ and Y_∞ be the one-point-compactifications of X and Y. Then f is a proper map if and only if it extends as a continuous map $f_\infty \colon X_\infty \to Y_\infty$ by letting $f_\infty|_X = f$ and $f_\infty(\infty) = \infty$.*

We now have all the lemmas needed to prove the theorem of Rothstein.

Proof of Theorem 1.4.4. Suppose we have a proper holomorphic map $f\colon \mathbb{D}^2 \to \mathbb{B}_2$. Fix some $e^{i\theta}$ in the boundary of the disc $\mathbb{D}$. Take a sequence $w_k \in \mathbb{D}$ such that $w_k \to e^{i\theta}$. The functions $g_k(\zeta) = f(\zeta, w_k)$ map the unit disc into $\mathbb{B}_2$. By the standard Montel's theorem, by passing to a subsequence we assume that the sequence of functions converges (uniformly on compact subsets) to a limit $g\colon \mathbb{D} \to \overline{\mathbb{B}}_2$. As $(\zeta, w_k) \to (\zeta, e^{i\theta}) \in \partial\mathbb{D}^2$, then by Lemma 1.4.7 we have that $g(\mathbb{D}) \subset \partial\mathbb{B}_2$ and hence g must be constant.

Let g_k' denote the derivative (we differentiate each component). The functions g_k' converge (uniformly on compacts) to $g' = 0$, so for every fixed $\zeta \in \mathbb{D}$, $\frac{\partial f}{\partial z_1}(\zeta, w_k) \to 0$. Notice that this limit holds for all $e^{i\theta}$ and a subsequence of an arbitrary sequence $\{w_k\}$ where $w_k \to e^{i\theta}$. The mapping that takes w to $\frac{\partial f}{\partial z_1}(\zeta, w)$ therefore extends continuously to $\partial\mathbb{D}$ and is zero on the boundary. We apply the maximum principle or the Cauchy formula and the fact that ζ was arbitrary to find $\frac{\partial f}{\partial z_1} \equiv 0$. By symmetry $\frac{\partial f}{\partial z_2} \equiv 0$. Therefore f is constant, which is a contradiction as f was proper. $\square$

We saw that the reason why there is not even a proper mapping is the fact that the boundary of the polydisc contained analytic discs, while the sphere did not. Similar proof extends to higher dimensions as well. In fact, it is not hard to prove the following theorem.

Theorem 1.4.8. *Let $U = U' \times U'' \subset \mathbb{C}^n \times \mathbb{C}^k$ and $V \subset \mathbb{C}^m$ be bounded domains such that ∂V contains no analytic discs. Then there exist no proper holomorphic mappings $f\colon U \to V$.*

***Exercise* 1.4.3:** *Prove Theorem 1.4.8.*

The key take-away from this section is that in several variables, when looking at which domains are equivalent, it is the geometry of the boundaries makes a difference, not just the topology of the domains.

There is a fun exercise in one dimension about proper maps of discs:

***Exercise* 1.4.4:** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a proper holomorphic map. Then*

$$f(z) = e^{i\theta} \prod_{j=1}^{k} \frac{z - a_j}{1 - \bar{a}_j z},$$

for some real θ and some $a_j \in \mathbb{D}$ (that is, f is a finite Blaschke product). Hint: Consider the set $f^{-1}(0)$.

In several dimensions when $\mathbb{D}$ is replaced by a ball, this question (what are the proper maps) becomes much more involved, and when the dimensions of the balls are different, it is not solved in general.

1.5 Cartan's uniqueness theorem

The following theorem is another analogue of Schwarz's lemma to several variables. It says that for a bounded domain, it is enough to know that a self mapping is the identity at a single point to show that it is the identity everywhere. As there are quite a few theorems named for Cartan, this one is often referred to as the *Cartan's uniqueness theorem*. It can be very useful in computing the automorphism groups of certain domains. An *automorphism* of U is a biholomorphic map from U to U. Automorphisms form a group under composition, called the *automorphism group*. As an exercise, you will use the theorem to compute the automorphism groups of $\mathbb{B}_n$ and $\mathbb{D}^n$.

Theorem 1.5.1 (Cartan)**.** *Suppose $U \subset \mathbb{C}^n$ is a bounded domain, $a \in U$, $f\colon U \to U$ is a holomorphic mapping, $f(a) = a$, and $Df(a)$ is the identity. Then $f(z) = z$ for all $z \in U$.*

***Exercise* 1.5.1:** *Find a counterexample if U is unbounded. Hint: For simplicity take $a = 0$ and $U = \mathbb{C}^n$.*

Before we get into the proof, let us write down the Taylor series of a function in a nicer way, splitting it up into parts of different degree.

A polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$ is *homogeneous* of degree d if

$$P(sz) = s^d P(z)$$

for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$. A homogeneous polynomial of degree d is a polynomial whose every monomial is of total degree d. For example, $z^2w - iz^3 + 9zw^2$ is homogeneous of degree 3 in the variables $(z,w) \in \mathbb{C}^2$. A polynomial vector-valued mapping is homogeneous, if each component is. If f is holomorphic near $a \in \mathbb{C}^n$, then write the power series of f at a as

$$\sum_{j=0}^{\infty} f_j(z-a),$$

where f_j is a homogeneous polynomial of degree j. The f_j is called the *degree d homogeneous part* of f at a. The f_j would be vector-valued if f is vector-valued, such as in the statement of the theorem. In the proof, we will require the vector-valued Cauchy estimates (exercise below)*.

***Exercise* 1.5.2:** *Prove a vector-valued version of the Cauchy estimates. Suppose $f\colon \overline{\Delta_r(a)} \to \mathbb{C}^m$ is continuous function holomorphic on a polydisc $\Delta_r(a) \subset \mathbb{C}^n$. Let T denote the distinguished boundary of Δ. Show that for any multi-index α we get*

$$\left\| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) \right\| \leq \frac{\alpha!}{r^\alpha} \sup_{z \in T} \|f(z)\| .$$

*The normal Cauchy estimates could also be used in the proof of Cartan by applying them componentwise.

Proof of Cartan's uniqueness theorem. Without loss of generality, assume $a = 0$. Write f as a power series at the origin, written in homogeneous parts:

$$f(z) = z + f_k(z) + \sum_{j=k+1}^{\infty} f_j(z),$$

where $k \geq 2$ is an integer such that $f_j(z)$ is zero for all $2 \leq j < k$. The degree 1 homogeneous part is simply the vector z as the derivative of the mapping at the origin is the identity. Compose f with itself ℓ times:

$$f^\ell(z) = \underbrace{f \circ f \circ \cdots \circ f}_{\ell \text{ times}}(z).$$

As $f(U) \subset U$, then f^ℓ is a holomorphic map of U to U. As U is bounded, there is an M such that $\|z\| \leq M$ for all $z \in U$. Therefore $\|f(z)\| \leq M$ for all $z \in U$, and $\|f^\ell(z)\| \leq M$ for all $z \in U$.

By direct computation we get

$$f^\ell(z) = z + \ell f_k(z) + \sum_{j=k+1}^{\infty} \tilde{f}_j(z),$$

for some other degree j homogeneous polynomials $\tilde{f}_j$. Suppose $\Delta_r(0)$ is a polydisc whose closure is in U. Via Cauchy estimates, for any multinomial α with $|\alpha| = k$,

$$\frac{\alpha!}{r^\alpha} M \geq \left\| \frac{\partial^{|\alpha|} f^\ell}{\partial z^\alpha}(0) \right\| = \ell \left\| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) \right\|.$$

The inequality holds for all $\ell \in \mathbb{N}$, and so $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) = 0$. Therefore $f_k \equiv 0$. Hence $f(z) = z$, as there is no other nonzero homogeneous part in the expansion of f. □

As an application, let us classify all biholomorphisms of all bounded circular domains that fix a point. A *circular domain* is a domain $U \subset \mathbb{C}^n$ such that if $z \in U$, then $e^{i\theta} z \in U$ for all $\theta \in \mathbb{R}$.

Corollary 1.5.2. *Suppose $U, V \subset \mathbb{C}^n$ are bounded circular domains with $0 \in U$, $0 \in V$, and $f \colon U \to V$ is a biholomorphic map such that $f(0) = 0$. Then f is linear.*

For example $\mathbb{B}_n$ is circular and bounded, so a biholomorphism of $\mathbb{B}_n$ (an automorphism) that fixes the origin is linear. Similarly a polydisc centered at zero is also circular and bounded.

Proof. The map $g(z) = f^{-1}\big(e^{-i\theta} f(e^{i\theta} z)\big)$ is an automorphism of U and via the chain-rule, $g'(0) = I$. Therefore $f^{-1}\big(e^{-i\theta} f(e^{i\theta} z)\big) = z$, or in other words

$$f(e^{i\theta} z) = e^{i\theta} f(z).$$

Write f near zero as $f(z) = \sum_{j=1}^{\infty} f_j(z)$ where f_j are homogeneous polynomials of degree j (notice $f_0 = 0$). Then

$$e^{i\theta} \sum_{j=1}^{\infty} f_j(z) = \sum_{j=1}^{\infty} f_j(e^{i\theta} z) = \sum_{j=1}^{\infty} e^{ij\theta} f_j(z).$$

By the uniqueness of the Taylor expansion, $e^{i\theta} f_j(z) = e^{ij\theta} f_j(z)$, or $f_j(z) = e^{i(j-1)\theta} f_j(z)$, for all j, all z, and all θ. If $j \neq 1$ we obtain that $f_j \equiv 0$, which proves the claim. $\square$

Exercise* 1.5.3*: *Show that every automorphism f of $\mathbb{D}^n$ (that is a biholomorphism $f \colon \mathbb{D}^n \to \mathbb{D}^n$) is given as*

$$f(z) = P\left(e^{i\theta_1}\frac{z_1 - a_1}{1 - \bar{a}_1 z_1}, e^{i\theta_2}\frac{z_2 - a_2}{1 - \bar{a}_2 z_2}, \ldots, e^{i\theta_n}\frac{z_n - a_n}{1 - \bar{a}_n z_n}\right)$$

for $\theta \in \mathbb{R}^n$, $a \in \mathbb{D}^n$, and a permutation matrix P.

Exercise* 1.5.4*: *Given $a \in \mathbb{B}_n$, define the linear map $P_a z = \frac{\langle z,a \rangle}{\langle a,a \rangle} a$ if $a \neq 0$ and $P_0 z = 0$. Let $s_a = \sqrt{1 - \lVert a \rVert^2}$. Show that every automorphism f of $\mathbb{B}_n$ (that is a biholomorphism $f \colon \mathbb{B}_n \to \mathbb{B}_n$) can be written as*

$$f(z) = U \frac{a - P_a z - s_a (I - P_a) z}{1 - \langle z, a \rangle}$$

for a unitary matrix U and some $a \in \mathbb{B}_n$.

Exercise* 1.5.5*: *Using the previous two exercises, show that $\mathbb{D}^n$ and $\mathbb{B}_n$, $n \geq 2$, are not biholomorphic via a method more in the spirit of what Poincaré used: Show that the groups of automorphisms of the two domains are different groups when $n \geq 2$.*

Exercise* 1.5.6*: *Suppose $U \subset \mathbb{C}^n$ is a bounded domain, $a \in U$, and $f \colon U \to U$ is a holomorphic mapping such that $f(a) = a$. Show that every eigenvalue λ of the matrix $Df(a)$ satisfies $\lvert \lambda \rvert \leq 1$.*

***Exercise* 1.5.7** (Tricky)*: Find a domain $U \subset \mathbb{C}^n$ such that the only biholomorphism $f \colon U \to U$ is the identity $f(z) = z$. Hint: Take the polydisc (or the ball) and remove some number of points (be careful in how you choose them). Then show that f extends to a biholomorphism of the polydisc. Then see what happens to those points you took out.*

1.6 Riemann extension theorem, zero sets, and injective maps

Let us extend a very useful theorem from one dimension to several dimensions. In one dimension if a function is holomorphic in $U \setminus \{p\}$ and locally bounded in U, in particular bounded near p, then the function extends holomorphically to U. In several variables the same theorem holds, and the analogue of a single point is the zero set of a holomorphic function.

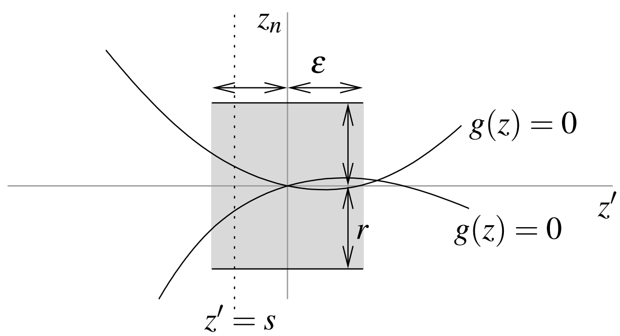
Theorem 1.6.1 (Riemann extension theorem). *Let $U \subset \mathbb{C}^n$ be a domain and let $g \in \mathcal{O}(U)$ that is not identically zero. Let $N = g^{-1}(0)$ be the zero set of g. Suppose $f \in \mathcal{O}(U \setminus N)$ is locally bounded in U. Then there exists a unique $F \in \mathcal{O}(U)$ such that $F|_{U\setminus N} = f$.*

The proof is an application of the Riemann extension theorem from one dimension.

Proof. Take any complex line L in $\mathbb{C}^n$, that is, an image of an affine mapping $\varphi \colon \mathbb{C} \to \mathbb{C}^n$ defined by $\varphi(\xi) = a\xi + b$, for two vectors $a, b \in \mathbb{C}^n$. If L goes through a point $p \in N$, that is say $b = p$, then $g \circ \varphi$ is holomorphic function of one variable. It either is identically zero, or the zero at $\xi = 0$ is isolated. If it would be identically zero for all lines going through p, then g would be identically zero in a neighborhood of p and hence everywhere in U. So there is one line L such that $L \cap N$ has p as an isolated point.

Write $z' = (z_1, \ldots, z_{n-1})$ and $z = (z', z_n)$. Without loss of generality suppose $p = 0$, and L is the line obtained by setting $z' = 0$. There is some small $r > 0$ such that g is nonzero on the set given by $|z_n|^2 = r$ and $z' = 0$. By continuity, g is never zero on the set given by $|z_n|^2 = r$ and $\|z'\| < \varepsilon$ for some $\varepsilon > 0$.

For any fixed small $s \in \mathbb{C}^{n-1}$, with $\|s\| < \varepsilon$, setting $z' = s$, the zeros of $\xi \mapsto g(s, \xi)$ are isolated.



For $\|z'\| < \varepsilon$ and $|z_n| < r$, write

$$F(z', z_n) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(z', \xi)}{\xi - z_n}\, d\xi.$$

The function $\xi \to f(z', \xi)$ extends holomorphically to the entire disc of radius r by the Riemann extension from one dimension. Therefore, F is equal to f on the points where they are both defined. By differentiating under the integral, the function F is holomorphic.

In a neighborhood of each point of N, f extends to a continuous (holomorphic in fact) function. So f uniquely extends continuously to the closure of $U \setminus N$ in the subspace topology, $\overline{(U \setminus N)} \cap U$. The set N has empty interior, so $\overline{(U \setminus N)} \cap U = U$. Hence, F is the unique continuous extension of f to U. $\square$

The set of zeros of a holomorphic function has nice structure at most points.

Theorem 1.6.2. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{O}(U)$ and f is not identically zero. Let $N = f^{-1}(0)$. Then there exists a open dense set of $p \in N$ such that near each such p, after possibly reordering variables, N can be locally written as*

$$z_n = g(z_1, \dots, z_{n-1})$$

for a holomorphic function g.

Proof. Once we show that one regular point p exists, then a whole neighborhood of p in N are regular points. We would be done since we can repeat the procedure on any neighborhood of any point of N.

Since f is not identically zero, then not all derivatives of f can vanish identically on N. So pick a derivative of order k such that all derivatives of order less than k vanish identically on N. We obtain a function $h \colon U \to \mathbb{C}$, holomorphic, and such that without loss of generality the z_n derivative does not vanish identically on N. Then there is some point $p \in N$ such that $\frac{\partial h}{\partial z_n}(p) \neq 0$. We apply the implicit function theorem at p to find g such that

$$h\bigl(z_1, \dots, z_{n-1}, g(z_1, \dots, z_{n-1})\bigr) = 0,$$

and the solution $z_n = g(z_1, \dots, z_{n-1})$ is the unique one in $h = 0$ near p.

Near p we have that the zero set of h contains the zero set of f, and we need to show equality. Write $p = (p', p_n)$. Then the function

$$\xi \mapsto f(p', \xi)$$

has a zero in a small disc around p_n. By Rouche's theorem $\xi \mapsto f(z', \xi)$ must have a zero for z' sufficiently close to p'. It follows that since g was giving the unique solution near p and the zeros of f are contained in the zeros of h, we are done. □

The zero set N of a holomorphic function is a so-called *analytic set* or a *subvariety*, although the general definition of an analytic set is a little more complicated, and includes more sets. See chapter 6. Points where N is written as a graph of a holomorphic mapping are called *regular points*. In particular, since N is a graph of a single holomorphic function, they are called regular points of (complex) dimension $n-1$, or (complex) codimension 1. The set of regular points is what is called an $n-1$ dimensional *complex submanifold*. It is also a real submanifold of real dimension $2n-2$. The points on an analytic set that are not regular are called *singular points*.

Exercise* 1.6.1*: *Find all the regular points of the analytic set $X = \{z \in \mathbb{C}^2 : z_1^2 = z_2^3\}$.*

Exercise* 1.6.2*: *Show that the complement of the zero set of a holomorphic function is connected.*

Let us now prove that a one-to-one holomorphic mapping is biholomorphic, a theorem definitely not true in the smooth setting: $x \mapsto x^3$ is smooth, one-to-one, onto map of $\mathbb{R}$ to $\mathbb{R}$, but the inverse is not differentiable.

Theorem 1.6.3. *Suppose $U \subset \mathbb{C}^n$ is a domain and $f : U \to \mathbb{C}^n$ is holomorphic and one-to-one. Then the Jacobian determinant is never equal to zero on U.*

In particular if $f : U \to V$ is one-to-one and onto for two domains $U, V \subset \mathbb{C}^n$, then f is biholomorphic.

The function f is locally biholomorphic, in particular f^{-1} is holomorphic, on the set where the Jacobian determinant J_f, that is the determinant

$$J_f(z) = \det Df(z) = \det \left[\frac{\partial f_j}{\partial z_k}(z) \right]_{jk},$$

is not zero. This follows from the inverse function theorem, which is just a special case of the implicit function theorem. The trick is to show that J_f happens to be nonzero everywhere.

In one complex dimension, every holomorphic function can in the proper local holomorphic coordinates written as z^d for $d = 0, 1, 2, \ldots$. Such a simple result does not hold in several variables in general, but if the mapping is locally one-to-one then the present theorem says that such a mapping can be locally written as the identity.

Proof. We proceed by induction. We know the theorem for $n = 1$, and suppose we know the theorem is true for dimension $n - 1$.

Suppose for contradiction that $J_f = 0$ somewhere. The Jacobian determinant cannot be identically zero. For example, by the classical theorem of Sard the set of critical values (the image of the set where the Jacobian determinant vanishes) is a null set.

Let us find a regular point q on the zero set of J_f. Write the zero set of J_f near q as

$$z_n = g(z_1, \ldots, z_{n-1})$$

for some holomorphic g. If we prove the theorem near q we are done. So without loss of generality we can assume that $q = 0$ and that U is a small neighborhood of 0. The map

$$F(z_1, \ldots, z_n) = \big(z_1, z_2, \ldots, z_{n-1}, z_n - g(z_1, \ldots, z_{n-1})\big)$$

takes the zero set of J_f to the set given by $z_n = 0$. Let us therefore assume that $J_f = 0$ precisely on the set given by $z_n = 0$.

We wish to show that all the derivatives of f in the $z_1, \ldots, z_{n-1}$ variables vanish whenever $z_n = 0$. This would clearly contradict f being one-to-one.

Suppose without loss of generality that $\frac{\partial f_1}{\partial z_1}$ is nonzero at the origin and $f(0) = 0$. The map

$$G(z_1, \ldots, z_n) = \big(f_1(z), z_2, \ldots, z_n\big)$$

is biholomorphic on a small neighborhood of the origin. The function $f \circ G^{-1}$ is holomorphic and one-to-one on a small neighborhood. Furthermore by definition of G,

$$f \circ G^{-1}(w_1, \ldots, w_n) = \big(w_1, h(w)\big).$$

The mapping
$$\varphi(w_2,\ldots,w_n) = h(0,w_2,\ldots,w_n)$$
is one-to-one holomorphic mapping of a neighborhood of the origin in $\mathbb{C}^{n-1}$ to $\mathbb{C}^{n-1}$. By induction hypothesis, the Jacobian determinant of φ is nowhere zero.

If we differentiate $f \circ G^{-1}$ we notice $D(f \circ G^{-1}) = Df \circ D(G^{-1})$ so at the origin
$$\det D(f \circ G^{-1}) = \bigl(\det Df\bigr)\bigl(\det D(G^{-1})\bigr) = 0.$$
We obtain a contradiction as at the origin
$$\det D(f \circ G^{-1}) = \det D\varphi \neq 0.$$
□

Exercise* 1.6.3*: *Take the analytic set $X = \{z \in \mathbb{C}^2 : z_1^2 = z_2^3\}$. Find a one-to-one holomorphic mapping $f \colon \mathbb{C} \to X$. Then note that the derivative of f vanishes at a certain point. So Theorem 1.6.3 has no analogue when the domain and range have different dimension.*

Exercise* 1.6.4*: *Find a continuous function $f \colon \mathbb{R} \to \mathbb{R}^2$ that is one-to-one but such that the inverse $f^{-1} \colon f(\mathbb{R}) \to \mathbb{R}$ is not continuous.*

We can now state a well-known and as yet unsolved conjecture (and most likely ridiculously hard to solve): the *Jacobian conjecture*. This conjecture is a converse to the above theorem in a special case: *Suppose $F \colon \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map (each component is a polynomial) and the Jacobian derivative J_F is never zero, then F is invertible with a polynomial inverse.* Clearly F would be locally one-to-one, but proving (or disproving) the existence of a global polynomial inverse is the content of the conjecture.

Exercise* 1.6.5*: *Prove the Jacobian conjecture for $n = 1$. That is, prove that if $F \colon \mathbb{C} \to \mathbb{C}$ is a polynomial such that F' is never zero, then F has an inverse, which is a polynomial.*

Exercise* 1.6.6*: *Let $F \colon \mathbb{C}^n \to \mathbb{C}^n$ be an injective polynomial map. Prove J_F is a nonzero constant.*

Exercise* 1.6.7*: *Prove that the Jacobian conjecture is false if "polynomial" is replaced with "entire holomorphic," even for $n = 1$.*

Exercise* 1.6.8*: *Prove that if a holomorphic $f \colon \mathbb{C} \to \mathbb{C}$ is injective then it is onto, and therefore $f(z) = az + b$ for $a \neq 0$.*

Let us also remark that while every injective holomorphic map of $f \colon \mathbb{C} \to \mathbb{C}$ is onto, the same is not true. In $\mathbb{C}^n$, $n \geq 2$, there exist so-called *Fatou-Bieberbach domains*, that is proper subsets of $\mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$.

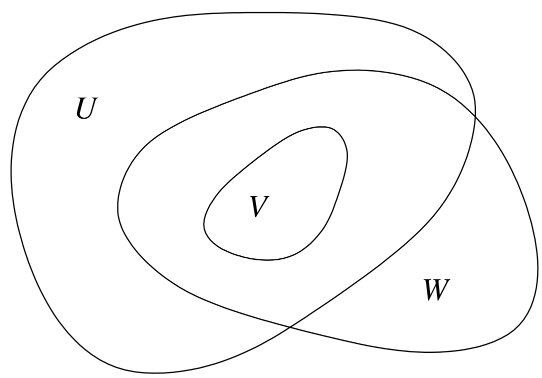
Chapter 2

Convexity and pseudoconvexity

2.1 Domains of holomorphy and holomorphic extensions

It turns out that not every domain in $\mathbb{C}^n$ is a natural domain for holomorphic functions.

Definition 2.1.1. Let $U \subset \mathbb{C}^n$ be a domain (connected open set). The set U is called a *domain of holomorphy* if there do not exist nonempty open sets V and W, with $V \subset U \cap W$, $W \not\subset U$, and W connected, such that for every $f \in \mathscr{O}(U)$ there exists an $\widetilde{f} \in \mathscr{O}(W)$ with $f(z) = \widetilde{f}(z)$ for all $z \in V$.



Example 2.1.2: The unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ is a domain of holomorphy. Proof: Suppose we have V, W, and $\widetilde{f}$ as in the definition. As W is connected and open, it is path connected. There exist points in W that are not in $\mathbb{B}_n$, so there is a path γ in W that goes from a point $q \in V$ to some $p \in \partial\mathbb{B}_n \cap W$. Without loss of generality (after composing with rotations, that is unitary matrices), we assume that $p = (1,0,0,\ldots,0)$. Take the function $f(z) = \frac{1}{1-z_1}$. The function $\widetilde{f}$ must agree with f on the component of $\mathbb{B}_n \cap W$ that contains q. But that component also contains p and so $\widetilde{f}$ must blow up (in particular it cannot be holomorphic) at p. The contradiction shows that no V and W exist.

In one dimension this notion has no real content. Every domain is a domain of holomorphy.

Exercise* 2.1.1** (Easy):*** *In $\mathbb{C}$, every domain is a domain of holomorphy.*

Exercise* 2.1.2*: *If $U_j \subset \mathbb{C}^n$ are domains of holomorphy, then the intersection*

$$\bigcap_j U_j$$

is either empty or every connected component is a domain of holomorphy.

Exercise* 2.1.3** (Easy):*** *Show that a polydisc in $\mathbb{C}^n$ is a domain of holomorphy.*

Exercise* 2.1.4*: *a) Given $p \in \partial\mathbb{B}_n$, find a function f holomorphic on $\mathbb{B}_n$, C^∞-smooth on $\overline{\mathbb{B}_n}$, that does not extend past p. Hint: For the principal branch of $\sqrt{\cdot}$ the function $\xi \mapsto e^{-1/\sqrt{\xi}}$ is holomorphic for* $\operatorname{Re}\xi > 0$ *and can be extended to be continuous (even smooth) on all of* $\operatorname{Re}\xi \geq 0$*.*
b) Find a function f holomorphic on $\mathbb{B}_n$ that does not extend past any point of $\partial\mathbb{B}_n$.

Exercise* 2.1.5*: *Show that a convex domain in $\mathbb{C}^n$ is a domain of holomorphy.*

In the following when we say $f \in \mathscr{O}(U)$ extends holomorphically to V where $U \subset V$, we will mean that there exists a function $\widetilde{f} \in \mathscr{O}(V)$ such that $f = \widetilde{f}$ on U.

Remark 2.1.3. Do note that the subtlety of the definition of domain of holomorphy is that it does not necessarily talk about functions extending to a larger set, since we must take into account single-valuedness. For example, let f be the principal branch of the logarithm defined on $U = \mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. We can define locally an extension from one side through the boundary of the domain, but we cannot define an extension on a larger set that contains U. This example should be motivation why we need V to possibly be a subset of $U \cap W$, and why W need not include all of U.

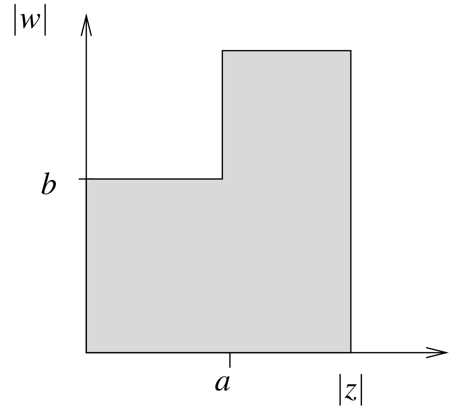
In several dimensions not every domain is a domain of holomorphy. We have the following theorem. The domain H in the theorem is called the *Hartogs figure*.

Theorem 2.1.4. *Let $(z,w) = (z_1,\ldots,z_m,w_1,\ldots,w_k) \in \mathbb{C}^m \times \mathbb{C}^k$ be the coordinates. For two numbers $0 < a,b < 1$, let the set $H \subset \mathbb{D}^{m+k}$ be defined by*

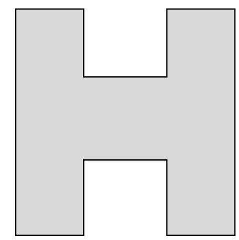
$$H = \{(z,w) \in \mathbb{D}^{m+k} : |z_j| > a \text{ for } j = 1,\ldots,m\} \cup \{(z,w) \in \mathbb{D}^{m+k} : |w_j| < b \text{ for } j = 1,\ldots,k\}.$$

If $f \in \mathscr{O}(H)$, then f extends holomorphically to $\mathbb{D}^{m+k}$.

In $\mathbb{C}^2$ if $m = 1$ and $k = 1$, the figure looks like:



In diagrams, often the Hartogs figure is drawn as:



Proof. Pick a $c \in (a,1)$. Let

$$\Gamma = \{ z \in \mathbb{D}^m : |z_j| = c \text{ for } j = 1,\ldots,m \}.$$

That is, Γ is the distinguished boundary of $c\mathbb{D}^m$, a polydisc centered at 0 of radius c in $\mathbb{C}^m$. We use the Cauchy formula in the first m variables and define the function F

$$F(z,w) = \frac{1}{(2\pi i)^m} \int_\Gamma \frac{f(\xi,w)}{\xi - z} \, d\xi.$$

Clearly F is well defined on all of

$$c\mathbb{D}^m \times \mathbb{D}^k$$

as ξ only ranges through Γ and so as long as $w \in \mathbb{D}^k$ then $(\xi,w) \in H$.

The function F is holomorphic in w as we can differentiate underneath the integral and f is holomorphic in w on H. Furthermore, F is holomorphic in z as the kernel $\frac{1}{\xi - z}$ is holomorphic in z as long as $z \in c\mathbb{D}^m$.

When $|w_j| < b$ for all j, then we know that $F(z,w) = f(z,w)$ for all $z \in c\mathbb{D}^m$. Therefore, F and f are equal on an open subset of H, and hence they are equal everywhere where their domains intersect. It is now easy to see that combining F and f we obtain a holomorphic function on all of $\mathbb{D}^{m+k}$ that extends f. $\square$

The theorem is used in many situations to extend holomorphic functions. We usually need to translate, scale, rotate (apply a unitary), and even take more general biholomorphic mappings of H, to place it wherever we need it. The corresponding polydisc—or the image of the polydisc under the appropriate biholomorphic mapping if one was used—to which all holomorphic functions on H extend is denoted by $\widehat{H}$ and is called the *hull* of H.

Let us state a simple but useful case of the so-called *Hartogs phenomenon*.

Corollary 2.1.5. *Let $U \subset \mathbb{C}^n$, $n \geq 2$, be a domain and $p \in U$. Then every $f \in \mathscr{O}(U \setminus \{p\})$ extends holomorphically to U.*

Proof. Without loss of generality, by translating and scaling (those operations are after all holomorphic), we can assume that $p = (\frac{3}{4}, 0, \ldots, 0)$ and the unit polydisc $\mathbb{D}^n$ is contained in U. We fit a Hartogs figure H in U by letting $m = 1$ and $k = n-1$, writing $\mathbb{C}^n = \mathbb{C}^1 \times \mathbb{C}^{n-1}$, and taking $a = b = \frac{1}{2}$. Then $H \subset U$, and $p \in \mathbb{D}^n \setminus H$. By applying the extension theorem we know that f extends to be holomorphic at p. $\square$

A consequence of this result is that holomorphic functions in several variables have no isolated zeros. That is, suppose $U \subset \mathbb{C}^n$, $n \geq 2$, and $f \in \mathscr{O}(U)$ with f being zero only at p, that is $f^{-1}(0) = \{p\}$. Then $\frac{1}{f}$ would be holomorphic in $U \setminus \{p\}$. It would not be possible to extend f through p (not even continuously let alone holomorphically), and we obtain a contradiction.

The extension works in an even more surprising fashion. We could take out a very large set, for example any geometrically convex* subset:

Exercise* 2.1.6*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, be a domain and $K \subset\subset U$ is a compact geometrically convex subset. If $f \in \mathscr{O}(U \setminus K)$ then f extends to be holomorphic in U. Hint: Find a nice point on ∂K and try extending a little bit. Then make sure your extension is single-valued.*

Convexity of K is not needed; we only need that $U \setminus K$ is connected, however, the proof is much harder. The singlevaluedness of the extension is the key point that makes the general proof harder.

Notice the surprising fact that any holomorphic function on

$$\mathbb{B}_n \setminus \overline{B_{1-\varepsilon}(0)} = \{z \in \mathbb{C}^n : 1 - \varepsilon < \|z\| < 1\}$$

for any $\varepsilon > 0$ automatically extends to a holomorphic function of $\mathbb{B}_n$. We need $n > 1$. The extension result decisively does not work in one dimension; for example take $1/z$. Notice that if $n \geq 2$, then if $f \in \mathscr{O}(\mathbb{B}_n)$ the set of its zeros must "touch the boundary" or be empty. If the set of zeros was in fact compact in $\mathbb{B}_n$, then we could try to extend the function $1/f$.

Exercise* 2.1.7** (Hartogs triangle):*** *Let*

$$T = \{(z_1, z_2) \in \mathbb{D}^2 : |z_2| < |z_1|\}.$$

Show that T is a domain of holomorphy. Then show that if

$$\widetilde{T} = T \cup B_\varepsilon(0)$$

for arbitrarily small $\varepsilon > 0$, then $\widetilde{T}$ is not a domain of holomorphy and in fact every function holomorphic on $\widetilde{T}$ extends to a holomorphic function of $\mathbb{D}^2$.

*By geometrically convex we mean the classical definition of convex: K is convex if given any two $x, y \in K$ and $t \in [0,1]$, then $tx + (1-t)y \in K$.

***Exercise* 2.1.8:** *Take the natural embedding of $\mathbb{R}^2 \subset \mathbb{C}^2$. Suppose $f \in \mathscr{O}(\mathbb{C}^2 \setminus \mathbb{R}^2)$. Show that f extends to be holomorphic in all of $\mathbb{C}^2$. Hint: Change coordinates before using Hartogs.*

***Exercise* 2.1.9:** *Suppose*

$$U = \{(z,w) \in \mathbb{D}^2 : 1/2 < |z|\}.$$

Draw U. Let $\gamma = \{z : |z| = 3/4\}$ oriented positively. If $f \in \mathscr{O}(U)$, then show that the function

$$F(z,w) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi,w)}{\xi - z}\, d\xi$$

is well defined in $\big((3/4)\mathbb{D}\big) \times \mathbb{D}$, holomorphic where defined, yet it is not necessarily true that $F = f$ on the intersections of their domains.

***Exercise* 2.1.10:** *Suppose $U \subset \mathbb{C}^n$ is an open set such that for every $z \in \mathbb{C}^n \setminus \{0\}$, there is a $\lambda \in \mathbb{C}$ such that $\lambda z \in U$. Let $f \colon U \to \mathbb{C}$ be holomorphic with $f(\lambda z) = f(z)$ whenever $z \in U$, $\lambda \in \mathbb{C}$ and $\lambda z \in U$. a) (easy) Prove that f is constant. b) (hard) Relax the requirement on f to being meromorphic, that is $f = g/h$ for holomorphic g and h, find a nonconstant example and prove that such an f must be rational (that is g and h must be polynomials).*

Example 2.1.6: By Exercise 2.1.8 we see that $U_1 = \mathbb{C}^2 \setminus \mathbb{R}^2$ is not a domain of holomorphy. On the other hand $U_2 = \mathbb{C}^2 \setminus \{z : z_2 = 0\}$ is a domain of holomorphy; simply use $f(z) = \frac{1}{z_2}$ as the function that cannot extend. Therefore U_1 and U_2 are very different as far as complex variables are concerned, yet they are the same set if we ignore the complex structure. They are both simply a 4 dimensional real vector space minus a 2 dimensional real vector subspace. That is, U_1 is defined by $\operatorname{Im} z_1 \neq 0$ and $\operatorname{Im} z_2 \neq 0$, while U_2 is defined by $\operatorname{Re} z_2 \neq 0$ and $\operatorname{Im} z_2 \neq 0$.

The condition of being a domain of holomorphy, will require something more than just some real geometric condition on the set. In particular we have shown that the image of a domain of holomorphy via an orthonormal real-linear mapping (so preserving distances, angles, straight lines, etc...) need not be a domain of holomorphy. Therefore, when we want to "rotate" in complex analysis we need to use a complex linear mapping, so a unitary.

2.2 Tangent vectors, the Hessian, and convexity

An exercise in the previous section showed that any convex domain is a domain of holomorphy. However, classical convexity is too strong.

***Exercise* 2.2.1:** *Show that if $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^k$ are both domains of holomorphy, then $U \times V$ is a domain of holomorphy.*

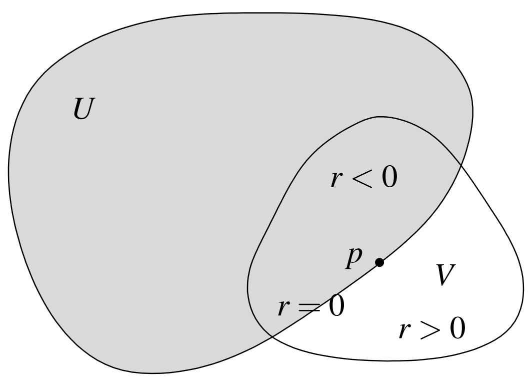
In particular, the exercise says that given any domain $U \subset \mathbb{C}$ and any domain $V \subset \mathbb{C}$, the domain $U \times V$ is a domain of holomorphy in two variables. The domains U and V, and therefore $U \times V$ can be spectacularly non-convex. But we should not discard convexity completely, there is a notion of *pseudoconvexity*, which vaguely means "convexity in the complex directions" that is the correct notion to classify which domains are domains of holomorphy.

Definition 2.2.1. A set $M \subset \mathbb{R}^n$ is a real C^k-smooth *hypersurface* if at each point $p \in M$, there exists a k-times continuously differentiable function $r \colon V \to \mathbb{R}$, defined in a neighborhood V of p with nonvanishing derivative such that $M \cap V = \{x \in V : r(x) = 0\}$. The function r is called the *defining function* (at p).

A domain U with *C^k-smooth boundary* is a domain where ∂U is a C^k-smooth hypersurface, where we further require for a defining function r of ∂U near some p, that $r < 0$ for points in U and $r > 0$ for points not in U.

If we say simply *smooth* we mean C^∞-smooth.

In fact for simplicity in these notes we generally deal with smooth (that is, C^∞) functions and hypersurfaces only. Dealing with C^k-smooth functions for finite k introduces technicalities that make certain theorems and arguments unnecessarily difficult.



Notice that the definition for a smooth boundary is not just that the boundary is a smooth hypersurface, that is not enough. It also says that one side of that hypersurface is in U and one side is not in U. That is because if the derivative of r never vanishes, then r must have different signs on different sides of $\{x : r(x) = 0\}$. The verification of this fact is left to the reader (Hint: look at where the gradient points to).

Same definition works for $\mathbb{C}^n$ where we simply treat $\mathbb{C}^n$ as $\mathbb{R}^{2n}$. For example the ball $\mathbb{B}_n$ is a domain with smooth boundary with defining function $r(z, \bar{z}) = \|z\|^2 - 1$. We can in fact always find a single global defining function, but we have no need of this.

Definition 2.2.2. For any $p \in \mathbb{R}^n$, the set of tangent vectors $T_p\mathbb{R}^n$ is given by

$$T_p\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p \right\}.$$

That is, a vector $X_p \in T_p\mathbb{R}^n$ is an object of the form

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p,$$

for real numbers a_j. An object $\frac{\partial}{\partial x_j}\big|_p$ is a linear functional on the space of smooth functions: when applied to a smooth function g it gives $\frac{\partial g}{\partial x_j}\big|_p$. Therefore X_p is also such a functional.

Let $M \subset \mathbb{R}^n$ be a smooth hypersurface, $p \in M$, and r is the defining function at p, then a vector $X_p \in T_p\mathbb{R}^n$ is tangent to M at p if

$$X_p r = 0, \qquad \text{or in other words} \qquad \sum_{j=1}^{n} a_j \frac{\partial r}{\partial x_j}\Big|_p = 0.$$

The space of tangent vectors to M is written as T_pM. Notice that the space T_pM is an $n-1$ dimensional real vector space. Also notice that the definition is independent of the choice of r by the next two exercises.

> ***Exercise* 2.2.2:** *If r and $\tilde{r}$ are two smooth defining functions for M at p, show that there exists a nonzero smooth function g such that $\tilde{r} = gr$. Hint: First assume that $r = x_n$, so that M is simply the set $x_n = 0$, then show that there exists a g such that $\tilde{r} = x_n g$. Then think about a local change of variables that makes M into x_n. Hint for the hint: Notice the basic calculus fact that if $f(0) = 0$ and f is smooth then $s\int_0^1 f'(ts)\,dt = f(s)$ and $\int_0^1 f'(ts)\,dt$ is a smooth function of s.*
>
> ***Exercise* 2.2.3:** *Show that T_pM is independent of which defining function we take. That is prove that if $\tilde{r}$ is another defining function for M at p, then $\sum a_j \frac{\partial r}{\partial x_j}\big|_p = 0$ if and only if $\sum a_j \frac{\partial \tilde{r}}{\partial x_j}\big|_p = 0$.*

The disjoint union

$$T\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$$

is called the *tangent bundle*. There is a natural identification $\mathbb{R}^n \times \mathbb{R}^n \cong T\mathbb{R}^n$, that is,

$$(p,a) \in \mathbb{R}^n \times \mathbb{R}^n \quad \mapsto \quad \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p \in T\mathbb{R}^n.$$

The topology and smooth structure on $T\mathbb{R}^n$ comes from this identification. The wording "bundle" comes from the natural projection $\pi\colon T\mathbb{R}^n \to \mathbb{R}^n$, where fibers are $\pi^{-1}(p) = T_p\mathbb{R}^n$.

A smooth *vector field* in $T\mathbb{R}^n$ is an object of the form

$$X = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}$$

where a_j are smooth functions. That is, X is a function $X : \mathbb{R}^n \to T\mathbb{R}^n$ such that $X(p) \in T_p\mathbb{R}^n$, and the vectors vary smoothly. Usually we write X_p rather than $X(p)$. To be more fancy we could say X is a *section* of $T\mathbb{R}^n$.

Similarly

$$TM = \bigcup_{p \in M} T_pM$$

is the tangent bundle of M. A vector field X in TM is a vector field such that $X_p \in T_pM$ for all $p \in M$.

Now that we know what tangent vectors are, let us define convexity for domains with smooth boundary.

Definition 2.2.3. Suppose $U \subset \mathbb{R}^n$ is a domain with smooth boundary, and r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U.

If for all nonzero $X_p \in T_p\partial U$,

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p,$$

we have

$$\sum_{j=1,\ell=1}^{n} a_j a_\ell \frac{\partial^2 r}{\partial x_j \partial x_\ell}\Big|_p \geq 0,$$

then U is said to be *convex* at p. If the inequality above is strict for all nonzero $X_p \in T_p\partial U$, then U is said to be *strongly convex* at p.

The domain U is *convex* if it is convex at all $p \in \partial U$. Similarly U is *strongly convex* if it is strongly convex at all $p \in \partial U$.

The matrix

$$\left[\frac{\partial^2 r}{\partial x_j \partial x_\ell}\Big|_p \right]_{j\ell}$$

is called the *the Hessian* of r at p. So, U is convex at $p \in \partial U$ if the Hessian of r at p as a bilinear form is positive definite (or positive semidefinite) when restricted to tangent vectors in $T_p\partial U$. This matrix is essentially the second fundamental form from Riemannian geometry in mild disguise (or perhaps it is the other way around).

Notice that we cheated above a little bit since we have not proved that the notion is well defined. In particular we may have more than one defining function.

***Exercise* 2.2.4:** *Show that the definition of convexity is independent of the defining function. Hint: If $\tilde{r}$ is another defining function near p then there is a function $g > 0$ such that $\tilde{r} = gr$.*

***Exercise* 2.2.5:** *Show that if a domain is strongly convex at a point, then it is strongly convex at all nearby points. On the other hand find an example of a domain that is convex at one point p, but not convex at points arbitrarily near p.*

Example 2.2.4: Let us look at an example. Let us prove that the unit disc in $\mathbb{R}^2$ is convex (actually strongly convex). Let x, y be our coordinates and then our defining function is $r(x,y) = x^2 + y^2 - 1$.

The tangent space to the circle is one dimensional, so we simply need to find a single nonzero tangent vector at each point. Notice that $\nabla r = (2x, 2y)$, so it is easy to check that

$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

is tangent to the circle, that is when $x^2 + y^2 = 1$. It is also nonzero on the circle.

The Hessian matrix of r is

$$\begin{bmatrix} \frac{\partial^2 r}{\partial x^2} & \frac{\partial^2 r}{\partial x \partial y} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial x^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Applying the vector $(y, -x)$ gets us

$$\begin{bmatrix} y & -x \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = 2y^2 + 2x^2 = 2 > 0.$$

So the domain given by $r < 0$ is strongly convex at all points.

***Exercise* 2.2.6:** *Show that the domain in $\mathbb{R}^2$ defined by $x^4 + y^4 < 1$ is convex, but not strongly convex. Find all the points where the domain is not strongly convex.*

***Exercise* 2.2.7:** *Show that the domain in $\mathbb{R}^3$ defined by $(x_1^2 + x_2^2)^2 < x_3$ is strongly convex at all points except the origin, where it is just convex (but not strongly).*

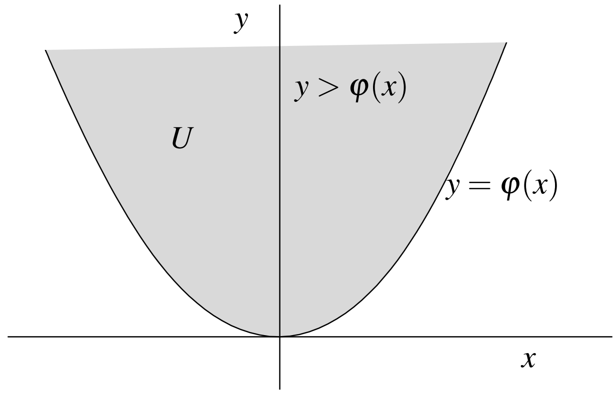
For computations it is often useful to use a more convenient defining function.

Lemma 2.2.5. *Suppose $M \subset \mathbb{R}^n$ is a smooth hypersurface, and $p \in M$. Then after a rotation and translation, p is the origin and near the origin M is defined by*

$$y = \varphi(x)$$

where $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ are our coordinates and φ is a smooth function that is $O(2)$ at the origin, that is $\varphi(0) = 0$ and $d\varphi(0) = 0$.

If M is the boundary of a domain U with smooth boundary and $r < 0$ on U, then the rotation can be chosen such that for points in U we have $y > \varphi(x)$.



Proof. Let r be a defining function at p. Take $v = \nabla r(p)$. By translating p to zero, and applying a rotation (an orthogonal matrix), we assume that $v = (0,0,\dots,0,v_n)$, where $v_n < 0$. Denote our coordinates by $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$. As $\nabla r(0) = v$, then $\frac{\partial r}{\partial y}(0) \neq 0$. We apply the implicit function theorem to find a smooth function φ such that $r\bigl(x,\varphi(x)\bigr) = 0$ for all x in a neighborhood of the origin, and in fact that $\{(x,y) : y = \varphi(x)\}$ are all the solutions to $r = 0$ near the origin.

What is left is to show that the derivative at 0 of φ vanishes. We have $r\bigl(x,\varphi(x)\bigr) = 0$ for all x in a neighborhood of the origin. For any $j = 1,\dots,n-1$,

$$0 = \frac{\partial}{\partial x_j}\Bigl[r\bigl(x,\varphi(x)\bigr)\Bigr] = \left(\sum_{\ell=1}^{n-1} \frac{\partial r}{\partial x_\ell}\frac{\partial x_\ell}{\partial x_j}\right) + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial x_j} = \frac{\partial r}{\partial x_j} + \frac{\partial r}{\partial y}\frac{\partial \varphi}{\partial x_j}$$

At the origin, $\frac{\partial r}{\partial x_j}(0,0) = 0$ and $\frac{\partial r}{\partial y}(0,0) = v_n \neq 0$ and therefore $\frac{\partial \varphi}{\partial x_j}(0) = 0$.

To prove the final statement suppose $r < 0$ on the domain. It is enough to check that r is negative for $(0,y)$ if $y > 0$ is small, which follows as $\frac{\partial r}{\partial y}(0,0) < 0$. $\square$

The advantage of this representation is that the tangent plane at p can be identified with the x coordinates for the purposes of computation. Let M be smooth for simplicity. We can write the hypersurface as

$$y = \frac{1}{2}x^t H x + E(x)$$

where H is the Hessian matrix of φ at the origin, that is $H = \left[\frac{\partial^2 \varphi}{\partial x_j \partial x_k}\Big|_0\right]_{jk}$, and E is $O(3)$. That is, $E(0) = 0$, and all first and second derivatives of E vanish. If we are dealing with a domain boundary ∂U, then we pick the rotation so that $y > \frac{1}{2}x^t H x + E(x)$ on U. It is an easy exercise to see that U is convex at p if H positive semidefinite and strongly convex if H is positive definite.

***Exercise* 2.2.8:** *Prove the above statement about H and convexity at p.*

Exercise* 2.2.9*: *M is convex from both sides at p if and only if for a defining function r for M at p, both the set given by $r > 0$ and the set given by $r < 0$ are convex at p. Prove that if a hypersurface $M \subset \mathbb{R}^n$ is convex from both sides at all points then it is locally just a hyperplane (the zero set of a real affine function).*

There is also a geometric notion of convexity, that is, U is *geometrically convex* if for every $p, q \in U$ the line between p and q is in U, or in other words $tp + (1-t)q \in U$ for all $t \in [0,1]$.

Exercise* 2.2.10*: *Suppose a domain with smooth boundary is geometrically convex. Show that it is convex.*

The other direction is considerably more complicated, and we will not worry about it here. Similar difficulties will be present once we move back to several complex variables and try to relate pseudoconvexity with domains of holomorphy.

2.3 Holomorphic vectors, the Levi-form, and pseudoconvexity

As $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$ using $z = x + iy$, we have $T_p\mathbb{C}^n = T_p\mathbb{R}^{2n}$. We write

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial y_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial y_n}\Big|_p \right\}.$$

That is, we simply replace all the real coefficients with complex ones. The space $\mathbb{C} \otimes T_p\mathbb{C}^n$ is a $2n$ dimensional complex vector space. Once we do that we notice that $\frac{\partial}{\partial z_j}\big|_p$, and $\frac{\partial}{\partial \bar{z}_j}\big|_p$ are both in $\mathbb{C} \otimes T_p\mathbb{C}^n$, and in fact:

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

Define

$$T_p^{(1,0)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \ldots, \frac{\partial}{\partial z_n}\Big|_p \right\} \qquad \text{and} \qquad T_p^{(0,1)}\mathbb{C}^n \overset{\text{def}}{=} \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_p, \ldots, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

The vectors in $T_p^{(1,0)}\mathbb{C}^n$ are the *holomorphic vectors* and vectors in $T_p^{(0,1)}\mathbb{C}^n$ are the *antiholomorphic vectors*. We decompose the full tangent space as

$$\mathbb{C} \otimes T_p\mathbb{C}^n = T_p^{(1,0)}\mathbb{C}^n \oplus T_p^{(0,1)}\mathbb{C}^n.$$

A holomorphic function is one that vanishes on $T_p^{(0,1)}\mathbb{C}^n$.

Let us note what holomorphic functions do to these spaces. Given a smooth mapping f from $\mathbb{C}^n$ to $\mathbb{C}^m$, its derivative at $p \in \mathbb{C}^n$ is a real-linear mapping $D_{\mathbb{R}} f(p) \colon T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$. Given the basis above, this mapping is represented by the standard real Jacobian matrix, that is a real $2m$ by $2n$ matrix which we before also wrote as $D_{\mathbb{R}} f(p)$.

Proposition 2.3.1. *Let $f \colon U \subset \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic function with $p \in U$. Suppose $D_{\mathbb{R}} f(p) \colon T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$ is the real derivative of f at p. Then we naturally extend the derivative to $D_{\mathbb{C}} f(p) \colon \mathbb{C} \otimes T_p\mathbb{C}^n \to \mathbb{C} \otimes T_{f(p)}\mathbb{C}^m$. Then*

$$D_{\mathbb{C}} f(p)\bigl(T_p^{(1,0)}\mathbb{C}^n\bigr) \subset T_{f(p)}^{(1,0)}\mathbb{C}^m \qquad \text{and} \qquad D_{\mathbb{C}} f(p)\bigl(T_p^{(0,1)}\mathbb{C}^n\bigr) \subset T_{f(p)}^{(0,1)}\mathbb{C}^m.$$

If f is a biholomorphism, then $D_{\mathbb{C}} f(p)$ restricted to $T_p^{(1,0)}\mathbb{C}^n$ is a vector space isomorphism. Similarly for $T_p^{(0,1)}\mathbb{C}^n$.

Exercise* 2.3.1*: *Prove the proposition. Hint: First start with $D_{\mathbb{R}} f(p)$ as a real $2m \times 2n$ matrix to show it extends (it is the same matrix if you think of it as a matrix). Think of $\mathbb{C}^n$ and $\mathbb{C}^m$ in terms of the zs and the $\bar{z}$s and think of f as a mapping*

$$(z,\bar{z}) \mapsto \bigl(f(z), \bar{f}(\bar{z})\bigr).$$

Write the derivative as a matrix in terms of the zs and the $\bar{z}$s and fs and $\bar{f}$s and the result will follow. That is just changing the basis.

When talking about only holomorphic functions and holomorphic vectors, when we say derivative of f, we will mean the holomorphic part of the derivative which we write as,

$$Df(p) \colon T_p^{(1,0)}\mathbb{C}^n \to T_{f(p)}^{(1,0)}\mathbb{C}^m.$$

That is, $Df(p)$ is the restriction of $D_{\mathbb{C}} f(p)$ to $T_p^{(1,0)}\mathbb{C}^n$. In other words, if we have specific coordinates in mind, the holomorphic derivative of $f \colon \mathbb{C}^n \to \mathbb{C}^m$ can be represented as the $m \times n$ Jacobian matrix

$$\left[\frac{\partial f_j}{\partial z_k}\right]_{jk},$$

that we have seen before and that we have before also represented by $Df(p)$.

Similarly as before we define the tangent bundles

$$\mathbb{C} \otimes T\mathbb{C}^n, \quad T^{(1,0)}\mathbb{C}^n, \quad \text{and} \quad T^{(0,1)}\mathbb{C}^n,$$

by taking the disjoint unions, and we have vector fields in these bundles.

Given a real smooth hypersurface $M \subset \mathbb{C}^n$ we can take $\mathbb{C} \otimes T_pM$. Let r be a real-valued defining function of M at p. A vector $X_p \in \mathbb{C} \otimes T_pM$ is a vector in $\mathbb{C} \otimes T_p\mathbb{C}^n$ such that $X_p r = 0$ at p. That is, write

$$X_p = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial z_j}\Big|_p + b_j \frac{\partial}{\partial \bar{z}_j}\Big|_p \right),$$

then $X_p \in \mathbb{C} \otimes T_pM$ if

$$\sum_{j=1}^{n} \left(a_j \frac{\partial r}{\partial z_j}\Big|_p + b_j \frac{\partial r}{\partial \bar{z}_j}\Big|_p \right) = 0,$$

for a defining function r of M at p. Therefore, $\mathbb{C} \otimes T_pM$ is a $2n-1$ dimensional complex vector space. We decompose $\mathbb{C} \otimes T_pM$ as

$$\mathbb{C} \otimes T_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M \oplus B_p,$$

where

$$T_p^{(1,0)}M \overset{\text{def}}{=} \big(\mathbb{C} \otimes T_pM\big) \cap \big(T_p^{(1,0)}\mathbb{C}^n\big), \qquad \text{and} \qquad T_p^{(0,1)}M \overset{\text{def}}{=} \big(\mathbb{C} \otimes T_pM\big) \cap \big(T_p^{(0,1)}\mathbb{C}^n\big).$$

The B_p is just the “left-over” and we really need to include it otherwise even the dimensions will not work out.

Do make sure that you understand what all the objects are. The space T_pM is a real vector space; $\mathbb{C} \otimes T_pM$, $T_p^{(1,0)}M$, $T_p^{(0,1)}M$, and B_p are complex vector spaces. Before we have that these are all vector bundles, we must have that their dimensions do not vary from point to point. The easiest way to see this fact is to write down convenient local coordinates. First, let us see what a biholomorphic map does to the holomorphic and antiholomorphic vectors. A biholomorphic map f is a diffeomorphism. And if a real hypersurface M is near p defined by a function r, then the image $f(M)$ is near $f(p)$ defined by $r \circ f^{-1}$.

Proposition 2.3.2. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$, and $U \subset \mathbb{C}^n$ is a neighborhood of p. Let $f\colon U \to \mathbb{C}^n$ be a biholomorphic map. Let $D_{\mathbb{C}} f(p)$ be the complexified real derivative as before. Then*

$$D_{\mathbb{C}} f(p)\big(T_p^{(1,0)}M\big) = T_{f(p)}^{(1,0)} f(M), \qquad D_{\mathbb{C}} f(p)\big(T_p^{(0,1)}M\big) = T_{f(p)}^{(0,1)} f(M).$$

That is, the spaces are isomorphic as complex vector spaces.

Proof. Without loss of generality assume that $M \subset U$. The proof is simply the application of Proposition 2.3.1. We have

$$D_{\mathbb{C}} f(p)\big(T_p^{(1,0)}\mathbb{C}^n\big) = T_{f(p)}^{(1,0)}\mathbb{C}^n, \quad D_{\mathbb{C}} f(p)\big(T_p^{(0,1)}\mathbb{C}^n\big) = T_{f(p)}^{(0,1)}\mathbb{C}^n, \quad \text{and}$$
$$D_{\mathbb{C}} f(p)\big(\mathbb{C} \otimes T_pM\big) = \mathbb{C} \otimes T_{f(p)} f(M).$$

Then it is clear that $D_{\mathbb{C}} f(p)$ must take $T_p^{(1,0)}M$ to $T_{f(p)}^{(1,0)} f(M)$ and $T_p^{(0,1)}M$ to $T_{f(p)}^{(0,1)} f(M)$. $\square$

In the next proposition it is important to note that a translation and applying a unitary matrix are biholomorphic changes of coordinates.

Proposition 2.3.3. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$. After a translation and rotation via a unitary matrix, $p = 0$ and near the origin M is written in variables $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ as*

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

with the $\varphi(0)$ and $d\varphi(0) = 0$. Consequently

$$\begin{aligned} T_0^{(1,0)} M &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_0, \ldots, \frac{\partial}{\partial z_{n-1}}\Big|_0 \right\}, \\ T_0^{(0,1)} M &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_0, \ldots, \frac{\partial}{\partial \bar{z}_{n-1}}\Big|_0 \right\}, \\ B_0 &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial (\operatorname{Re} w)}\Big|_0 \right\}. \end{aligned}$$

In particular, $\dim_{\mathbb{C}} T_p^{(1,0)} M = \dim_{\mathbb{C}} T_p^{(0,1)} M = n-1$ *and* $\dim_{\mathbb{C}} B_p = 1$.

If M is the boundary of a domain U with smooth boundary, the rotation can be chosen so that $\operatorname{Im} w > \varphi(z, \bar{z}, \operatorname{Re} w)$ *on U.*

Proof. We apply a translation to put $p = 0$ and in the same manner as in Lemma 2.2.5 apply a unitary matrix to make sure that ∇r is in the direction $-\frac{\partial}{\partial (\operatorname{Im} w)}\big|_0$. That $\varphi(0) = 0$ and $d\varphi(0) = 0$ follows as before. As a translation and a unitary matrix are holomorphic and in fact biholomorphic, then via Proposition 2.3.1 we obtain that the tangent spaces are all transformed correctly.

The rest of the proposition follows at once as $\frac{\partial}{\partial (\operatorname{Im} w)}\big|_0$ is the normal vector to M at 0. □

Remark 2.3.4. When M is of smaller dimension than $2n-1$ (no longer a hypersurface, but a higher codimension submanifold), then the proposition above does not hold. That is, we would still have $\dim_{\mathbb{C}} T_p^{(1,0)} M = \dim_{\mathbb{C}} T_p^{(0,1)} M$, but this number need not be constant from point to point. Fortunately, when talking about smoothly bounded domains where the boundaries are hypersurfaces, this complication does not arise.

Definition 2.3.5. Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary, and suppose r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U.

If for all nonzero $X_p \in T_p^{(1,0)} \partial U$,

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}\Big|_p,$$

we have

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \geq 0,$$

then U is said to be *pseudoconvex* at p (or *Levi pseudoconvex*). If the inequality above is strict for all nonzero $X_p \in T_p^{(1,0)} \partial U$, then U is said to be *strongly pseudoconvex*. If U is pseudoconvex, but not strongly pseudoconvex at p, then we say that U is *weakly pseudoconvex*.

The domain U is pseudoconvex if it is pseudoconvex at all $p \in \partial U$. Similarly U is strongly pseudoconvex if it is strongly pseudoconvex at all $p \in \partial U$.

For $X_p \in T_p^{(1,0)} \partial U$, the expression

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p$$

is called the *Levi-form* at p. So U is pseudoconvex at $p \in \partial U$ if the Levi-form is positive (semi)definite at p.

The matrix

$$\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \right]_{j\ell}$$

is called the *the complex Hessian* of r at p. So, U is pseudoconvex at $p \in \partial U$ if the Hessian of r at p as a sesquilinear form is positive definite (or positive semidefinite) when restricted to tangent vectors in $T_p^{(1,0)} \partial U$.

Notice that the complex Hessian is not the full Hessian. Let us write down the full Hessian, using the basis of $\frac{\partial}{\partial z}$s and $\frac{\partial}{\partial \bar{z}}$s. Then the full Hessian is the symmetric matrix

$$\begin{bmatrix}
\frac{\partial^2 r}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial z_n} & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_1 \partial \bar{z}_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial z_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial z_n} & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n} \\
\frac{\partial^2 r}{\partial \bar{z}_1 \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_1 \partial \bar{z}_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 r}{\partial \bar{z}_n \partial z_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n} & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 r}{\partial \bar{z}_n \partial \bar{z}_n}
\end{bmatrix}.$$

So the complex Hessian is the lower left, or the transpose of the upper right, block. In particular it is a smaller matrix, and we also apply it only to a subspace of the complexified tangent space.

Let us illustrate the change of basis on one dimension. The change of variables is left to student for higher dimensions. Therefore let $z = x + iy$ be in $\mathbb{C}$:

$$T = \begin{bmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{bmatrix}, \qquad T^t \begin{bmatrix} \frac{\partial^2 r}{\partial x \partial x} & \frac{\partial^2 r}{\partial x \partial y} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial y \partial y} \end{bmatrix} T = \begin{bmatrix} \frac{\partial^2 r}{\partial z \partial z} & \frac{\partial^2 r}{\partial z \partial \bar{z}} \\ \frac{\partial^2 r}{\partial \bar{z} \partial z} & \frac{\partial^2 r}{\partial \bar{z} \partial \bar{z}} \end{bmatrix}.$$

Let us also mention how a complex linear change of variables acts on the Hessian matrix. A complex linear change of variables is not an arbitrary $2n \times 2n$ matrix. If the Hessian is in the basis of $\frac{\partial}{\partial z}$s and $\frac{\partial}{\partial \bar{z}}$s, an $n \times n$ complex linear matrix A acts on the Hessian as $A \oplus \overline{A}$, that is $\left[\begin{smallmatrix} A & 0 \\ 0 & \overline{A} \end{smallmatrix}\right]$. Write the real Hessian as $\left[\begin{smallmatrix} X & L^t \\ L & \overline{X} \end{smallmatrix}\right]$, where L is the complex Hessian. Then the complex linear change of variables A transforms the Hessian as

$$\begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}^t \begin{bmatrix} X & L^t \\ L & \overline{X} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} = \begin{bmatrix} A^tXA & (A^*LA)^t \\ A^*LA & \overline{A^tXA} \end{bmatrix},$$

where $A^* = \overline{A}^t$ is the conjugate transpose of A. In other words if H is the ma If we let L be the complex Hessian, then we see that A transforms L as A^*LA, that is, by $*$-congruence.

Exercise* 2.3.2*: *If r is real-valued, then the complex Hessian of r is Hermitian, that is, the matrix is equal to its conjugate transpose.*

Exercise* 2.3.3*: *Show that pseudoconvexity is not dependent on the defining function.*

Exercise* 2.3.4*: *Show that a convex domain is pseudoconvex, and show that strongly convex domain is strongly pseudoconvex.*

Exercise* 2.3.5*: *Show that if a domain is strongly pseudoconvex at a point, it is strongly pseudoconvex at all nearby points.*

In particular the exercise says that the unit ball $\mathbb{B}_n$ is strongly pseudoconvex as it is strongly convex. We are generally interested what happens under a holomorphic change of variables, that is, a biholomorphic mapping. And as far as pseudoconvexity is concerned we are interested in local changes of coordinates as pseudoconvexity is a local property.

Example 2.3.6: Let us change variables to show how we write $\mathbb{B}_n$ in different local holomorphic coordinates where the Levi-form is displayed nicely. Let $\mathbb{B}_n$ be defined in the variables $Z = (Z_1, \ldots, Z_n) \in \mathbb{C}^n$ by $\|Z\| = 1$.

Let us change variables to $(z_1, \ldots, z_{n-1}, w)$ where

$$z_j = \frac{Z_1}{1-Z_n} \quad \text{for all } j = 1, \ldots, n-1, \qquad w = i\frac{1+Z_n}{1-Z_n}.$$

This change of variables is a biholomorphic mapping from the set where $Z_n \neq 1$ to the set where $w \neq -i$ (exercise). For us it is sufficient to notice that the map is invertible near $(0, \ldots, 0, -1)$, which follows by simply computing the derivative. Notice that the last component is the inverse of the Cayley transform (that takes the disc to the upper half plane).

We claim that the mapping takes the unit sphere given by $\|Z\| = 1$, to the set defined by

$$\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2,$$

and that it takes $(0,\ldots,0,-1)$ to the origin (this part is trivial). Let us check:

$$
\begin{aligned}
|z_1|^2 + \cdots + |z_{n-1}|^2 - \operatorname{Im} w &= \left| \frac{Z_1}{1-Z_n} \right|^2 + \cdots + \left| \frac{Z_{n-1}}{1-Z_n} \right|^2 - \frac{i\frac{1+Z_n}{1-Z_n} - \overline{i\frac{1+Z_n}{1-Z_n}}}{2i} \\
&= \frac{|Z_1|^2}{|1-Z_n|^2} + \cdots + \frac{|Z_{n-1}|^2}{|1-Z_n|^2} - \frac{1+Z_n}{2(1-Z_n)} - \frac{1+\bar{Z}_n}{2(1-\bar{Z}_n)} \\
&= \frac{|Z_1|^2 + \cdots + |Z_{n-1}|^2 + |Z_n|^2 - 1}{|1-Z_n|^2}.
\end{aligned}
$$

Therefore $|Z_1|^2 + \cdots + |Z_n|^2 = 1$ if and only if $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$. As the map takes the point $(0,\ldots,0,-1)$ to the origin, we can think of

$$\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$$

as the local holomorphic coordinates at $(0,\ldots,0,-1)$ (by symmetry of the sphere we could have done this at any point by rotation). The inside of the sphere is taken to

$$\operatorname{Im} w > |z_1|^2 + \cdots + |z_{n-1}|^2.$$

In these new coordinates, the Levi-form is just the identity matrix at the origin. In particular the domain is strictly pseudoconvex. We have not yet proved that pseudoconvexity is a biholomorphic invariant, but when we do, it will also mean that the ball is strictly pseudoconvex.

Of course not the entire sphere gets transformed, the points where $Z_n = 1$ get sent to infinity. The hyper surface $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$ is sometimes called the *Lewy hypersurface*, and in the literature some even say it *is* the sphere*.

As an aside, the hypersurface $\operatorname{Im} w = |z_1|^2 + \cdots + |z_{n-1}|^2$ is also often called the *Heisenberg group*. The group in this case is the group defined on the parameters $(z, \operatorname{Re} w)$ of this manifold with the group law defined by $(z, \operatorname{Re} w)(z', \operatorname{Re} w') = (z + z', \operatorname{Re} w + \operatorname{Re} w' + 2 \operatorname{Im} z \cdot z')$.

> ***Exercise* 2.3.6*:*** *Prove the assertion in the example about the mapping being biholomorphic on the sets described above.*

Let us compute what happens to the Hessian of r under a biholomorphic change of coordinates. That is, let $f \colon U \to V$ be a biholomorphic map between two domains in $\mathbb{C}^n$, and let $r \colon V \to \mathbb{R}$ be a smooth function with nonvanishing derivative. Let us compute the Hessian of $r \circ f$. Let us compute first what happens to the non-mixed derivatives. As we have to apply chain rule twice let us write the derivatives as functions. That is, r is a function of z and $\bar{z}$, f is a function of z, and $\bar{f}$ is a function of $\bar{z}$.

*That is not in fact completely incorrect. If you think of the sphere in the complex projective space, we have simply looked a the sphere in a different coordinate patch.

$$
\begin{aligned}
\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k}(z,\bar{z}) &= \frac{\partial}{\partial z_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \frac{\partial f_\ell}{\partial z_k}(z) + \frac{\partial r}{\partial \bar{z}_\ell}\big(f(z),\bar{f}(\bar{z})\big) \cancelto{0}{\frac{\partial \bar{f}_\ell}{\partial z_k}(\bar{z})} \right) \\
&= \sum_{\ell,m=1}^{n} \left(\frac{\partial^2 r}{\partial z_m \partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \frac{\partial f_m}{\partial z_j}(z) \frac{\partial f_\ell}{\partial z_k}(z) + \frac{\partial^2 r}{\partial \bar{z}_m \partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \cancelto{0}{\frac{\partial \bar{f}_m}{\partial z_j}(\bar{z})} \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&\qquad + \sum_{\ell=1}^{n} \frac{\partial r}{\partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \frac{\partial^2 f_\ell}{\partial z_j \partial z_k}(z) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial z_m \partial z_\ell} \frac{\partial f_m}{\partial z_j} \frac{\partial f_\ell}{\partial z_k} + \sum_{\ell=1}^{n} \frac{\partial r}{\partial z_\ell} \frac{\partial^2 f_\ell}{\partial z_j \partial z_k}.
\end{aligned}
\tag{2.1}
$$

In particular, the matrix $\left[\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}\right]$ can have different eigenvalues than the matrix $\left[\frac{\partial^2 r}{\partial z_j \partial z_k}\right]$. In fact if r has nonvanishing gradient, then using the second term, we can (locally) choose f in such a way as to make the matrix $\left[\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}\right]$ be the zero matrix at a certain point since we can just choose the second derivatives of f arbitrarily at a point. See the exercise below. So nothing about the matrix $\left[\frac{\partial^2 r}{\partial z_j \partial z_k}\right]$ is preserved under a biholomorphic map. And that is precisely why it does not appear in the definition of pseudoconvexity. The story for $\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial \bar{z}_k}\right]$ is exactly the same.

Exercise 2.3.7: *Given a real function r with nonvanishing gradient at $p \in \mathbb{C}^n$. Find a local change of coordinates f at p (so f ought to be a holomorphic mapping with an invertible derivative at p) such that $\left[\frac{\partial^2 (r\circ f)}{\partial z_j \partial z_k}\Big|_p\right]$ and $\left[\frac{\partial^2 (r\circ f)}{\partial \bar{z}_j \partial \bar{z}_k}\Big|_p\right]$ are just the zero matrices.*

Let us look at the mixed derivatives:

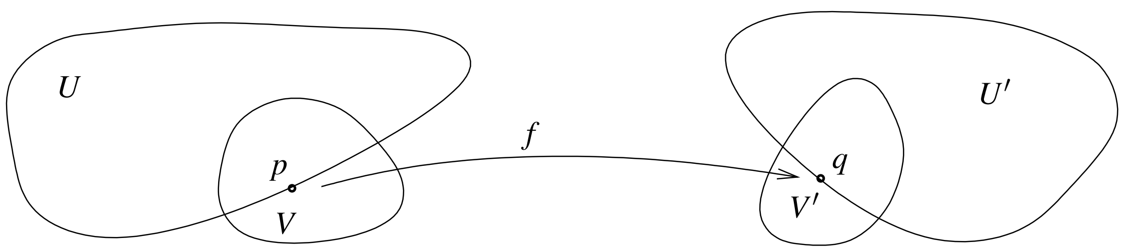
$$
\begin{aligned}
\frac{\partial^2 (r \circ f)}{\partial \bar{z}_j \partial z_k}(z,\bar{z}) &= \frac{\partial}{\partial \bar{z}_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{z}_m \partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \frac{\partial \bar{f}_m}{\partial \bar{z}_j}(\bar{z}) \frac{\partial f_\ell}{\partial z_k}(z) + \sum_{\ell=1}^{n} \frac{\partial r}{\partial z_\ell}\big(f(z),\bar{f}(\bar{z})\big) \cancelto{0}{\frac{\partial^2 f_\ell}{\partial \bar{z}_j \partial z_k}(z)} \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{z}_m \partial z_\ell} \frac{\partial \bar{f}_m}{\partial \bar{z}_j} \frac{\partial f_\ell}{\partial z_k}.
\end{aligned}
$$

The complex Hessian of $r \circ f$ is simply the complex Hessian H of r conjugated as D^*HD where D is the holomorphic derivative matrix of f at z and D^* is the conjugate transpose. Sylvester's Law of Inertia from linear algebra then says that the number of positive, negative, and zero eigenvalues of D^*HD is the same as that for H. The eigenvalues might have changed, but their sign did not.

In particular if H is positive definite, then D^*HD is positive definite. If a smooth hypersurface M is given by $r = 0$, then $f^{-1}(M)$ is a smooth hypersurface given by $r \circ f = 0$. The holomorphic derivative of f (given by D) takes the $T_z^{(1,0)} f^{-1}(M)$ space isomorphically to $T_{f(z)}^{(1,0)} M$. So H is positive semidefinite (resp. positive definite) on $T_{f(z)}^{(1,0)} M$ if and only if D^*HD is positive semidefinite (resp. positive definite) on $T_z^{(1,0)} f^{-1}(M)$. We have essentially proved the following theorem. That is, pseudoconvexity is a biholomorphic invariant.

Theorem 2.3.7. *Suppose $U, U' \subset \mathbb{C}^n$ are domains with smooth boundary, $p \in \partial U$, $V \subset \mathbb{C}^n$ a neighborhood of p, $q \in \partial U'$, $V' \subset \mathbb{C}^n$ a neighborhood of q, and $f \colon V \to V'$ a biholomorphic map with $f(p) = q$, such that $f(U \cap V) = U' \cap V'$.*

Then U is pseudoconvex at p if and only if U' is pseudoconvex at q. Similarly U is strongly pseudoconvex at p if and only if U' is strongly pseudoconvex at q.



The only thing left is to observe that if $f(U \cap V) = U' \cap V'$ then $f(\partial U \cap V) = \partial U' \cap V'$.

***Exercise* 2.3.8:** *Find an example of a bounded domain with smooth boundary that is not convex, but that is pseudoconvex.*

In fact we proved a stronger result. We proved that the inertia of the Levi-form is invariant under a biholomorphic change of coordinates. Let us put this together with the other observations we made above. We find the normal form for the quadratic part of the defining equation for a smooth real hypersurface under biholomorphic transformations. It is possible to do better than the following lemma, but it is not possible to get rid of the dependence on $\operatorname{Re} w$ except in the quadratic terms.

We will use the *big-oh notation*. A smooth function is $O(k)$ (or "$O(k)$ at the origin" to be precise), if the function and its first, second, …, $(k-1)$th derivatives vanish at the origin. For example, if E is $O(3)$ at the origin then $E(0) = 0$, and its first and second derivatives vanish. Often to simplify notation we write $O(3)$ instead of E to denote any function in $O(3)$ to simplify notation.

Lemma 2.3.8. *Let M be a smooth real hypersurface in $\mathbb{C}^n$ and $p \in M$. Then there exists a local holomorphic change of coordinates taking p to the origin and M to the manifold given by*

$$\operatorname{Im} w = \sum_{j=1}^{\alpha} |z_j|^2 - \sum_{j=\alpha+1}^{\alpha+\beta} |z_j|^2 + E(z,\bar{z},\operatorname{Re} w),$$

where E is $O(3)$ at the origin. Here α is the number of positive eigenvalues of the Levi-form at p, β is the number of negative eigenvalues, and $\alpha+\beta \leq n-1$.

Proof. Change coordinates so that M is given by $\operatorname{Im} w = \varphi(z,\bar{z},\operatorname{Re} w)$, where φ is $O(2)$. Apply Taylor's theorem to φ up to the second order:

$$\varphi(z,\bar{z},\operatorname{Re} w) = q(z,\bar{z}) + (\operatorname{Re} w)(Lz + \overline{Lz}) + a(\operatorname{Re} w)^2 + O(3),$$

where q is quadratic, $L\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is linear, and $a \in \mathbb{R}$. If $L \neq 0$, do a linear change of coordinates in the z only to make $Lz = z_1$. So assume $Lz = \varepsilon z_1$ where $\varepsilon = 0$ or $\varepsilon = 1$.

Change variables by letting $w = w' + bw'^2 + cw'z_1$. Let us ignore $q(z,\bar{z})$ for a moment as this change of coordinates does not affect it. Also let us only look up to second order.

$$\begin{aligned}
-\operatorname{Im} w + \varepsilon(\operatorname{Re} w)(z_1+\bar{z}_1) + a(\operatorname{Re} w)^2 &= -\frac{w-\bar{w}}{2i} + \varepsilon\frac{w+\bar{w}}{2}(z_1+\bar{z}_1) + a\left(\frac{w+\bar{w}}{2}\right)^2 \\
&= -\frac{w'+bw'^2+cw'z_1-\bar{w}'-\bar{b}\bar{w}'^2-\bar{c}\bar{w}'\bar{z}_1}{2i} \\
&\quad + \varepsilon\frac{w'+bw'^2+cw'z_1+\bar{w}'+\bar{b}\bar{w}'^2+\bar{c}\bar{w}'\bar{z}_1}{2}(z_1+\bar{z}_1) \\
&\quad + \frac{(w'+bw'^2+cw'z_1+\bar{w}'+\bar{b}\bar{w}'^2+\bar{c}\bar{w}'\bar{z}_1)^2}{4} \\
&= -\frac{w'-\bar{w}'}{2i} \\
&\quad + \frac{\big((\varepsilon i - c)w' + \varepsilon i\bar{w}'\big)z_1 + \big((\varepsilon i+\bar{c})\bar{w}' + \varepsilon i w'\big)\bar{z}_1}{2i} \\
&\quad + \frac{(ai-2b)w'^2 + (ai+2\bar{b})\bar{w}'^2 + 2iaw'\bar{w}'}{4i} + O(3).
\end{aligned}$$

We cannot quite get rid of all the quadratic terms in this equation, but we can set b and c to make the second order terms not depend on $\operatorname{Re} w'$. Setting $b = ai$ and $c = 2\varepsilon i$, and adding $q(z,\bar{z}) + O(3)$ into the mix we obtain

$$\begin{aligned}
&-\operatorname{Im} w + q(z,\bar{z}) + \varepsilon(\operatorname{Re} w)(z_1+\bar{z}_1) + a(\operatorname{Re} w)^2 + O(3) \\
&\quad = -\frac{w'-\bar{w}'}{2i} + q(z,\bar{z}) - \varepsilon i\frac{w'-\bar{w}'}{2i}(z_1-\bar{z}_1) + a\left(\frac{w'-\bar{w}'}{2i}\right)^2 + O(3) \\
&\quad = -\operatorname{Im} w' + q(z,\bar{z}) - \varepsilon i(\operatorname{Im} w')(z_1-\bar{z}_1) + a(\operatorname{Im} w')^2 + O(3).
\end{aligned}$$

Now the right hand side depends on $\operatorname{Im} w'$ so we have to apply the implicit function theorem to write the hypersurface as a graph again. We must solve for $\operatorname{Im} w'$. The expression for $\operatorname{Im} w'$ is $O(2)$, and therefore $-i\varepsilon(\operatorname{Im} w')(z_1 - \bar{z}_1)$ and $a(\operatorname{Im} w')^2$ are (at least) $O(3)$. Therefore we write M as a graph:

$$\operatorname{Im} w' = q(z,\bar{z}) + E(z,\bar{z},\operatorname{Re} w'),$$

where E is $O(3)$.

Next we apply the computation in (2.1). We again change variables in the w', that is, we fix the zs and we set $w' = w'' + g(z)$, where g is $O(2)$. That is, the biholomorphic mapping is $f_j(z,w'') = z_j$ and $f_n(z,w'') = w'' + g(z)$. We let $r = -\operatorname{Im} w' + q(z,\bar{z}) + E(z,\bar{z},\operatorname{Re} w')$, so r is a function of $(z_1,\ldots,z_{n-1},w')$ and f and $(r \circ f)$ are functions of $(z_1,\ldots,z_{n-1},w'')$

The only holomorphic derivative of r that does not vanish at the origin is the w' derivative. Also the second order derivatives of r involving w' or $\bar{w}'$ all vanish at the origin. Using (2.1) at the origin for $j,k = 1,\ldots,n-1$ we get

$$\frac{\partial^2 (r \circ f)}{\partial z_j \partial z_k}\Big|_0 = \sum_{\ell,m=1}^{n-1} \frac{\partial^2 r}{\partial z_m \partial z_\ell}\Big|_0 \delta_m^j \delta_\ell^k + \frac{\partial r}{\partial w'}\Big|_0 \frac{\partial^2 g}{\partial z_j \partial z_k}\Big|_0 = \frac{\partial^2 q}{\partial z_j \partial z_k}\Big|_0 + \frac{1}{2i}\frac{\partial^2 g}{\partial z_j \partial z_k}\Big|_0.$$

Where δ_j^k is the Kronecker delta, that is, $\delta_j^j = 1$, and $\delta_j^k = 0$ if $j \neq k$. Notice the q on the right hand side. Pick the $z_j z_k$ coefficient in g such that $\frac{\partial^2 g}{\partial z_k \partial z_j}\big|_0 = \frac{-1}{2i}\frac{\partial^2 q}{\partial z_k \partial z_j}\big|_0$ making the expression vanish. The left hand side of the equation are the coefficients of quadratic the holomorphic terms in z of $r \circ f$, that is, the holomorphic terms of the new q. This change of coordinates sets all the holomorphic terms of q to zero. It is left as an exercise that as q is real-valued, the coefficient of $\bar{z}_j \bar{z}_k$ in q also becomes zero. Therefore, after this change of coordinates we may assume

$$q(z,\bar{z}) = \sum_{j,k=1}^{n-1} c_{jk} z_j \bar{z}_k.$$

That is, q is a sesquilinear form. Since q is real-valued the matrix $C = [c_{jk}]$ must be Hermitian. In linear algebra notation, $q(z,\bar{z}) = z^* C z$, where the $*$ denotes the conjugate transpose, and we think of z as a column vector. If T is a linear transformation on the z variables we obtain $(Tz)^* C T z = z^*(T^* C T) z$. Thus, we normalize C up to $*$-congruence. A Hermitian matrix is $*$-congruent to a diagonal matrix with only 1s, -1s, and 0s on the diagonal. Writing out what that means is precisely the conclusion of the proposition. □

Lemma 2.3.9 (Narasimhan's lemma*). *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary. If U is strongly pseudoconvex at $p \in \partial U$, then there exists a local holomorphic change of coordinates fixing p such that in these new coordinates U is strongly convex in a neighborhood of p.*

---

*A statement essentially of Narasimhan's lemma was already used by Helmut Knesser in 1936.

***Exercise* 2.3.9:** *Prove the above lemma. Hint: See the proof of Lemma 2.3.8.*

The Narasimhan lemma only works at points of strong pseudoconvexity. For weakly pseudoconvex points the situation is far more complicated.

Let us prove an easy direction of the famous *Levi-problem*. The Levi-problem was a long standing problem* in several complex variables to classify domains of holomorphy in $\mathbb{C}^n$. The answer is that a domain is a domain of holomorphy if and only if it is pseudoconvex. Just as the problem of trying to show that the classical geometric convexity is the same as convexity as we have defined it, the Levi-problem has an easier direction and a harder direction. The easier direction is to show that a domain of holomorphy is pseudoconvex, and the harder direction is to show that a pseudoconvex domain is a domain of holomorphy.

Theorem 2.3.10 (Tomato can principle)**.** *If $U \subset \mathbb{C}^n$ is a smoothly bounded domain and at some point $p \in \partial U$, the Levi-form has a negative eigenvalue, then U is not a domain of holomorphy. In particular, every holomorphic function on U extends to a neighborhood of p.*

Pseudoconvex at p means that all eigenvalues of the Levi-form are nonnegative. Therefore a domain of holomorphy must be pseudoconvex. The naming comes from the proof, and sometimes other theorems which use a similar proof using a "tomato can" of analytic discs are also called tomato can principle. The general statement of the principle is that "an analytic function holomorphic in a neighborhood of the sides and the bottom of a tomato can extends to the inside."

Proof. Applying what we know, we change variables so that $p = 0$, and near p, U is given by

$$\operatorname{Im} w > -|z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w),$$

where $\varepsilon_j = -1, 0, 1$, E is $O(3)$, and $z' = (z_2, \ldots, z_{n-1})$. We embed an analytic disc via $\xi \overset{\varphi}{\mapsto} (\lambda \xi, 0, 0, \ldots, 0)$ for some small $\lambda > 0$. Clearly $\varphi(0) = 0 \in \partial U$. For $\xi \neq 0$ near the origin

$$-\lambda^2 |\xi|^2 + \sum_{j=2}^{n-1} \varepsilon_j |0|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) = -\lambda^2 |\xi|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) < 0.$$

That is because by second derivative test the function above has a strict minimum at $\xi = 0$. Therefore for ξ near the origin, but not zero, we have $\varphi(\xi) \in U$. By picking λ small enough, we have $\varphi(\overline{\mathbb{D}} \setminus \{0\}) \subset U$.

As $\varphi(\partial \mathbb{D})$ is compact we can "wiggle it a little" and still stay in U. In particular, for small enough $s > 0$, the disc

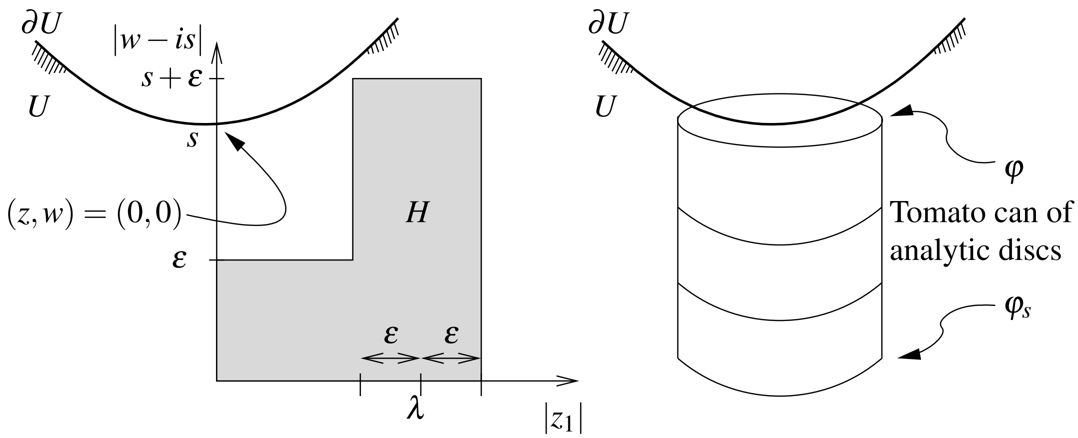
$$\xi \overset{\varphi_s}{\mapsto} (\lambda \xi, 0, 0, \ldots, 0, is)$$

*E. E. Levi stated the problem in 1911, but it was not completely solved until the 1950s, by Oka and others.

is entirely inside U (that is for slightly positive $\operatorname{Im} w$). Suppose $\varepsilon > 0$ is small and $\varepsilon < s$. Define the Hartogs figure

$$
\begin{aligned}
H = &\{(z,w) : \lambda - \varepsilon < |z_1| < \lambda + \varepsilon \text{ and } |z_j| < \varepsilon \text{ for } j = 2,\dots,n-1, \text{ and } |w - is| < s + \varepsilon\} \\
&\cup \{(z,w) : |z_1| < \lambda + \varepsilon, \text{ and } |z_j| < \varepsilon \text{ for } j = 2,\dots,n-1, \text{ and } |w - is| < \varepsilon\}.
\end{aligned}
$$

For small enough $\varepsilon > 0$, $H \subset U$. That is because the set where $|z_1| = \lambda$, $z' = 0$ and $|w| \leq s$, is inside U for all small enough s. So we can take an ε-neighborhood of that. Further for $w = is$ the whole disc where $|z_1| \leq \lambda$ is in U. So we can take an ε-neighborhood of that. We are really just taking a Hartogs figure in the z_1, w variables, and then "fattening it up" to the z' variables. We picture the Hartogs figure in the $|z_1|$ and $|w - is|$ variables. The boundary ∂U and U are only pictured diagrammatically. Also we make a "picture" the analytic discs giving the "tomato can". In the picture the U is below its boundary ∂U, unlike usually.



The origin is in the hull of H, and so every function holomorphic in U, and so in H, extends through the origin. Hence U is not a domain of holomorphy. □

Exercise* 2.3.10*: *Take $U \subset \mathbb{C}^2$ defined by $\operatorname{Im} w > |z|^2(\operatorname{Re} w)$. Find all the points in ∂U where U is weakly pseudoconvex and all the points where it is strongly pseudoconvex.*

Exercise* 2.3.11*: *Let $U \subset \mathbb{C}^n$ be a smoothly bounded domain that is strongly pseudoconvex at $p \in \partial U$. Show that there exists a neighborhood W of p and a smooth function $f \colon \overline{W \cap U} \to \mathbb{C}$ that is holomorphic on $W \cap U$ such that $f(p) = 1$ and $|f(z)| < 1$ for all $z \in \overline{W \cap U} \setminus \{p\}$.*

Exercise* 2.3.12*: *Suppose $U \subset \mathbb{C}^n$ is a smoothly bounded domain. Suppose for $p \in \partial U$, there is a neighborhood W of p and a holomorphic function $f \colon W \to \mathbb{C}$ such that $df(p) \neq 0$, $f(p) = 0$, but f is never zero on $W \cap U$. Show that U is pseudoconvex at p. Hint: you may need the holomorphic implicit function theorem, see Theorem 1.3.6. Note: the result does not require the derivative of f to not vanish, but is much harder to prove without that hypothesis.*

2.4 Harmonic, subharmonic, and plurisubharmonic functions

Definition 2.4.1. A C^2-smooth function $f \colon U \subset \mathbb{R}^n \to \mathbb{R}$ is called *harmonic* if*

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad \text{on } U.$$

A function $f \colon U \subset \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is called *subharmonic* if it is upper-semicontinuous† and for every ball $B_\rho(a)$ with $\overline{B_\rho(a)} \subset U$, and every function φ harmonic on $B_\rho(a)$ and continuous on $\overline{B_\rho(a)}$ such that $f(x) \leq \varphi(x)$ for $x \in \partial B_\rho(a)$, then

$$f(x) \leq \varphi(x), \qquad \text{for all } x \in B_\rho(a).$$

In other words, a subharmonic function is a function that is less than any harmonic function on every ball. We will generally look at harmonic and subharmonic functions in $\mathbb{C} \cong \mathbb{R}^2$. Let us go through some basic results on harmonic and subharmonic functions that you have seen in detail in your one-variable class. Consequently we leave some of these results as exercises. Notice that in this section (and not just here) we will often write $f(z)$ for a function even if it is not holomorphic.

Exercise* 2.4.1*: *An upper-semicontinuous function achieves a maximum on compact sets.*

Exercise* 2.4.2*: *Show that for a C^2 function $f \colon U \subset \mathbb{C} \to \mathbb{R}$,*

$$\frac{\partial^2}{\partial \bar{z} \partial z} f = \frac{1}{4} \nabla^2 f.$$

Use this fact to show that f is harmonic if and only if it is (locally) the real or imaginary part of a holomorphic function. Hint: Key is to be able to find an antiderivative of a holomorphic function.

It follows from the exercise that a harmonic function is infinitely differentiable, and by applying the Cauchy formula on a disc we obtain the following proposition.

Proposition 2.4.2 (Mean-value property and sub-mean-value property)**.** *A continuous function $f \colon U \subset \mathbb{C} \to \mathbb{R}$ is harmonic if and only if whenever $\overline{\Delta_r(a)} \subset U$ then*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) \, d\theta.$$

An upper-semicontinuous function $f \colon U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if whenever $\overline{\Delta_r(a)} \subset U$ then

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) \, d\theta.$$

*Recall the operator ∇^2, sometimes also written Δ, is called the *Laplacian*. It is the trace of the Hessian matrix.

†Recall f is *upper-semicontinuous* if $\limsup_{t \to x} f(t) \leq f(x)$ for all x.

Do note that for the sub-mean-value property we may have to use Lebesgue integral to be able to integrate a upper-semicontinuous function. On first reading, you can only think of continuous subharmonic functions and not much is lost.

***Exercise* 2.4.3:** *Fill in the details of the proof of the proposition.*

***Exercise* 2.4.4:** *Show that if $f: U \subset \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is subharmonic then for $z \in U$ we have*

$$\limsup_{w \to z} f(w) = f(z).$$

Proposition 2.4.3 (Maximum principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f: U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic. If f attains a maximum in U then f is constant.*

Proof. If $\overline{\Delta_r(a)} \subset U$ then

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta.$$

In particular $f = f(a)$ almost everywhere on $\partial \Delta_r(a)$. By upper-semicontinuity it is true everywhere. This was true for all r with $\overline{\Delta_r(a)} \subset U$, so $f = f(a)$ on $\Delta_r(a)$ and the set where $f = f(a)$ is open. The set where an upper-semicontinuous function attains a maximum is closed, so $f = f(a)$ on U as U is connected. □

Proposition 2.4.4. *Suppose $U \subset \mathbb{C}$ and $f: U \to \mathbb{R}$ is a C^2 function. The function f is subharmonic if and only if $\nabla^2 f \geq 0$.*

Proof. We have a C^2-smooth function on a subset of $\mathbb{C} \cong \mathbb{R}^2$ with $\nabla^2 f \geq 0$ and we wish to show that it is subharmonic. Take a disc $\Delta_\rho(a)$ such that f is continuous on the closure, and take a harmonic function g on the closure $\overline{\Delta_\rho(a)}$ such that $f \leq g$ on the boundary. Because $\nabla^2(f-g) = \nabla^2 f \geq 0$, we assume $g = 0$ and $f \leq 0$ on the boundary of $\Delta_\rho(a)$. We can also assume that $f = 0$ on at least one point on the boundary.

First suppose $\nabla^2 f > 0$ at all points on $\Delta_\rho(a)$. Suppose f attains a maximum in $\Delta_\rho(a)$, call this point p. $\nabla^2 f$ is the trace of the Hessian matrix, but for f to have a maximum, the Hessian must have only nonpositive eigenvalues at the critical points, which is a contradiction as the trace is the sum of the eigenvalues. So f has no maximum inside, and therefore $f \leq 0$ on all of $\overline{\Delta_\rho(a)}$.

Next suppose $\nabla^2 f \geq 0$. Let M be the maximum of $x^2 + y^2$ on $\overline{\Delta_\rho(a)}$. Take $f_n(x,y) = f(x,y) + \frac{1}{n}(x^2 + y^2) - \frac{1}{n}M$. Clearly $\nabla^2 f_n > 0$ everywhere on $\Delta_\rho(a)$ and $f_n \leq 0$ on the boundary, so $f_n \leq 0$ on all of $\overline{\Delta_\rho(a)}$. As $f_n \to f$ we obtain that $f \leq 0$ on all of $\overline{\Delta_\rho(a)}$.

The other direction is left as an exercise. □

***Exercise* 2.4.5:** *Finish the proof of the above proposition.*

Proposition 2.4.5. *Suppose $U \subset \mathbb{C}$ is a domain and $f_\alpha \colon U \to \mathbb{R} \cup \{-\infty\}$ is a family of subharmonic functions. Let*

$$\varphi(z) = \sup_\alpha f_\alpha(z).$$

If the family is finite then φ is subharmonic. If the family is infinite and we assume that $\varphi(z) \neq \infty$ for all z and that φ is upper-semicontinuous, then φ is subharmonic.

Proof. Suppose $\overline{\Delta_r(a)} \subset U$. For any α,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{i\theta})\,d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} f_\alpha(a+re^{i\theta})\,d\theta \geq f_\alpha(a).$$

Taking the supremum on the right over α obtains the results. □

***Exercise* 2.4.6:** *Prove that if $\varphi \colon \mathbb{R} \to \mathbb{R}$ is a monotonically increasing convex function and $f \colon U \subset \mathbb{C} \to \mathbb{R}$ is subharmonic, then $\varphi \circ f$ is subharmonic.*

There are too many harmonic functions in $\mathbb{C}^n$. To get the real and imaginary parts of holomorphic functions in $\mathbb{C}^n$ we require a smaller class of functions than all harmonic functions.

Definition 2.4.6. Twice differentiable function $f \colon U \subset \mathbb{C}^n \to \mathbb{R}$ is called *pluriharmonic* if for every $a, b \in \mathbb{C}^n$, the function

$$\xi \mapsto f(a+b\xi)$$

is harmonic (on the set of $\xi \in \mathbb{C}$ where $a+b\xi \in U$). That is, f is harmonic on every complex line.

A function $f \colon U \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic*, sometimes *plush* or *psh* for short, if it is upper-semicontinuous and for every $a, b \in \mathbb{C}^n$, the function

$$\xi \mapsto f(a+b\xi)$$

is subharmonic (whenever $a+b\xi \in U$). The notation $PSH(U)$ is used to denote these functions.

***Exercise* 2.4.7:** *A C^2-smooth function $f \colon U \subset \mathbb{C}^n \to \mathbb{R}$ is pluriharmonic if and only if*

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} = 0 \qquad \text{on } U \text{ for all } j,k = 1,\ldots,n.$$

***Exercise* 2.4.8:** *Show that a pluriharmonic function is harmonic. On the other hand, find an example of a harmonic function that is not pluriharmonic.*

***Exercise* 2.4.9:** *Show that a $f: U \subset \mathbb{C}^n \to \mathbb{R}$ is pluriharmonic if and only if it is locally the real or imaginary part of a holomorphic function. Hint: Using a previous exercise $\frac{\partial f}{\partial z_k}$ is holomorphic for all k. Assume that U is simply connected and $f(z^0) = 0$. Consider the line integral from $z^0 \in U$ to a nearby $z \in U$:*

$$F(z) = \int_{z^0}^{z} \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z)\, dz_k.$$

Prove that it is path independent, compute derivatives of F, and find out what is $f - F$.

***Exercise* 2.4.10:** *Prove the maximum principle for plurisubharmonic functions. That is, if $U \subset \mathbb{C}^n$ is a domain and $f: U \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic and achieves a maximum at $p \in U$, then f is constant.*

Proposition 2.4.7. *A C^2-smooth function $f: U \subset \mathbb{C}^n \to \mathbb{R}$ is plurisubharmonic if and only if the complex Hessian matrix*

$$\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\right]_{jk}$$

is positive semidefinite at every point.

Proof. Fix a point p, and after translation assume $p = 0$. After a holomorphic linear change of variables assume that the complex Hessian $\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\Big|_0\right]_{jk}$ is diagonal. If the complex Hessian has a negative eigenvalue, then one of the diagonal entries is negative. Without loss of generality suppose $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1} < 0$ at the origin. The function $z_1 \mapsto f(z_1, 0, \ldots, 0)$ has a negative Laplacian and therefore is not subharmonic, and thus f itself is not plurisubharmonic.

For the other direction, suppose the complex Hessian is positive semidefinite at all points. After a linear change of coordinates assume that the line $\xi \mapsto a + b\xi$ is simply setting all but the first variable to zero. As the complex Hessian is positive semidefinite we have $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1} \geq 0$ for all points $(z_1, 0, \ldots, 0)$. We proved above that $\nabla^2 g \geq 0$ implies g is subharmonic, and we are done. $\square$

***Exercise* 2.4.11:** *Suppose $f: U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic.*
a) Show that $\log|f(z)|$ *is plurisubharmonic. In fact it is pluriharmonic away from the zeros of f.*
b) Show that $|f(z)|^\eta$ is plurisubharmonic for all $\eta > 0$.

***Exercise* 2.4.12:** *Show that the set of plurisubharmonic functions on a domain $U \subset \mathbb{C}^n$ is a cone in the sense that if $a, b > 0$ are constants and $f, g: U \to \mathbb{R} \cup \{-\infty\}$ are plurisubharmonic, then $af + bg$ is plurisubharmonic.*

Theorem 2.4.8. *Suppose $U \subset \mathbb{C}^n$ is a domain and $f \colon U \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic. For every $\varepsilon > 0$, let $U_\varepsilon \subset U$ be the set of points further than ε away from ∂U. Then there exists a smooth plurisubharmonic function $f_\varepsilon \colon U_\varepsilon \to \mathbb{R}$ such that $f_\varepsilon(z) \geq f(z)$, and*

$$f(z) = \lim_{\varepsilon \to 0} f_\varepsilon(z).$$

That is, f is a limit of smooth plurisubharmonic functions. The idea of the proof is important and useful in many other contexts.

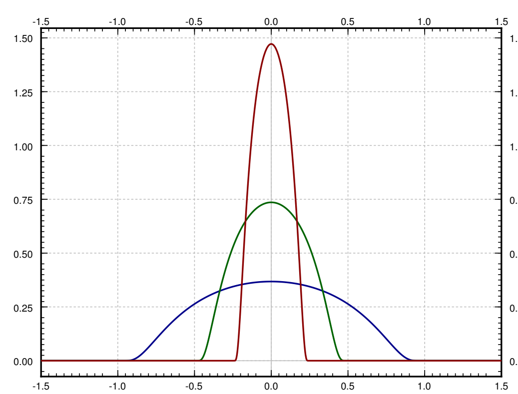
Proof. We smooth f out by convolving with so-called *mollifiers*. Many different mollifiers work, but let us use a very specific one for concreteness. For $\varepsilon > 0$, define

$$g(z) = \begin{cases} Ce^{-1/(1-\|z\|^2)} & \text{if } \|z\| < 1, \\ 0 & \text{if } \|z\| \geq 1, \end{cases} \qquad \text{and} \qquad g_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} g(z/\varepsilon).$$

It is left as an exercise that g, and therefore g_ε, is smooth. The function g clearly has compact support as it is only nonzero inside the unit ball. The support of g_ε is the ε-ball. Both are nonnegative. Choose C so that

$$\int_{\mathbb{C}^n} g \, dV = 1, \qquad \text{and therefore} \qquad \int_{\mathbb{C}^n} g_\varepsilon \, dV = 1.$$

The function g only depends on $\|z\|$. To get an idea of how these functions look, consider the following graphs of the functions $e^{-1/(1-x^2)}$, $\frac{1}{0.5} e^{-1/\left(1-(x/0.5)^2\right)}$, and $\frac{1}{0.25} e^{-1/\left(1-(x/0.25)^2\right)}$.



First f is bounded above on compact sets as it is upper semicontinuous. If f is not bounded below, we replace f with $\max\{f, 1/\varepsilon\}$, which is still plurisubharmonic. Therefore, without loss of generality we assume that f is locally bounded.

For $z \in U_\varepsilon$, we define f_ε as the convolution with g_ε:

$$f_\varepsilon(z) = (f * g_\varepsilon)(z) = \int_{\mathbb{C}^n} f(w) g_\varepsilon(z-w) \, dV(w) = \int_{\mathbb{C}^n} f(z-w) g_\varepsilon(w) \, dV(w).$$

The two forms of the integral follow easily via change of variables. We are perhaps abusing notation a bit since f is only defined on U, but it is not a problem as long as $z \in U_\varepsilon$. By differentiating the first form under the integral, f_ε is smooth. Let us show that f_ε is plurisubharmonic. We need to restrict to a line $\xi \mapsto a + b\xi$. We may translate and rotate by unitary matrices without affecting the setup. So without loss of generality, suppose that $a = 0$, $b = (1, 0, \dots, 0)$, and that we are testing subharmonicity on a disc of radius r around $\xi = 0$.

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f_\varepsilon(re^{i\theta}, 0, \dots, 0)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}^n} f(re^{i\theta} - w_1, -w_2, \dots, -w_n) g_\varepsilon(w)\, dV(w)\, d\theta \\
&= \int_{\mathbb{C}^n} \left(\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta} - w_1, -w_2, \dots, -w_n)\, d\theta \right) g_\varepsilon(w)\, dV(w) \\
&\geq \int_{\mathbb{C}^n} f(-w_1, -w_2, \dots, -w_n) g_\varepsilon(w)\, dV(w) = f_\varepsilon(0).
\end{aligned}
$$

For the inequality we used $g_\varepsilon \geq 0$. So f_ε is plurisubharmonic.

As $g_\varepsilon(w)$ only depends on $|w_1|, \dots, |w_n|$, we notice that $g_\varepsilon(w_1, \dots, w_n) = g_\varepsilon(|w_1|, \dots, |w_n|)$. Without loss of generality we consider $z = 0$:

$$
\begin{aligned}
f_\varepsilon(0) &= \int_{\mathbb{C}^n} f(-w) g_\varepsilon(|w_1|, \dots, |w_n|)\, dV(w) \\
&= \int_0^\varepsilon \cdots \int_0^\varepsilon \left(\int_0^{2\pi} \cdots \int_0^{2\pi} f(-r_1 e^{i\theta_1}, \dots, -r_n e^{i\theta_n})\, d\theta_1 \cdots d\theta_n \right) \\
&\qquad g_\varepsilon(r_1, \dots, r_n)\, r_1 \cdots r_n\, dr_1 \cdots dr_n \\
&\geq \int_0^\varepsilon \cdots \int_0^\varepsilon \left(\int_0^{2\pi} \cdots \int_0^{2\pi} (2\pi) f(0, -r_2 e^{i\theta_2}, \dots, -r_n e^{i\theta_n})\, d\theta_2 \cdots d\theta_n \right) \\
&\qquad g_\varepsilon(r_1, \dots, r_n)\, r_1 \cdots r_n\, dr_1 \cdots dr_n \\
&\geq f(0) \int_0^\varepsilon \cdots \int_0^\varepsilon (2\pi)^n g_\varepsilon(r_1, \dots, r_n)\, r_1 \cdots r_n\, dr_1 \cdots dr_n \\
&= f(0) \int_{\mathbb{C}^n} g_\varepsilon(w) d(w) \\
&= f(0).
\end{aligned}
$$

The second equality above follows because integral of g_ε only needs to be done over the polydisc of radius ε. The penultimate equality follows from the fact that $2\pi = \int_0^{2\pi} d\theta$.

We have $\limsup_{w \to z} f(w) = f(z)$ for subharmonic, and therefore for plurisubharmonic functions. Hence for any $\delta > 0$ find an $\varepsilon > 0$ so that for $w \in B_\varepsilon(0)$ we get $f(w) - f(0) \leq \delta$.

$$
\begin{aligned}
f_\varepsilon(0) - f(0) &= \int_{B_\varepsilon(0)} \big(f(-w) - f(0) \big)\, g_\varepsilon(w)\, dV(w) \\
&\leq \delta \int_{B_\varepsilon(0)} g_\varepsilon(w)\, dV(w) = \delta.
\end{aligned}
$$

Again we used that g_ε is nonnegative. ☐

Exercise* 2.4.13*: *Show that g in the proof above is smooth on all of $\mathbb{C}^n$.*

Exercise* 2.4.14*: *a) Show that for a subharmonic function $\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$ is a monotone function of r (Hint: try a C^2 function first and use Green's theorem). b) Use this fact to show that $f_\varepsilon(z)$ from Theorem 2.4.8 are monotone decreasing in ε.*

Exercise* 2.4.15*: *If $g\colon U \subset \mathbb{C}^n \to V \subset \mathbb{C}^m$ is holomorphic and $f\colon V \to \mathbb{R}$ is a C^2 plurisubharmonic function, then $f \circ g$ is plurisubharmonic. Then use this to show that this holds for all plurisubharmonic functions (Hint: monotone convergence).*

Exercise* 2.4.16*: *Show that plurisubharmonicity is a local property, that is, f is plurisubharmonic if and only if f is plurisubharmonic in some neighborhood of each point.*

Exercise* 2.4.17*: *Use the computation from Theorem 2.4.8 to show that if f is pluriharmonic, then $f_\varepsilon = f$ (where that makes sense), therefore obtaining another proof that a pluriharmonic function is C^∞.*

Exercise* 2.4.18*: *Let the f in Theorem 2.4.8 be continuous and suppose $K \subset\subset U$, in particular for small enough $\varepsilon > 0$, $K \subset U_\varepsilon$. Show that f_ε converges uniformly to f on K.*

Exercise* 2.4.19*: *Let the f in Theorem 2.4.8 be C^k for some $k \geq 0$. Show that all derivatives of f_ε up to order k converge uniformly on compact sets to the corresponding derivatives of f. See also previous exercise.*

It is often useful to find a harmonic function given boundary values. The proof of the following special case is contained in the exercises following the theorem.

Theorem 2.4.9. *Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc. Let the* Poisson kernel *for the unit disc be defined by*

$$P_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta}.$$

Given any continuous function $u\colon \partial\mathbb{D} \to \mathbb{C}$, the function $\tilde{u}\colon \mathbb{D} \to \mathbb{C}$ defined by

$$\tilde{u}(re^{i\theta}) = \int_{-\pi}^{\pi} u(e^{i\varphi})P_r(\theta-\varphi)\,d\varphi \tag{2.2}$$

is harmonic. Further as $z \in \mathbb{D}$ tends to $z_0 \in \partial\mathbb{D}$, $\tilde{u}(z)$ tends to $u(z_0)$.

Exercise* 2.4.20*: *a) Prove $P_r(\theta) > 0$ for all $0 \leq r < 1$ and all θ.*
b) Prove $\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$.
c) Prove for any given $\delta > 0$, $\sup\{P_r(\theta) : \delta \leq |\theta| \leq \pi\} \to 0$ as $r \to 1$.

Exercise* 2.4.21*: *Prove the above theorem. Hint: First, the Poisson kernel is harmonic, and second it acts like an approximate identity as $r \to 1$. Use the above properties.*

The Poisson kernel can sometimes be used as a reproducing kernel for holomorphic functions, as holomorphic functions are harmonic. A Poisson kernel also exists for higher dimensions, and has analogous properties.

As an application of harmonic functions, let us prove another useful theorem about zero sets of holomorphic functions, the theorem of Radó. It is sometimes covered in a one variable course, but it is often skipped. It is very useful in several variables and the proof is a direct application of the one variable theory. It is really a complementary theorem to the Riemann extension theorem, when on the one hand you know that the function is continuous and vanishes on the set you wish to extend across, but on the other hand you know nothing about this set.

Theorem 2.4.10 (Radó)**.** *Let $U \subset \mathbb{C}^n$ be a domain and $f : U \to \mathbb{C}$ be a continuous function which is holomorphic on the set*

$$U' = \{z \in U : f(z) \neq 0\}.$$

Then $f \in \mathscr{O}(U)$.

Proof. First assume $n = 1$. As the theorem is local, it is enough to prove it for a small disc Δ such that f is continuous on the closure $\overline{\Delta}$, let Δ' be the part of the disc where f is nonzero as before. If Δ' is empty then we are done as f is just identically zero and hence holomorphic.

Let u be the real part of f. On Δ', u is a harmonic function. Let Pu be the Poisson integral of u. Hence Pu has the same boundary values as u and is harmonic in all of Δ. Consider the function $Pu(z) - u(z)$ on $\overline{\Delta}$. The function is zero on $\partial\Delta$ and it is harmonic on Δ'. By rescaling f we can without loss of generality assume that $|f(z)| < 1$ for all $z \in \overline{\Delta}$. For any $t > 0$, the function $z \mapsto t \log|f(z)|$ is plurisubharmonic on Δ' and upper-semicontinuous on $\overline{\Delta'}$. Further, it is negative on $\partial\Delta$. The function $z \mapsto -t\log|f(z)|$ is plurisuperharmonic on Δ', lower-semicontinuous on $\overline{\Delta'}$ and positive on $\partial\Delta$. On the set where f is zero, the two functions are $-\infty$ and ∞ respectively. Therefore we have

$$t\log|f(z)| \leq Pu(z) - u(z) \leq -t\log|f(z)|.$$

on $\overline{\Delta}$ for all $t > 0$. Fixing $t \in \Delta'$ and letting $t \to 0$ shows that $Pu = u$ on Δ'. In the same way we find that if v is the imaginary part of f, $Pv = v$ on Δ'. So we find that $\tilde{f} = Pu + iPv$ equals f on $\overline{\Delta'}$ which includes $\partial\Delta$. Let $W = \Delta \setminus \overline{\Delta'}$. If this open set is empty we are finished. If it is not empty, the continuous function $\tilde{f} - f$ is zero on $\overline{\Delta'}$, which includes the boundary of Δ, so $\tilde{f} - f$ is zero on ∂W. As f is zero on W then $\tilde{f} - f$ is holomorphic on W and by the maximum principle it must be zero on W. Thus $\tilde{f} = f$ everywhere and we are done. The theorem is proved for $n = 1$.

The extension of the proof to several variables is left as an exercise. □

***Exercise* 2.4.22:** *Use the one variable result to extend the theorem to several variables.*

2.5 Hartogs pseudoconvexity

It is worth it to mention explicitly that by the above exercises, plurisubharmonicity is preserved under holomorphic mappings. That is if g is holomorphic and f is plurisubharmonic, then $f \circ g$ is plurisubharmonic. In particular if $\varphi \colon \mathbb{D} \to \mathbb{C}^n$ is an analytic disc and f is plurisubharmonic in a neighborhood of $\varphi(\mathbb{D})$, then $f \circ \varphi$ is subharmonic.

Definition 2.5.1. Let $\mathscr{F}$ be a class of (extended*)-real-valued functions defined on $U \subset \mathbb{R}^n$. If $K \subset U$, define $\widehat{K}$, the *hull* of K with respect to $\mathscr{F}$, as the set

$$\widehat{K} \overset{\text{def}}{=} \left\{ z \in U : f(z) \leq \sup_{w \in K} f(w) \text{ for all } f \in \mathscr{F} \right\}.$$

A domain U is said to be *convex with respect to* $\mathscr{F}$ if for every $K \subset\subset U$, the hull $\widehat{K} \subset\subset U$.†

Clearly $K \subset \widehat{K}$. The key is to show that $\widehat{K}$ is not "too large" for U. Keep in mind that the functions in $\mathscr{F}$ are defined on U, so $\widehat{K}$ depends on U not just on K. A common mistake is to consider functions defined on a larger set, obtaining a smaller $\mathscr{F}$ and hence a larger $\widehat{K}$. Sometimes it may be useful to use $\widehat{K}_{\mathscr{F}}$ to denote the dependence on $\mathscr{F}$, especially when talking about several different hulls.

***Exercise* 2.5.1:** *Show that a domain $U \subset \mathbb{R}^n$ is geometrically convex (that is, the line segment between any two points in U is contained in U) if and only if it is convex with respect to the convex functions on U.*

***Exercise* 2.5.2:** *Show that any domain $U \subset \mathbb{R}^n$ is convex with respect to real polynomials.*

Theorem 2.5.2 (Kotinuitätssatz—Continuity principle)**.** *Suppose $U \subset \mathbb{C}^n$ is convex with respect to plurisubharmonic functions, then given any collection of closed analytic discs Δ_α such that $\bigcup_\alpha \partial\Delta_\alpha \subset\subset U$, we have $\bigcup_\alpha \Delta_\alpha \subset\subset U$.*

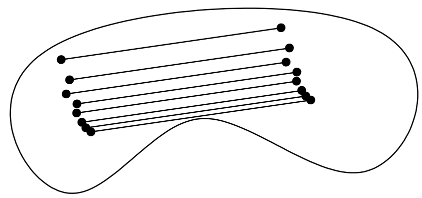
Various similar theorems are named the *continuity principle*. Generally what they have in common is the family of analytic discs whose boundaries stay inside the domain, and whose conclusion has to do with extension of holomorphic functions, or domains of holomorphy.

Proof. Let f be a plurisubharmonic function on U. If $\varphi_\alpha \colon \overline{\mathbb{D}} \to U$ is the holomorphic (in $\mathbb{D}$) mapping giving the closed analytic disc, then $f \circ \varphi_\alpha$ is subharmonic. By the maximum principle, f on Δ_α must be less than or equal to the supremum of f on $\partial\Delta_\alpha$, so $\overline{\Delta_\alpha}$ is in the hull of $\partial\Delta_\alpha$. In other words $\bigcup_\alpha \Delta_\alpha$ is in the hull of $\bigcup_\alpha \partial\Delta_\alpha$ and therefore $\bigcup_\alpha \Delta_\alpha \subset\subset U$ by convexity. ☐

*By extended reals we mean $\mathbb{R} \cup \{-\infty, \infty\}$.

†Recall that $\subset\subset$ means relatively compact.

Let us illustrate the failure of the continuity principle. If the domain is not convex with respect to plurisubharmonic functions then you could have discs (denoted by straight line segments) that approach the boundary as in the following picture. In the diagram the boundaries of the discs are denoted by the dark dots at the end of the segments.



Definition 2.5.3. Let $U \subset \mathbb{C}^n$ be a domain. An $f: U \to \mathbb{R}$ is an *exhaustion function* for U if

$$\{z \in U : f(z) < r\} \subset\subset U \qquad \text{for every } r \in \mathbb{R}.$$

A domain $U \subset \mathbb{C}^n$ is *Hartogs pseudoconvex* if there exists a continuous plurisubharmonic exhaustion function. The set $\{z \in U : f(z) < r\}$ is called the *sublevel set* of f, or the *r-sublevel set*.

Example 2.5.4: The unit ball $\mathbb{B}_n$ is Hartogs pseudoconvex. The continuous function

$$-\log(1 - \|z\|)$$

is an exhaustion function, and it is easy to check directly that it is plurisubharmonic.

Example 2.5.5: The entire $\mathbb{C}^n$ is Hartogs pseudoconvex as $\|z\|^2$ is a continuous plurisubharmonic exhaustion function. Also, because $\|z\|^2$ is plurisubharmonic, then given any $K \subset\subset \mathbb{C}^n$, the hull $\widehat{K}$ with respect to plurisubharmonic functions must be bounded. In other words, $\mathbb{C}^n$ is convex with respect to plurisubharmonic functions.

Theorem 2.5.6. *Suppose $U \subsetneq \mathbb{C}^n$ is a domain. The following are equivalent:*

(i) $-\log \rho(z)$ is plurisubharmonic, where $\rho(z)$ is the distance from z to ∂U.

(ii) U has a continuous plurisubharmonic exhaustion function, that is, U is Hartogs pseudoconvex.

(iii) U is convex with respect to plurisubharmonic functions defined on U.

Proof. (i) $\Rightarrow$ (ii): If U is bounded, the function $-\log \rho(z)$ is clearly a continuous exhaustion function. If U is unbounded, take $z \mapsto \max\{-\log \rho(z), \|z\|^2\}$.

(ii) $\Rightarrow$ (iii): Suppose f is a continuous plurisubharmonic exhaustion function. If $K \subset\subset U$, then for some r we have $K \subset \{z \in U : f(z) < r\} \subset\subset U$. But then by definition of the hull $\widehat{K}$ we have $\widehat{K} \subset \{z \in U : f(z) < r\} \subset\subset U$.

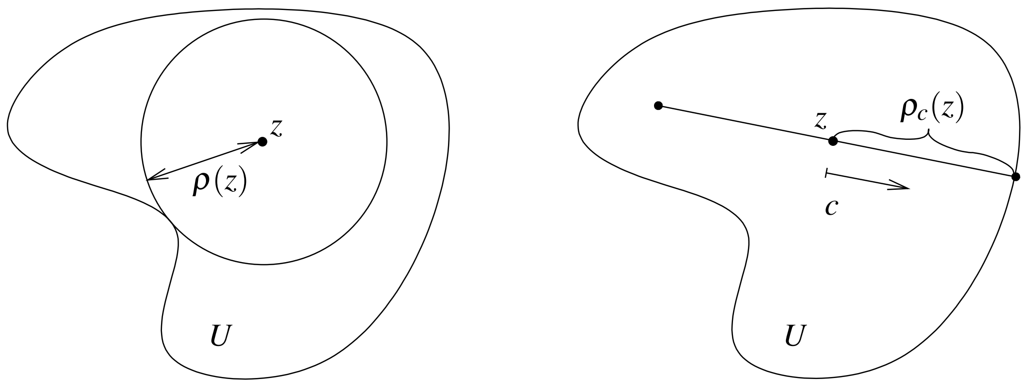
(iii) $\Rightarrow$ (i): For $c \in \mathbb{C}^n$ with $\|c\| = 1$, let us define

$$\rho_c(z) = \sup\{\lambda > 0 : z + \lambda t c \in U \text{ for all } |t| < 1\}.$$

So $\rho_c(z)$ is the radius of the largest affine disc centered at z in the direction c that still lies in U. As $\rho(z) = \inf_c \rho_c(z)$,

$$-\log \rho(z) = \sup_{\|c\|=1} \bigl(-\log \rho_c(z)\bigr).$$

If we prove that for any a, b, c the function $\xi \mapsto -\log \rho_c(a + b\xi)$ is subharmonic, then $\xi \mapsto -\log \rho(a+b\xi)$ is subharmonic and we are done. Here is the setup, the disc is drawn as a line:



Suppose $\Delta \subset \mathbb{C}$ is a disc such that for all $\xi \in \overline{\Delta}$, $a + b\xi \in U$. We need to show that if there is a harmonic function u on Δ continuous up to the boundary such that $-\log \rho_c(a+b\xi) \leq u(\xi)$ on $\partial\Delta$, then the inequality holds on Δ. Let $u = \operatorname{Re} f$ for a holomorphic function f. For $\xi \in \partial\Delta$ we have $-\log \rho_c(a+b\xi) \leq \operatorname{Re} f(\xi)$, or in other words

$$\rho_c(a+b\xi) \geq e^{-\operatorname{Re} f(\xi)} = \left|e^{-f(\xi)}\right|.$$

Using the definition of $\rho_c(a+b\xi)$, the statement above is equivalent to saying that whenever $|t| < 1$ then

$$(a+b\xi) + cte^{-f(\xi)} \in U.$$

This statement holds when $\xi \in \partial\Delta$. If we prove that it also holds for $\xi \in \Delta$ then we are finished.

We think of $\varphi_t(\xi) = (a+b\xi) + cte^{-f(\xi)}$ as a closed analytic disc with boundary inside U. We have a family of analytic discs, parametrized by t, whose boundaries are in U for all t with $|t| < 1$. For $t = 0$ the entire disc is inside U. Take $t_0 < 1$ such that $\varphi_t(\Delta) \subset U$ for all t with $|t| < t_0$. Then

$$\bigcup_{|t|<t_0} \varphi_t(\partial\Delta) \subset \bigcup_{|t|\leq t_0} \varphi_t(\partial\Delta) \subset\subset U,$$

because continuous functions take compact sets to compact sets. Convexity with respect to plurisubharmonic functions implies

$$\bigcup_{|t|<t_0} \varphi_t(\Delta) \subset\subset U.$$

Again by continuity we have $\varphi_t(\Delta) \subset\subset U$ for all t with $|t| = t_0$, and consequently it is true when $|t|$ is even slightly larger than t_0. Hence $\varphi_t(\mathbb{D}) \subset U$ for all t with $|t| < 1$. Thus $(a+b\xi) + cte^{-f(\xi)} \in U$

for all $\xi \in \Delta$ and all $|t| < 1$. And this implies $\rho_c(a+b\xi) \geq e^{-\operatorname{Re} f(\xi)}$, which in turn implies $-\log \rho_c(a+b\xi) \leq \operatorname{Re} f(\xi) = u(\xi)$, and therefore $-\log \rho_c(a+b\xi)$ is subharmonic. $\square$

***Exercise* 2.5.3:** *Show that if domains $U_1 \subset \mathbb{C}^n$ and $U_2 \subset \mathbb{C}^n$ are Hartogs pseudoconvex then so are all the topological components of $U_1 \cap U_2$.*

***Exercise* 2.5.4:** *Show that if domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are Hartogs pseudoconvex then so is $U \times V$.*

***Exercise* 2.5.5:** *Show that every domain $U \subset \mathbb{C}$ is Hartogs pseudoconvex.*

***Exercise* 2.5.6:** *Show that the union $\bigcup_j U_j$ of a nested sequence of Hartogs pseudoconvex domains $U_{j-1} \subset U_j \subset \mathbb{C}^n$ is Hartogs pseudoconvex.*

***Exercise* 2.5.7:** *Let $\mathbb{R}^2 \subset \mathbb{C}^2$ be naturally embedded (that is, it is the set where z_1 and z_2 are real). Show that the set $\mathbb{C}^2 \setminus \mathbb{R}^2$ is not Hartogs pseudoconvex.*

***Exercise* 2.5.8:** *Suppose $U, V \subset \mathbb{C}^n$ are biholomorphic domains. Prove that U is Hartogs pseudoconvex if and only if V is Hartogs pseudoconvex.*

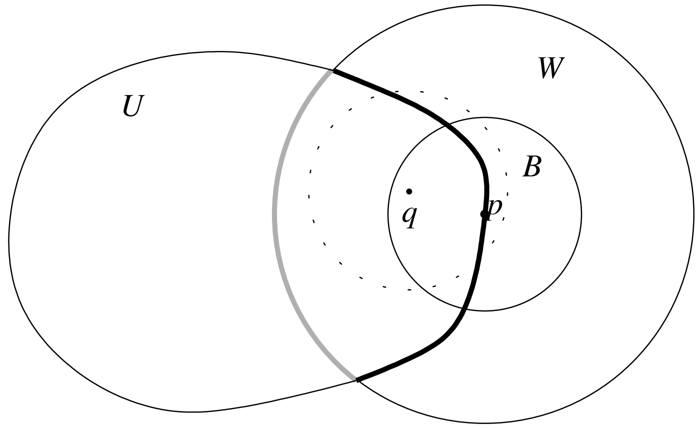
The statement corresponding to Exercise 2.5.6 on nested unions for domains of holomorphy is the *Behnke-Stein theorem*, which follows using this exercise and the solution of the Levi-problem. Although historically Behnke-Stein was proved independently and used to solve the Levi-problem.

Exercise 2.5.8 says that (Hartogs) pseudoconvexity is a biholomorphic invariant. That is a good indication that we are looking at a correct notion. It also allows us to change variables to more convenient ones when proving a specific domain is (Hartogs) pseudoconvex.

It is not immediately clear from the definition, but Hartogs pseudoconvexity is a local property.

Lemma 2.5.7. *A domain $U \subset \mathbb{C}^n$ is Hartogs pseudoconvex if and only if for every point $p \in \partial U$ there exists a neighborhood W of p such that $W \cap U$ is Hartogs pseudoconvex.*

Proof. One direction is trivial, so let us consider the other direction. For $p \in \partial U$ let W be such that $U \cap W$ is Hartogs pseudoconvex. By intersecting with a ball, which is Hartogs pseudoconvex, we assume $W = B_r(p)$ (a ball centered at p). Let $B = B_{r/2}(p)$. For any $z \in B \cap U$, the distance from z to the boundary of $W \cap U$ is the same as the distance to ∂U. The setup is illustrated in the following figure. The part of the boundary ∂U in W is marked by a thick black line, the part of the boundary of $\partial(W \cap U)$ that arises as the boundary of W is marked by a thick gray line. A point $q \in B$ is marked and a ball of radius $r/2$ around q is dotted.



No point of distance $r/2$ from q can be in ∂W, and the distance of q to ∂U is at most $r/2$ as $p \in \partial U$ and p is the center of B. Let $\operatorname{dist}(x,y)$ denote the distance function. Then for $z \in B \cap U$

$$-\log \operatorname{dist}(z, \partial U) = -\log \operatorname{dist}\bigl(z, \partial(U \cap W)\bigr).$$

We know the right hand side is plurisubharmonic. We have such a ball B of positive radius around every $p \in \partial U$, so we have a plurisubharmonic exhaustion function near the boundary.

If U is bounded then ∂U is compact. So there is some $\varepsilon > 0$ such that $-\log \operatorname{dist}(z, \partial U)$ is plurisubharmonic if $\operatorname{dist}(z, \partial U) < 2\varepsilon$. The function

$$\varphi(z) = \max\{-\log \operatorname{dist}(z, \partial U), -\log \varepsilon\}$$

is a continuous plurisubharmonic exhaustion function. The proof for unbounded U requires some function of $\|z\|^2$ rather than a constant ε, and is left as an exercise. □

***Exercise* 2.5.9:** *Finish the proof of the lemma for unbounded domains.*

It may seem that we defined a totally different concept, but it turns out that Levi and Hartogs pseudoconvexity are one and the same on domains where both concepts make sense.

Theorem 2.5.8. *Let $U \subset \mathbb{C}^n$ be a domain with smooth boundary. Then U is Hartogs pseudoconvex if and only if U is Levi pseudoconvex.*

As a consequence of this theorem we say simply "pseudoconvex" and there is no ambiguity.

Proof. Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary that is not Levi pseudoconvex at $p \in \partial U$. As in Theorem 2.3.10, change coordinates so that $p = 0$ and U is defined by

$$\operatorname{Im} z_n > -|z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + O(3).$$

For some small fixed $\lambda > 0$, the analytic discs defined by $\varphi(\xi) = (\lambda\xi, 0, \cdots, 0, is)$ are in U for all small enough $s > 0$. As the origin is in their limit set, Kontinuitätssatz is not satisfied, and U is not convex with respect to the plurisubharmonic functions. Therefore U is not Hartogs pseudoconvex.

Next suppose U is Levi pseudoconvex. Take any $p \in \partial U$. After translation and rotation by a unitary, assume $p = 0$ and write the defining function r as

$$r(z,\bar{z}) = \varphi(z', \bar{z}', \operatorname{Re} z_n) - \operatorname{Im} z_n,$$

where $z' = (z_1, \ldots, z_{n-1})$ and φ is $O(2)$ at the origin. The condition of Levi pseudoconvexity says

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\Big|_q = 0, \tag{2.3}$$

for all $q \in \partial U$ near 0. If we translate ∂U slightly in the $\operatorname{Im} z_n$ direction we still have a Levi pseudoconvex hypersurface. That is, we look at the surface $r = s$ for small real s and the condition is satisfied for $r - s$. As $\frac{\partial r}{\partial z_j} = \frac{\partial (r-s)}{\partial z_j}$ for all j and the complex Hessians of r and $r - s$ are equal, condition (2.3) holds for r for all $q \in U$ near 0. We will use r to manufacture a plurisubharmonic exhaustion function, that is one with a semidefinite Hessian. Therefore, we already have what we need in all but one direction.

Let $\nabla_z r(q) = \left(\frac{\partial r}{\partial z_1}\Big|_q, \ldots, \frac{\partial r}{\partial z_n}\Big|_q \right)$ denote the gradient of r in the holomorphic directions only. Given $q \in U$ near 0, decompose an arbitrary $c \in \mathbb{C}^n$ as $c = a + b$, where $a = (a_1, \ldots, a_n)$ satisfies

$$\sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\Big|_q = \left\langle a, \overline{\nabla_z r(q)} \right\rangle = 0.$$

Taking the orthogonal decomposition, b is a scalar multiple of $\overline{\nabla_z r(q)}$. Then by Cauchy-Schwarz we find

$$\left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right| = \left| \sum_{j=1}^{n} b_j \frac{\partial r}{\partial z_j}\Big|_q \right| = \left| \left\langle b, \overline{\nabla_z r(q)} \right\rangle \right| = \|b\| \|\nabla_z r(q)\|.$$

As $\nabla_z r(0) = (0, \ldots, 0, -1/2i)$, then for q sufficiently near 0 we have that $\|\nabla_z r(q)\| \geq 1/3$. Therefore for such q,

$$\|b\| = \frac{\left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right|}{\|\nabla_z r(q)\|} \leq 3 \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right|.$$

As $c = a + b$ is the orthogonal decomposition we have that $\|c\| \geq \|b\|$.

Next, let $M \geq 0$ be the operator norm of the Hessian matrix of r. Then again using Cauchy-

Schwarz

$$\begin{aligned}
\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q &= \sum_{j=1,\ell=1}^{n} (\bar{a}_j + \bar{b}_j)(a_\ell + b_\ell) \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&= \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&\quad + \sum_{j=1,\ell=1}^{n} \bar{b}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q + \sum_{j=1,\ell=1}^{n} \bar{c}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q - \sum_{j=1,\ell=1}^{n} \bar{b}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \\
&\geq \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q - M\|b\|\|c\| - M\|c\|\|b\| - M\|b\|^2 \\
&\geq -3M\|c\|\|b\|.
\end{aligned}$$

Together with what we know about $\|b\|$ we get:

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q \geq -3M\|c\|\|b\| \geq -3^2 M\|c\| \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right|.$$

For $z \in U$ sufficiently close to 0 define

$$f(z) = -\log\bigl(-r(z)\bigr) + A\|z\|^2,$$

where $A > 0$ is some constant we will choose later. The log is there to make the function blow up as we approach the boundary and the $A\|z\|^2$ is adding a constant diagonal matrix to the complex Hessian of f, which we hope is enough to make it positive semidefinite. Let us compute:

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell} = \frac{1}{r^2} \frac{\partial r}{\partial \bar{z}_j} \frac{\partial r}{\partial z_\ell} - \frac{1}{r} \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} + A\delta_j^\ell,$$

where δ_j^ℓ is the Kronecker delta*. Apply the complex Hessian of f to c (recall that r is negative on U and so for $q \in U$, $-r = |r|$):

$$\begin{aligned}
\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell}\Big|_q &= \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_q \right|^2 + \frac{1}{|r|} \sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q + A\|c\|^2 \\
&\geq \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\Big|_q \right|^2 - \frac{3^2 M}{|r|} \|c\| \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\Big|_q \right| + A\|c\|^2.
\end{aligned}$$

*Recall $\delta_j^\ell = 0$ if $j \neq \ell$ and $\delta_j^\ell = 1$ if $j = \ell$.

Now comes a somewhat funky trick. As a quadratic polynomial in $\|c\|$, the right hand side of the inequality is always nonnegative if $A > 0$ and the discriminant is negative or zero. Let us set the discriminant to zero:

$$0 = \left(\frac{3^2 M}{|r|} \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j} \Big|_q \right| \right)^2 - 4A \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell} \Big|_q \right|^2 .$$

All the nonconstant terms go away and $A = \frac{3^4 M^2}{4}$. Thus

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell} \Big|_q \geq 0.$$

In other words, the complex Hessian of f is positive semidefinite at all points $q \in U$ near 0. The function $f(z)$ goes to infinity as z approaches ∂U, so for every $t \in \mathbb{R}$ the t-sublevel set (set where $f(z) < t$) must be a positive distance away from ∂U near 0.

We have a local continuous plurisubharmonic exhaustion function for U near p. If we intersect with a small ball B centered at p we get that $U \cap B$ is Hartogs pseudoconvex. This is true at all $p \in \partial U$, so U is Hartogs pseudoconvex. □

2.6 Holomorphic convexity

Definition 2.6.1. Let $U \subset \mathbb{C}^n$ be a domain. For a set $K \subset U$, define the *holomorphic hull*

$$\widehat{K}_U \overset{\text{def}}{=} \left\{ z \in U : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathscr{O}(U) \right\}.$$

The domain U is *holomorphically convex* if whenever $K \subset\subset U$, then $\widehat{K}_U \subset\subset U$. In other words, U is holomorphically convex if it is convex with respect to moduli of holomorphic functions on U.*

It is a simple exercise (see below) to show that a holomorphically convex domain is Hartogs pseudoconvex. The converse is the Levi problem for Hartogs pseudoconvex domains and is considerably more difficult. The thing is that there are lots of plurisubharmonic functions and they are easy to construct; we can even construct them locally and then piece them together as we have already seen. There are far fewer holomorphic functions, and clearly we cannot just construct them locally and expect the pieces to somehow fit together.

Exercise* 2.6.1*: *Prove that a holomorphically convex domain is Hartogs pseudoconvex. See Exercise 2.4.11.*

*It is common to use $\widehat{K}_U$ rather than just $\widehat{K}$ to emphasize the dependence on U.

Exercise* 2.6.2*: *Compute the hull $\widehat{K}_{\mathbb{D}^n}$ of the set $K = \{z \in \mathbb{D}^n : |z_j| = \lambda_j \text{ for } j = 1,\ldots,n\}$ where $0 \leq \lambda_j < 1$. Prove that the unit polydisc is holomorphically convex.*

Exercise* 2.6.3*: *Prove that a geometrically convex domain $U \subset \mathbb{C}^n$ is holomorphically convex.*

Exercise* 2.6.4*: *Prove that the Hartogs figure is not holomorphically convex.*

Exercise* 2.6.5*: *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{O}(U)$. Show that if U is holomorphically convex then $\widetilde{U} = U \setminus \{z : f(z) = 0\}$ is holomorphically convex. Hint: first see Exercise 1.6.2.*

Exercise* 2.6.6*: *Suppose $U,V \subset \mathbb{C}^n$ are biholomorphic domains. Prove that U is holomorphically convex if and only if V is holomorphically convex.*

Exercise* 2.6.7*: *In the definition of holomorphic hull of K, replace U with $\mathbb{C}^n$ and $\mathscr{O}(U)$ with holomorphic polynomials on $\mathbb{C}^n$, to get the* polynomial hull *of K. Prove that the polynomial hull of $K \subset\subset \mathbb{C}^n$ is the same as the holomorphic hull $\widehat{K}_{\mathbb{C}^n}$.*

Exercise* 2.6.8*: *a) Prove the Hartogs triangle T (see Exercise 2.1.7) is holomorphically convex. b) Prove $T \cup B_\varepsilon(0)$ (for a small enough $\varepsilon > 0$) is not holomorphically convex.*

Exercise* 2.6.9*: *Show that if domains $U_1 \subset \mathbb{C}^n$ and $U_2 \subset \mathbb{C}^n$ are holomorphically convex then so are all the topological components of $U_1 \cap U_2$.*

Exercise* 2.6.10** (Behnke-Stein again):*** *Show that the union $\bigcup_j U_j$ of a nested sequence of holomorphically convex domains $U_{j-1} \subset U_j \subset \mathbb{C}^n$ is holomorphically convex.*

Using an above exercise we see that $\mathbb{C}^n$ is both holomorphically convex and a domain of holomorphy. In fact, these two notions are equivalent for all other domains in $\mathbb{C}^n$.

Theorem 2.6.2 (Cartan-Thullen)**.** *Let $U \subsetneq \mathbb{C}^n$ be a domain. The following are equivalent:*

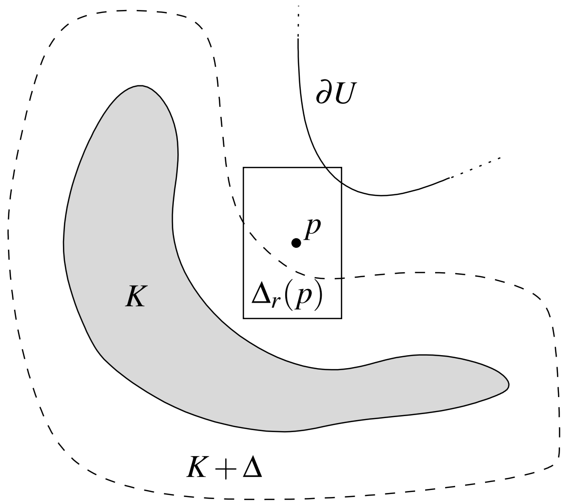
(i) U is a domain of holomorphy.

(ii) For all $K \subset\subset U$, $\operatorname{dist}(K,\partial U) = \operatorname{dist}(\widehat{K}_U,\partial U)$.

(iii) U is holomorphically convex.

Proof. Let us start with (i) $\Rightarrow$ (ii). Suppose there is a $K \subset\subset U$ with $\operatorname{dist}(K,\partial U) > \operatorname{dist}(\widehat{K}_U,\partial U)$. After possibly a rotation by a unitary, there exists a point $p \in \widehat{K}_U$ and a polydisc $\Delta = \Delta_r(0)$ with polyradius $r = (r_1,\ldots,r_n)$ such that $p + \Delta = \Delta_r(p)$ contains a point of ∂U, but

$$K + \Delta = \bigcup_{q \in K} \Delta_r(q) \subset\subset U.$$



If $f \in \mathscr{O}(U)$, then there is an $M > 0$ such that $|f| \leq M$ on $K + \Delta$ as that is a relatively compact set. By the Cauchy estimates for each $q \in K$ we get

$$\left| \frac{\partial^\alpha f}{\partial z^\alpha}(q) \right| \leq \frac{M\alpha!}{r^\alpha}.$$

This inequality therefore holds on $\widehat{K}_U$ and hence at p. The series

$$\sum_\alpha \frac{1}{\alpha} \frac{\partial^\alpha f}{\partial z^\alpha}(p)(z-p)^\alpha$$

converges in $\Delta_r(p)$. Hence f extends to all of $\Delta_r(p)$ and $\Delta_r(p)$ contains points outside of U, in other words, U is not a domain of holomorphy.

The implication (ii) $\Rightarrow$ (iii) is easy.

Finally let us prove (iii) $\Rightarrow$ (i). Suppose U is holomorphically convex. Let $p \in \partial U$. By convexity we choose nested compact sets $K_{j-1} \subsetneq K_j \subset\subset U$ such that $\bigcup_j K_j = U$, and $\widehat{(K_j)}_U = K_j$. We pick the sequence of K_j in such a way that there exists a sequence of points $p_j \in K_j \setminus K_{j-1}$ such that $\lim_{j\to\infty} p_j = p$.

Since p_j is not in the hull of K_{j-1}, we find a function $f_j \in \mathscr{O}(U)$ such that $|f_j| < 2^{-j}$ on K_{j-1}, but such that

$$|f_j(p_j)| > j + \left| \sum_{k=1}^{j-1} f_k(p_j) \right|.$$

Finding such a function is left as an exercise below. The series $\sum_{k=1}^\infty f_k(z)$ converges uniformly on K_j as for all $k > j$, $|f_k| < 2^{-k}$ on K_j. As the K_j exhaust U the series converges uniformly on compact subsets of U. Consequently,

$$f(z) = \sum_{k=1}^\infty f_k(z)$$

is a holomorphic function on U. We bound

$$|f(p_j)| \geq |f_j(p_j)| - \left|\sum_{k=1}^{j-1} f_k(p_j)\right| - \left|\sum_{k=j+1}^{\infty} f_k(p_j)\right| \geq j - \sum_{k=j+1}^{\infty} 2^{-k} \geq j-1.$$

So $\lim_{j\to\infty} f(p_j) = \infty$. Clearly there cannot be any open $W \subset \mathbb{C}^n$ containing p to which f extends (see definition of domain of holomorphy). As any connected open W such that $W \setminus U \neq \emptyset$ must contain a point of ∂U, we are done. □

Exercise 2.6.6 says that holomorphic convexity is a biholomorphic invariant. Therefore, being a domain of holomorphy is also a biholomorphic invariant. This fact is not easy to prove directly from the definition of a domain of holomorphy, as the biholomorphism is only defined on the interior of our domains.

Holomorphic convexity is an intrinsic notion; that is, it does not require knowing anything about points outside of U. Therefore, it is a much better way to think about domains of holomorphy. In fact holomorphic convexity generalizes easily to more complicated complex manifolds*, while the notion of a domain of holomorphy only makes sense in $\mathbb{C}^n$.

Exercise* 2.6.11*: *Find the function $f_j \in \mathscr{O}(U)$ as indicated in the proof above.*

Exercise* 2.6.12*: *Extend the proof to show that if $U \subset \mathbb{C}^n$ is holomorphically convex then there exists a single function $f \in \mathscr{O}(U)$, that does not extend through any point $p \in \partial U$.*

*Manifolds with complex structure, that is, "manifolds with multiplication by i on the tangent space".

Chapter 3

CR geometry

3.1 Real analytic functions and complexification

Definition 3.1.1. Let $U \subset \mathbb{R}^n$ be open. A function $f: U \to \mathbb{C}$ is said to be *real-analytic* (sometimes just *analytic* if clear from context) if at each point $p \in U$, the function f has a convergent power series that converges (absolutely) to f in some neighborhood of p. A common notation for real-analytic is C^ω.

Before discuss the connection to holomorphic functions let us prove a simple lemma.

Lemma 3.1.2. *Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion and suppose $U \subset \mathbb{C}^n$ is a domain such that $U \cap \mathbb{R}^n \neq \emptyset$. Suppose $f, g: U \to \mathbb{C}$ be holomorphic functions such that $f = g$ on $U \cap \mathbb{R}^n$. Then $f = g$ on all of U.*

Proof. By taking $f - g$ we can assume that $g = 0$. Let $z = x + iy$ as usual and $\mathbb{R}^n$ is given by $y = 0$, and let us assume that $f = 0$ on $y = 0$. At every point $p \in \mathbb{R}^n \cap U$,

$$0 = \frac{\partial f}{\partial x_j} = -i \frac{\partial f}{\partial y_j}.$$

Therefore, on $y = 0$,

$$\frac{\partial f}{\partial z_j} = 0.$$

Since the holomorphic function $\frac{\partial f}{\partial z_j} = 0$ on $y = 0$, then by induction all derivatives of f at p vanish, it has a zero power series. Hence f is identically zero in a neighborhood of p in $\mathbb{C}^n$ and by the identity theorem it is zero on all of U. $\square$

Let us return to $\mathbb{R}^n$ for a moment. We write a power series in $\mathbb{R}^n$ in multinomial notation as usual. Suppose that for some $a \in \mathbb{R}^n$ and some polyradius $r = (r_1, \ldots, r_n)$, the series

$$\sum_\alpha c_\alpha (x - a)^\alpha$$

converges whenever $|x_j - a_j| \leq r_j$ for all j. Here convergence is absolute convergence. That is,

$$\sum_\alpha |c_\alpha| |x-a|^\alpha$$

converges. If we replace $x_j \in \mathbb{R}$ with $z_j \in \mathbb{C}$ such that $|z_j - a_j| \leq |x_j - a_j|$, then the series still converges. Hence the series

$$\sum_\alpha c_\alpha (z-a)^\alpha$$

converges absolutely in $\Delta_r(a) \subset \mathbb{C}^n$.

Proposition 3.1.3 (Complexification part I)**.** *Suppose $U \subset \mathbb{R}^n$ is a domain and $f : U \to \mathbb{C}$ is real-analytic. Let $\mathbb{R}^n \subset \mathbb{C}^n$ be the natural inclusion. Then there exists a domain $V \subset \mathbb{C}^n$ such that $U \subset V$ and a unique holomorphic function $F : V \to \mathbb{C}$ such that $F|_U = f$.*

In particular, among many other things that follow from this proposition, we can now conclude that a real-analytic function is C^∞. Be careful and notice that U is a domain in $\mathbb{R}^n$, but it is not an open set when considered as a subset of $\mathbb{C}^n$. Furthermore, V may be a very “thin” neighborhood around U. There is no way of finding V just from knowing U. You need to also know f. As an example, consider $f(x) = \frac{1}{\varepsilon^2 + x^2}$ for $\varepsilon > 0$, which is real-analytic on $\mathbb{R}$, but the complexification is not holomorphic at $\pm \varepsilon i$.

Proof. We already proved the local version. But we must prove that if we extend our f near every point, we always get the same function. That follows from the lemma above; any two such functions are equal on $\mathbb{R}^n$, and hence equal. There is a subtle topological technical point in this, so let us elaborate. A key topological fact is that we define V as a union of the polydiscs where the series converges. If there was a point where we get two distinct values, then this point must be in two distinct such polydiscs. The intersection of two polydiscs is always connected, so we can apply the lemma above. ∎

Recall that a polynomial $P(x)$ in n real variables $(x_1, \ldots, x_n)$ is homogeneous of degree d if $P(sx) = s^d P(x)$ for all $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$. That is, a homogeneous polynomial of degree d is a polynomial whose every monomial is of total degree d. If f is real-analytic near $a \in \mathbb{R}^n$, then write the power series of f at a as

$$\sum_{j=0}^\infty f_j(x-a),$$

where f_j is a homogeneous polynomial of degree j. The f_j is then called the *degree d homogeneous part* of f at a.

When dealing with real-analytic functions in $\mathbb{C}^n$, there is usually a better way to complexify. Suppose $U \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, and suppose $f : U \to \mathbb{C}$ is real-analytic. Let us assume that $a = 0$ for simplicity. Writing $z = x + iy$,

$$f(x,y) = \sum_{j=0}^\infty f_j(x,y) = \sum_{j=0}^\infty f_j \left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i} \right).$$

The polynomial f_j becomes a homogeneous polynomial of degree j in the variables z and $\bar{z}$. Therefore the entire series becomes a power series in z and $\bar{z}$. As we mentioned before, we simply write $f(z,\bar{z})$, and when we consider the power series representation it will be in z and $\bar{z}$ rather than in x and y. In multinomial notation we write a power series at $a \in \mathbb{C}^n$ as

$$\sum_{\alpha,\beta} c_{\alpha,\beta}(z-a)^{\alpha}(\bar{z}-\bar{a})^{\beta}.$$

Notice that a holomorphic function is real-analytic, but not vice-versa. A holomorphic function is a real-analytic function that does not depend on $\bar{z}$.

Before we discuss complexification in terms of z and $\bar{z}$ we need the following lemma.

Lemma 3.1.4. *Let $U \subset \mathbb{C}^n \times \mathbb{C}^n$ be a domain, let the coordinates be $(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n$, let*

$$D = \{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C}^n : \zeta = \bar{z}\},$$

and suppose $D \cap U \neq \emptyset$. Suppose $f,g \colon U \to \mathbb{C}$ be holomorphic functions such that $f = g$ on $D \cap U$. Then $f = g$ on all of U.

The set D is sometimes called the *diagonal*.

Proof. Again assume without loss of generality $g = 0$. Whenever $(z,\bar{z}) \in U$ we have $f(z,\bar{z}) = 0$, that is really f composed with the map that takes z to $(z,\bar{z})$. Using the chain rule

$$0 = \frac{\partial}{\partial \bar{z}_j}\big(f(z,\bar{z})\big) = \frac{\partial f}{\partial \zeta_j}(z,\bar{z}).$$

Let us do this again with the z_j

$$0 = \frac{\partial}{\partial z_j}\big(f(z,\bar{z})\big) = \frac{\partial f}{\partial z_j}(z,\bar{z}).$$

Each time we get another holomorphic function that is zero on D. By induction, for all α and β we get

$$0 = \frac{\partial^{|\alpha|+|\beta|}}{\partial z^{\alpha}\partial \bar{z}^{\beta}}\big(f(z,\bar{z})\big) = \frac{\partial^{|\alpha|+|\beta|} f}{\partial z^{\alpha}\partial \zeta^{\beta}}(z,\bar{z}).$$

Therefore all holomorphic derivatives of f are zero on every point $(z,\bar{z})$. So f must be identically zero in a neighborhood of any point $(z,\bar{z})$. The lemma follows by the identity theorem. $\square$

Let f be a real-analytic function. Suppose the series (in multinomial notation)

$$f(z,\bar{z}) = \sum_{\alpha,\beta} c_{\alpha,\beta}(z-a)^{\alpha}(\bar{z}-\bar{a})^{\beta}$$

converges in a polydisc $\Delta_r(a) \subset \mathbb{C}^n$. By convergence we mean absolute convergence as we discussed before: that is,

$$\sum_{\alpha,\beta} |c_{\alpha,\beta}| |z-a|^{\alpha} |\bar{z}-\bar{a}|^{\beta}$$

converges. Therefore the series still converges if we replace $\bar{z}_j$ with ζ_j where $|\zeta_j - \bar{a}| \leq |\bar{z}_j - \bar{a}|$. So the series

$$F(z,\zeta) = \sum_{\alpha,\beta} c_{\alpha,\beta} (z-a)^{\alpha} (\zeta - \bar{a})^{\beta}$$

converges for all $(z,\zeta) \in \Delta_r(a) \times \Delta_r(\bar{a})$.

Putting together the above with the lemma we obtain.

Proposition 3.1.5 (Complexification part II). *Suppose $U \subset \mathbb{C}^n$ is a domain and $f\colon U \to \mathbb{C}$ is real-analytic. Then there exists a domain $V \subset \mathbb{C}^n \times \mathbb{C}^n$ such that*

$$\{(z,\zeta) : \zeta = \bar{z} \text{ and } z \in U\} \subset V,$$

and a unique holomorphic function $F\colon V \to \mathbb{C}$ such that $F(z,\bar{z}) = f(z,\bar{z})$ for all $z \in U$.

The function f can be thought of as the restriction of F to the set where $\zeta = \bar{z}$. We will abuse notation and write simply $f(z,\zeta)$ both for f and its extension.

Remark 3.1.6. The domain V above is not simply U times the conjugate of U. In general it is much smaller. For example a real-analytic $f\colon \mathbb{C}^n \to \mathbb{C}$ does not necessarily complexify to all of $\mathbb{C}^n \times \mathbb{C}^n$. That is because the domain of convergence for a real-analytic function on $\mathbb{C}^n$ is not necessarily all of $\mathbb{C}^n$. For example, in one dimension the function

$$f(z,\bar{z}) = \frac{1}{1+|z|^2}$$

is real-analytic on $\mathbb{C}$, but it is not a restriction to the diagonal of a holomorphic function on all of $\mathbb{C}^2$. The problem is that the complexified function

$$f(z,\zeta) = \frac{1}{1+z\zeta}$$

cannot be defined along the set where $z\zeta = -1$, which by a fluke never happens when $\zeta = \bar{z}$.

Remark 3.1.7. This form of complexification is sometimes called *polarization* due to its relation to the polarization identities*. That is, suppose A is a Hermitian matrix, we recover A and therefore the sesquilinear form $\langle Az, w \rangle$ for $z,w \in \mathbb{C}^n$, by simply knowing the values of

$$\langle Az, z \rangle = z^* A z = \sum_{j,k=1}^{n} a_{jk} \bar{z}_j z_k$$

for all $z \in \mathbb{C}^n$. In fact, under the hood Proposition 3.1.5 is polarization in an infinite dimensional Hilbert space, but we digress.

*Such as $4\langle z,w \rangle = \|z+w\|^2 - \|z-w\|^2$.

The idea of treating $\bar{z}$ as a separate variable is very powerful, and as we have just seen it is completely natural when speaking about real-analytic functions. This is one of the reasons why real-analytic functions play a special role in several complex variables.

Example 3.1.8: Not every C^∞ smooth function is real-analytic. For example, on the real line

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function $f : \mathbb{R} \to \mathbb{R}$ is C^∞, $f^{(k)}(0) = 0$ for all k, and so its Taylor series at the origin does not converge to f in any neighborhood of the origin. It converges to the zero function but not to f.

> ***Exercise* 3.1.1*:*** *Prove the statements of the above example.*

Definition 3.1.9. A real hypersurface $M \subset \mathbb{R}^n$ is said to be real-analytic if locally at every point it is the graph of a real-analytic function. That is near every point (that is, locally), after perhaps a rotation M can be written as

$$y = \varphi(x),$$

where φ is real-analytic.

Compare this definition to Definition 2.2.1. In fact we could define a real-analytic hypersurface as in Definition 2.2.1 and then prove an analogue of Lemma 2.2.5 to show that this would be identical to the definition above. The definition we gave will be sufficient and so we avoid the complication and leave it to the interested reader.

> ***Exercise* 3.1.2*:*** *Show that the definition above is equivalent to an analogue of Definition 2.2.1. That is, state the alternative definition of real-analytic hypersurface and then prove the analogue of Lemma 2.2.5.*

A mapping to $\mathbb{R}^m$ is real-analytic if all the components are real-analytic functions. Via complexification we give a simple proof of the following result.

Proposition 3.1.10. *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ be domains and let $f : U \to V$ and $g : V \to \mathbb{R}^m$ be real-analytic. Then $g \circ f$ is real-analytic.*

Proof. Let $x \in \mathbb{R}^n$ be our coordinates in U and $y \in \mathbb{R}^k$ be our coordinates in V. We complexify $f(x)$ and $g(y)$ by allowing x to be a complex vector in a small neighborhood of U in $\mathbb{C}^n$ and y to be a complex vector in a small neighborhood of V in $\mathbb{C}^k$. So we treat f and g as holomorphic functions. On a certain neighborhood of U in $\mathbb{C}^n$, the composition $f \circ g$ makes sense and it is holomorphic as composition of holomorphic mappings is holomorphic. Restricting the complexified $f \circ g$ back to $\mathbb{R}^n$ we obtain a real-analytic function. $\square$

The proof demonstrates a very simple application of complexification. Many properties of holomorphic functions are easy to prove because holomorphic functions are solutions to certain PDE (the Cauchy-Riemann equations). However, there is no PDE that defines real-analytic functions, so complexification provides a useful tool to transfer certain properties of holomorphic functions to real-analytic functions. We must be careful however, hypotheses on real-analytic functions only give us hypotheses on certain points of the complexified holomorphic functions.

***Exercise* 3.1.3:** *Suppose $\varphi \colon U \to \mathbb{R}$ is a pluriharmonic function for $U \subset \mathbb{C}^n$. Knowing that φ is real-analytic, let $z_0 \in U$ be fixed. Using complexification, write down a formula for a holomorphic function near z_0 whose real part is φ.*

3.2 CR functions

We first need to know what it means for a function $f \colon X \to \mathbb{C}$ to be smooth if X is not an open set, for example a hypersurface.

Definition 3.2.1. Let $X \subset \mathbb{R}^n$ be a set. The function $f \colon X \to \mathbb{C}$ is smooth (resp. real-analytic) if for each point $p \in X$ there is a neighborhood $U \subset \mathbb{R}^n$ of p and a smooth (resp. real-analytic) $F \colon U \to \mathbb{C}$ such that $F(q) = f(q)$ for $q \in X \cap U$.

For an arbitrary set X, issues surrounding this definition can be very subtle. It is very natural, however, if X is nice, such as a hypersurface, or if X is a closure of a domain with smooth boundary.

Proposition 3.2.2. *If $M \subset \mathbb{R}^n$ is a smooth (resp. real-analytic) real hypersurface, then $f \colon M \to \mathbb{C}$ is smooth (resp. real-analytic) if and only if whenever near some point we write M as*

$$y = \varphi(x)$$

for a smooth (resp. real-analytic) function φ, then the function $f\big(x, \varphi(x)\big)$ is a smooth (resp. real-analytic) function of x.

***Exercise* 3.2.1:** *Prove the proposition above.*

***Exercise* 3.2.2:** *Prove that in the definition if X is a smooth or real-analytic hypersurface, then the function F from the definition is never unique, even for a fixed neighborhood U.*

Definition 3.2.3. Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface. Then a smooth function $f \colon M \to \mathbb{C}$ is a *smooth CR function* if

$$X_p f = 0$$

for all $p \in M$ and all vectors $X_p \in T_p^{(0,1)} M$.

Remark 3.2.4. Of course one only needs one-derivative in the above definition. One can also define a continuous CR function if the derivative is taken in the distribution sense, but we digress.

Remark 3.2.5. When $n = 1$, a real hypersurface $M \subset \mathbb{C}$ is a curve and $T_p^{(0,1)}M$ is trivial. Therefore, all functions are CR functions.

Proposition 3.2.6. *Let $M \subset U$ be a smooth (resp. real-analytic) real hypersurface in a domain $U \subset \mathbb{C}^n$. Suppose $F: U \to \mathbb{C}$ is a holomorphic function, then the restriction $f = F|_M$ is a smooth (resp. real-analytic) CR function.*

Proof. First let us prove that f is smooth. Given any $p \in M$ write M as $\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)$ for a smooth φ. Then $F\big(z, \operatorname{Re} w + i\varphi(z, \bar{z}, \operatorname{Re} w)\big)$ is clearly smooth as it is a composition of smooth functions. If both M and f are real-analytic we obtain that $F\big(z, \operatorname{Re} w + i\varphi(z, \bar{z}, \operatorname{Re} w)\big)$ is real-analytic, which we could also prove directly by complexifying as before.

Let us show it is CR. We have $X_p F = 0$ for all $X_p \in T_p^{(0,1)}\mathbb{C}^n$. As $T_p^{(0,1)}M \subset T_p^{(0,1)}\mathbb{C}^n$ we have $X_p f = 0$ for all $X_p \in T_p^{(0,1)}M$. $\square$

Not every smooth CR function is a restriction of a holomorphic function.

Example 3.2.7: Take the smooth function $f: \mathbb{R} \to \mathbb{R}$ we defined before that is not real-analytic at the origin. Take $M \subset \mathbb{C}^2$ be the set defined by $\operatorname{Im} z_2 = 0$, this is a real-analytic real hypersurface. Clearly $T_p^{(0,1)}M$ is one complex dimensional and at each point $\frac{\partial}{\partial \bar{z}_1}$ is tangent and therefore spans $T_p^{(0,1)}M$. Define $g: M \to \mathbb{C}$ by

$$g(z_1, z_2, \bar{z}_1, \bar{z}_2) = f(\operatorname{Re} z_2).$$

Then g is CR as it is independent of $\bar{z}_1$. If $G: U \subset \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function where U is some open set containing the origin, then G restricted to M must be real-analytic (a power series in $\operatorname{Re} z_1$, $\operatorname{Im} z_1$, and $\operatorname{Re} z_2$) and therefore G cannot equal to g on M.

***Exercise* 3.2.3:** *Suppose $M \subset \mathbb{C}^n$ is a smooth hypersurface and $f: M \to \mathbb{C}$ a CR function that is a restriction of a holomorphic function $F: U \to \mathbb{C}$ defined in some neighborhood $U \subset \mathbb{C}^n$ of M. Show that F is unique, that is if $G: U \to C$ is another holomorphic function such that $G|_M = f = F|_M$, then $G = F$.*

***Exercise* 3.2.4:** *Show that there is no maximum principle of CR functions. In fact, find a smooth hypersurface $M \subset \mathbb{C}^n$, $n \geq 2$, and a smooth CR function f on M such that $|f|$ attains a strict maximum at a point.*

***Exercise* 3.2.5:** *Suppose $M \subset \mathbb{C}^n$, $n \geq 2$, is the hypersurface given by $\operatorname{Im} z_n = 0$. Show that any smooth CR function on M is holomorphic in the variables $z_1, \ldots, z_{n-1}$. Use this to show that for no smooth CR function f on M can $|f|$ attain a* strict *maximum on M. But show that there do exist functions such that $|f|$ attains a (nonstrict) maximum M.*

Real-analytic CR functions on a real-analytic hypersurface M always extend to holomorphic functions of a neighborhood of M. Before we prove that fact, let us find a convenient way to write the defining equation for a real-analytic hypersurface.

Proposition 3.2.8. *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface and $p \in M$. Then there are holomorphic coordinates near p taking p to 0, such that locally M is given by*

$$\bar{w} = \Phi(z, \bar{z}, w),$$

for a holomorphic function Φ defined on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1} \times \mathbb{C}$ that is $O(2)$ at the origin. Furthermore, a local basis for $T^{(0,1)}M$ vector fields is given by

$$\frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{w}}, \qquad j = 1, \ldots, n-1.$$

Proof. Find coordinates such that M is given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w).$$

Write the defining function as $r(z,w,\bar{z},\bar{w}) = -(1/2i)(w - \bar{w}) + \varphi\big(z, \bar{z}, (1/2)(w + \bar{w})\big)$. Complexifying, write $r(z,w,\zeta,\omega)$ as a holomorphic function of $2n$ variables, and the derivative in ω (that is $\bar{w}$) does not vanish near the origin. Use the implicit function theorem for holomorphic functions to write $r = 0$ as

$$\omega = \Phi(z, \zeta, w).$$

Restricting to the diagonal, $\bar{w} = \omega$ and $\bar{z} = \zeta$, we get the result. The statement about the CR vector fields then follows since those vector fields annihilate the defining function $\Phi(z,\bar{z},w) - \bar{w}$. □

Proposition 3.2.9 (Severi)**.** *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface and $p \in M$. For any real-analytic CR function $f \colon M \to \mathbb{C}$, there exists a holomorphic function $F \in \mathscr{O}(U)$ for a neighborhood U of p such that $F|_{M \cap U} = f$.*

Proof. Write M near p as $\bar{w} = \Phi(z,\bar{z},w)$. Let $\mathscr{M}$ be the set in the $2n$ variables (z,w,ζ,ω) given by $\omega = \Phi(z,\zeta,w)$. Take $f(z,w,\bar{z},\bar{w})$ and consider any real-analytic extension to a neighborhood. Complexify as before to $f(z,w,\zeta,\omega)$. On $\mathscr{M}$ we have $f(z,w,\zeta,\omega) = f\big(z,w,\zeta,\Phi(z,\zeta,w)\big)$. Let

$$F(z,w,\zeta) = f\big(z,w,\zeta,\Phi(z,\zeta,w)\big).$$

Clearly $F(z,w,\bar{z})$ equals f on M. As f is a CR function, it is annihilated by $\frac{\partial}{\partial \bar{z}_j} + \frac{\partial \Phi}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{w}}$ on M. So

$$\frac{\partial F}{\partial \zeta_j} + \frac{\partial \Phi}{\partial \zeta_j} \frac{\partial F}{\partial \omega} = \frac{\partial F}{\partial \zeta_j} = 0$$

on the subset of $\mathscr{M}$ where $\bar{z} = \zeta$ (and $\bar{w} = \omega$). Therefore the equation holds on all of $\mathscr{M}$ or in other words for all z, ζ, w in a neighborhood. Consequently

$$\frac{\partial F(z,w,\bar{z})}{\partial \bar{z}_j} = 0$$

for all j, and F is actually a holomorphic function in z and w only. □

CR functions can often be considered as boundary values of holomorphic functions.

Proposition 3.2.10. *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary. Suppose $f : \overline{U} \to \mathbb{C}$ is smooth and holomorphic on U. Then $f|_{\partial U}$ is a smooth CR function.*

Proof. The function $f|_{\partial U}$ is clearly smooth.

Suppose $p \in \partial U$. If $X_p \in T_p^{(0,1)} \partial U$ is such that

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}\Big|_p ,$$

take $\{q_k\}$ in U that approaches p, then take

$$X_{q_k} = \sum_{j=1}^{n} a_j \frac{\partial}{\partial \bar{z}_j}\Big|_{q_k} .$$

If $X_{q_k} f = 0$ for all k and by continuity $X_p f = 0$. □

The boundary values of a holomorphic function define the function uniquely. In particular we have the following result you know from one variable.

Proposition 3.2.11. *Suppose $U \subset \mathbb{C}^n$ is a domain with smooth boundary and $f : \overline{U} \to \mathbb{C}$ is continuous and holomorphic on U. If $f = 0$ on an open subset of ∂U, then $f = 0$ on all of U.*

Proof. Take $p \in \partial U$ such that $f = 0$ on a neighborhood of p in ∂U. Near p write U as

$$\operatorname{Im} w > \varphi(z, \bar{z}, \operatorname{Re} w)$$

for $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, and where φ is $O(2)$. Next fix any small z. Considering f as a function of w defined on $\operatorname{Im} w \geq \varphi(z, \bar{z}, \operatorname{Re} w)$ we obtain from the corresponding one dimensional result that f is identically zero inside the domain. As this held for every fixed z it holds in an open set of U and by identity it holds everywhere. □

***Exercise* 3.2.6:** *Find a domain $U \subset \mathbb{C}^n$, $n \geq 2$, with smooth boundary and a smooth CR function f on ∂U such that there is no holomorphic function on U or $\mathbb{C}^n \setminus U$ whose boundary values are f.*

***Exercise* 3.2.7:** *a) Suppose $U \subset \mathbb{C}^n$ is a bounded domain with smooth boundary. Suppose $f : \overline{U} \to \mathbb{C}$ is a continuous function holomorphic in U. Suppose $f|_{\partial U}$ is real-valued. Show that f is constant. b) Find a counterexample to the statement if you allow U to be unbounded.*

***Exercise* 3.2.8:** *Find a smooth CR function on the sphere $S^{2n-1} \subset \mathbb{C}^n$ that is not a restriction of a holomorphic function of a neighborhood of S^{2n-1}.*

3.3 Approximation of CR functions

The following theorem (proved circa 1980) holds in much more generality, but we state its simplest version. One of the simplifications we make is that we consider only smooth CR functions here, although the theorem holds even for continuous CR functions where the CR conditions are interpreted in the sense of distributions.

Theorem 3.3.1 (Baouendi-Trèves)**.** *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface. Let $p \in M$ be fixed and let $z = (z_1, \dots, z_n)$ be holomorphic coordinates near p. Then there exists a compact neighborhood $K \subset M$ of p, such that for any smooth CR function $f : M \to \mathbb{C}$, there exists a sequence $\{p_j\}$ of polynomials such that*

$$p_j(z) \to f(z) \qquad \textit{uniformly in } K.$$

A key point is that K cannot be chosen arbitrarily, it depends on p and M. On the other hand it does not depend on f. So given M and $p \in M$ there is a K such that *every* CR function on M is approximated uniformly on K by polynomials. The theorem also applies in $n = 1$ (in one dimension), although in that case it follows from the more general Mergelyan theorem.

Example 3.3.2: Let us show that K cannot possibly be arbitrary. Let us give an example in one dimension. Let $S^1 \subset \mathbb{C}$ be the unit circle (boundary of the disc), then any smooth function on S^1 is a smooth CR function. So pick let us say a nonconstant real function such as $\operatorname{Re} z$. Let us suppose for contradiction that we could take $K = S^1$. Then $\operatorname{Re} z$ would be uniformly approximated by holomorphic polynomials on S^1. By the maximum principle, the polynomials would converge on $\mathbb{D}$ to a holomorphic function on $\mathbb{D}$ continuous on $\overline{\mathbb{D}}$ this function would have nonconstant real boundary values, which is impossible. Clearly K cannot be the entire circle.

The example is easily extended to $\mathbb{C}^n$ by considering $M = S^1 \times \mathbb{C}^{n-1}$, then $\operatorname{Re} z_1$ is a smooth CR function on M that cannot be approximated uniformly by holomorphic polynomials on $S^1 \times \{0\}$.

The technique of the above example will be used later in a more general situation, to extend CR functions using Baouendi-Trèves.

Remark 3.3.3. It is important to note the difference between Baouendi-Trèves (and similar theorems in complex analysis) and the Weierstrass approximation theorem. In Baouendi-Trèves we obtain approximation by holomorphic polynomials, while Weierstrass gives us polynomials in the real variables, or in z and $\bar{z}$. For example, via Weierstrass, any continuous function is uniformly approximable on S^1 via polynomials in $\operatorname{Re} z$ and $\operatorname{Im} w$, and therefore by polynomials in z and $\bar{z}$, but these polynomials will not in general converge anywhere but on S^1.

Exercise* 3.3.1*: *Let $z = x + iy$ as usual in $\mathbb{C}$. Find a sequence of polynomials in x and y that converge uniformly to e^{x-y} on S^1, but diverge everywhere else.*

The proof is an ingenious use of the standard technique used to prove the Weierstrass approximation theorem. Also, as we have seen mollifiers before, the technique will not be completely foreign even to the reader who does not know the Weierstrass approximation theorem. Basically what we do is use the standard convolution argument, this time against a holomorphic function. Letting $z = x + iy$ we only do the convolution in the x variables keeping $y = 0$. Then we use the fact that the function is CR to show that we get an approximation even for other y.

In the formulas below, given any vector $v = (v_1, \ldots, v_n)$, it will be useful to write

$$[v]^2 \overset{\text{def}}{=} v_1^2 + \cdots + v_n^2.$$

The following lemma is a neat application of ideas from several complex variables to solve a problem that does not at first seems to involve holomorphic functions.

Lemma 3.3.4. *Let W be the set of $n \times n$ complex matrices A such that*

$$\|(\operatorname{Im} A)x\| < \|(\operatorname{Re} A)x\|$$

for all nonzero $x \in \mathbb{R}^n$ and $\operatorname{Re} A$ is positive definite. Then for all $A \in W$,

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx = \pi^{n/2}.$$

Proof. Suppose A has real entries and A is positive definite (so A is also invertible). By a change of coordinates

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx = \int_{\mathbb{R}^n} e^{-[x]^2} \, dx = \left(\int_{\mathbb{R}} e^{-x_1^2} \, dx_1 \right) \cdots \left(\int_{\mathbb{R}} e^{-x_n^2} \, dx_n \right) = \left(\sqrt{\pi}\right)^n.$$

Next suppose A is any matrix in W. There is some $\varepsilon > 0$ such that $\|(\operatorname{Im} A)x\|^2 \leq (1 - \varepsilon^2)\|(\operatorname{Re} A)x\|^2$ for all $x \in \mathbb{R}^n$. That is because we only need to check this for x in the unit sphere, which is compact (exercise). Also note that by reality of $\operatorname{Re} A$, $\operatorname{Im} A$, and x we get $[(\operatorname{Re} A)x]^2 = \|(\operatorname{Re} A)x\|^2$ and $[(\operatorname{Im} A)x]^2 = \|(\operatorname{Im} A)x\|^2$.

$$\left| e^{-[Ax]^2} \right| \leq e^{-[(\operatorname{Re} A)x]^2 + [(\operatorname{Im} A)x]^2} \leq e^{-\varepsilon^2 [(\operatorname{Re} A)x]^2}.$$

Therefore the integral exists for all A in W.

The expression

$$\int_{\mathbb{R}^n} e^{-[Ax]^2} \det A \, dx$$

is a well-defined holomorphic function in the entries of A thinking of W as a domain (see exercises below) in $\mathbb{C}^{n^2}$. We have a holomorphic function that is constantly equal to $\pi^{n/2}$ on $W \cap \mathbb{R}^{n^2}$ and hence it is equal to $\pi^{n/2}$ everywhere on W. $\square$

***Exercise* 3.3.2:** *Prove the existence of $\varepsilon > 0$ in the proof above.*

***Exercise* 3.3.3:** *Show that $W \subset \mathbb{C}^{n^2}$ in the proof above is a domain (open and connected).*

***Exercise* 3.3.4:** *Prove that we can really differentiate under the integral to show that the integral is holomorphic in the entries of A.*

***Exercise* 3.3.5:** *Show that some hypotheses are needed for the lemma. In particular take $n = 1$ and find the exact set of A (now just a complex number) for which the theorem is true.*

Below for an $n \times n$ matrix A, we use the standard operator norm

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \sup_{v \in \mathbb{C}^n, v \neq 0} \frac{\|Av\|}{\|v\|}.$$

***Exercise* 3.3.6:** *Let W be as in Lemma 3.3.4. Let B be an $n \times n$ real matrix such that $\|B\| < 1$. Show that $I + iB \in W$.*

Proof of the theorem of Baouendi-Trèves. Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, and without loss of generality suppose $p = 0 \in M$. Let $z = (z_1, \ldots, z_n)$ be the holomorphic coordinates, write $z = x + iy$, $y = (y', y_n)$, and suppose M is given by

$$y_n = \psi(x, y'),$$

where ψ is $O(2)$. Note that (x, y') parametrize M near 0. In other words,

$$z_j = x_j + iy_j, \qquad \text{for } j < n, \text{ and } \qquad z_n = x_n + i\psi(x, y').$$

Write the mapping

$$\varphi(x, y') = \big(y_1, \ldots, y_{n-1}, \psi(x, y')\big).$$

We then write $z = x + i\varphi(x, y')$ as our parametrization.

Let $r > 0$ and $d > 0$ be small numbers to be determined later. We assume they are small enough such that f and φ are defined on some neighborhood of the set where $\|x\| \leq r$ and $\|y'\| \leq d$.

There exists a smooth function $g \colon \mathbb{R}^n \to [0, 1]$ such that $g \equiv 1$ on $B_{r/2}(0)$ and $g \equiv 0$ outside of $B_r(0)$. Explicit formula can be given, or we can also obtain such a function by use of mollifiers on the function that is identically one on $B_{3r/4}(0)$ and zero elsewhere. Such a g is commonly called a *cutoff function*.

***Exercise* 3.3.7:** *Find an explicit formula for g without using mollifiers.*

Let
$$K' = \{(x,y') : \|x\| \leq r/4, \|y'\| \leq d\}.$$
Let $K = z(K')$, that is the image of K' under the mapping $z(x,y')$, as we will think of z as a function of (x,y').

Let us consider the CR function f to be a function of (x,y') and write $f(x,y')$. For $\ell \in \mathbb{N}$, let α_ℓ be a differential n-form defined (thinking of $w \in \mathbb{C}^n$ as a constant parameter) by

$$\begin{aligned}\alpha_\ell(x,y') &= \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z]^2} g(x) f(x,y')\, dz \\ &= \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-x-i\varphi(x,y')]^2} g(x) f(x,y') \\ &\qquad\qquad (dx_1 + i dy_1) \wedge \cdots \wedge (dx_{n-1} + i dy_{n-1}) \wedge \big(dx_n + i d\psi(x,y')\big).\end{aligned}$$

The key is the exponential, which looks like the bump function mollifier, except that now we have w and z possibly complex. The exponential is holomorphic in w and that is key. And as long as we do not stray far in the y' direction it should go to zero quickly for $w \neq z$.

Fix y' with $0 < \|y'\| < d$ and let D be defined by

$$D = \{(x,s) \in \mathbb{R}^n \times \mathbb{R}^{n-1} : \|x\| < r \text{ and } s = ty' \text{ for } t \in (0,1)\}.$$

D is an $n+1$ dimensional "cylinder." That is, we take a ball in the x direction and then take a single fixed direction y'. We orient D in the standard way as if it sat in the (x,t) variables in $\mathbb{R}^n \times \mathbb{R}$. Via Stokes' theorem we get
$$\int_D d\alpha_\ell(x,s) = \int_{\partial D} \alpha_\ell(x,s).$$
Since $g(x) = 0$ if $\|x\| \geq r$ then

$$\begin{aligned}\int_{\partial D} \alpha_\ell(x,s) =& \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,y')]^2} g(x) f(x,y')\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,y')\big) \\ &- \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x\in\mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,0)]^2} g(x) f(x,0)\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x\psi(x,0)\big),\end{aligned} \tag{3.1}$$

where d_x means the derivative in the x directions only. That is, $d_x\psi = \frac{\partial\psi}{\partial x_1}dx_1 + \cdots + \frac{\partial\psi}{\partial x_n}dx_n$. As is usual in these types of arguments the integral extends to all of $\mathbb{R}^n$ because of g. We can ignore that f and φ are undefined where g is identically zero.

We will show that the left hand side of (3.1) goes to zero uniformly for $w \in K$ and the first term on the right hand side will go to $f(\tilde{x}, y')$ if $w = z(\tilde{x}, y')$ is in M. We define entire functions that we will show approximate f

$$f_\ell(w) = \left(\frac{\ell}{\pi}\right)^{n/2} \int_{x \in \mathbb{R}^n} e^{-\ell[w-x-i\varphi(x,0)]^2} g(x) f(x,0)\, dx_1 \wedge \cdots \wedge dx_{n-1} \wedge \big(dx_n + i d_x \psi(x,0)\big).$$

Clearly each f_ℓ is holomorphic and defined for all $w \in \mathbb{C}^n$.

In the next claim it is important that f is a CR function.

Claim 3.3.5. *We have*

$$d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s),$$

and for sufficiently small $r > 0$ and $d > 0$,

$$\lim_{\ell \to \infty} \left(\frac{\ell}{\pi}\right)^{n/2} \int_{(x,s) \in D} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s) = 0$$

uniformly as a function of $w \in K$ and $y' \in B_d(0)$ (recall that D depends on y').

Proof. First we claim that at each point df is a linear combination of dz_1 through dz_n (recall that we are considering f and $z_1, \ldots, z_n$ as functions on M). After a complex affine change of coordinates we simply need to show this at the origin. Let the new holomorphic coordinates be $\xi_1, \ldots, \xi_n$, and suppose the $T_0^{(1,0)} M$ tangent space is spanned by $\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_{n-1}}$, and such that $\frac{\partial}{\partial \operatorname{Re} \xi_n}$ is tangent and $\frac{\partial}{\partial \operatorname{Im} \xi_n}$ is normal. At the origin the CR conditions give us

$$df(0) = \frac{\partial f}{\partial \xi_1}(0)\, d\xi_1(0) + \cdots + \frac{\partial f}{\partial \xi_{n-1}}(0)\, d\xi_{n-1}(0) + \frac{\partial f}{\partial \operatorname{Re} \xi_n}(0)\, d(\operatorname{Re} \xi_n)(0).$$

But at the origin $d\xi_n(0) = d(\operatorname{Re} \xi_n)(0)$. As ξ is an affine function of z, then $d\xi_j$ are linear combinations of dz_1 through dz_n, an the claim follows. So for any CR function f we get that $d(f\, dz) = df \wedge dz = 0$ since $dz_j \wedge dz_j = 0$ of course.

The function $e^{-\ell[w-z(x,s)]^2}$ is CR as a function of (x,s), and so is $f(x,s)$. Therefore

$$d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s).$$

Since dg is zero for $\|x\| \leq {}^{r}/_{2}$, we get that

$$\int_D d\alpha_\ell(x,s) = \left(\frac{\ell}{\pi}\right)^{n/2} \int_D e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s)$$

is only evaluated for the subset of D where $\|x\| > {}^r\!/2$.

Suppose $w \in K$ and $(x,s) \in D$ with $\|x\| > {}^r\!/2$. We need to estimate

$$\left|e^{-\ell[w-z(x,s)]^2}\right| = e^{-\ell\,\mathrm{Re}\,[w-z(x,s)]^2}.$$

Let $w = z(\tilde{x},\tilde{s})$. Then

$$-\mathrm{Re}\,[w-z]^2 = -\|\tilde{x}-x\|^2 + \|\varphi(\tilde{x},\tilde{s}) - \varphi(x,s)\|^2.$$

By the mean value theorem

$$\|\varphi(\tilde{x},\tilde{s}) - \varphi(x,s)\| \leq \|\varphi(\tilde{x},\tilde{s}) - \varphi(x,\tilde{s})\| + \|\varphi(x,\tilde{s}) - \varphi(x,s)\| \leq a\|\tilde{x}-x\| + A\|\tilde{s}-s\|,$$

where a and A are

$$a = \sup_{\|\hat{x}\|\leq r, \|\hat{y}'\|\leq d} \left\| \frac{\partial \varphi}{\partial x}(\hat{x},\hat{y}') \right\|, \qquad A = \sup_{\|\hat{x}\|\leq r, \|\hat{y}'\|\leq d} \left\| \frac{\partial \varphi}{\partial y'}(\hat{x},\hat{y}') \right\|.$$

Here $\left[\frac{\partial \varphi}{\partial x}\right]$ and $\left[\frac{\partial \varphi}{\partial y'}\right]$ are the derivatives (matrices) of φ with respect to x and y' respectively, and the norm we are taking is the operator norm. Because $\left[\frac{\partial \varphi}{\partial x}\right]$ is zero at the origin, we pick r and d small enough (and hence K small enough) so that $a \leq {}^1\!/4$. We furthermore pick d possibly even smaller to ensure that $d \leq \frac{r}{32A}$. We have that ${}^r\!/2 \leq \|x\| \leq r$, but $\|\tilde{x}\| \leq {}^r\!/4$ (recall $w \in K$), so

$$\frac{r}{4} \leq \|\tilde{x}-x\| \leq \frac{5r}{4}.$$

We also have $\|\tilde{s}-s\| \leq 2d$ by triangle inequality.

Therefore,

$$\begin{aligned}
-\mathrm{Re}\,[w-z(x,s)]^2 &\leq -\|\tilde{x}-x\|^2 + a^2\|\tilde{x}-x\|^2 + A^2\|\tilde{s}-s\|^2 + 2aA\|\tilde{x}-x\|\|\tilde{s}-s\| \\
&\leq \frac{-15}{16}\|\tilde{x}-x\|^2 + A^2\|\tilde{s}-s\|^2 + \frac{A}{2}\|\tilde{x}-x\|\|\tilde{s}-s\| \\
&\leq \frac{-r^2}{64}.
\end{aligned}$$

In other words

$$\left|e^{-\ell[w-z(x,s)]^2}\right| \leq e^{-\ell r^2/64}$$

or

$$\left| \left(\frac{\ell}{\pi}\right)^{n/2} \int_{(x,s)\in D} e^{-\ell[w-z(x,s)]^2} f(x,s)\, dg(x) \wedge dz(x,s) \right| \leq C\ell^{n/2} e^{-\ell r^2/64}.$$

For a constant C. Do notice that D depends on y'. By looking at all y' with $\|y'\| \leq d$, which is a compact set, we can make C large enough to not depend on the y' that was chosen. The claim follows. □

Claim 3.3.6. *For the given $r > 0$ and $d > 0$,*

$$\lim_{\ell\to\infty}\left(\frac{\ell}{\pi}\right)^{n/2}\int_{x\in\mathbb{R}^n} e^{-\ell[\tilde{x}+i\varphi(\tilde{x},y')-x-i\varphi(x,y')]^2} g(x)f(x,y')dx_1\wedge\cdots\wedge dx_{n-1}\wedge\big(dx_n+id_x\psi(x,y')\big) = f(\tilde{x},y')$$

uniformly in $(\tilde{x},y')\in K'$.

That is, we look at (3.1) and we plug in $w = z(\tilde{x},y')\in K$. Notice that the g (as usual) makes sure we never evaluate f, ψ, or ϕ at points where they are not defined.

Proof. The change of variables formula implies

$$dx_1\wedge\cdots\wedge dx_{n-1}\wedge\big(dx_n+id_x\psi(x,y')\big) = d_x z(x,y') = \det\left[\frac{\partial z}{\partial x}(x,y')\right]dx, \tag{3.2}$$

where $\left[\frac{\partial z}{\partial x}(x,y)\right]$ is the matrix corresponding to the derivative of the mapping z with respect to the x variables evaluated at (x,y').

Let us change variables of integration via $\xi = \sqrt{\ell}(x-\tilde{x})$:

$$\begin{aligned}\left(\frac{\ell}{\pi}\right)^{n/2}&\int_{x\in\mathbb{R}^n} e^{-\ell[\tilde{x}+i\varphi(\tilde{x},y')-x-i\varphi(x,y')]^2} g(x)f(x,y')\det\left[\frac{\partial z}{\partial x}(x,y')\right]dx \\ &= \left(\frac{1}{\pi}\right)^{n/2}\int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2} \\ &\qquad g\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}}\right)f\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)\det\left[\frac{\partial z}{\partial x}\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)\right]d\xi.\end{aligned}$$

We now wish to take a limit as $\ell\to\infty$ and for this we need to apply the dominated convergence theorem. So we need to dominate the integrand.

As a function of ξ,

$$g\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}}\right)f\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)\det\left[\frac{\partial z}{\partial x}\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)\right]$$

is globally bounded independent of ℓ, because it has compact support and it is continuous.

Hence it is enough to worry about the exponential term. We also only need to consider those ξ where the integrand is not zero. Recall that r and d are small enough that

$$\sup_{\|\hat{x}\|\leq r,\|\hat{y}'\|\leq d}\left\|\frac{\partial\varphi}{\partial x}(\hat{x},\hat{y}')\right\|\leq\frac{1}{4},$$

and as $\|\tilde{x}\| \leq {}^{r}/_{4}$ (as $(\tilde{x},y') \in K$) and $\left\|\tilde{x}+\frac{\xi}{\sqrt{\ell}}\right\| \leq r$ (because g is zero otherwise) then

$$\left\|\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right\| \leq \frac{1}{4}\left\|\tilde{x}+\frac{\xi}{\sqrt{\ell}}-\tilde{x}\right\| = \frac{\|\xi\|}{4\sqrt{\ell}}.$$

So under the same conditions we have

$$\begin{aligned}\left|e^{-\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2}\right| &= e^{-\operatorname{Re}\left[\xi+i\sqrt{\ell}\left(\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right)\right]^2} \\ &= e^{-\|\xi\|^2+\ell\left\|\varphi\left(\tilde{x}+\frac{\xi}{\sqrt{\ell}},y'\right)-\varphi(\tilde{x},y')\right\|^2} \\ &\leq e^{-(15/16)\|\xi\|^2}.\end{aligned}$$

Therefore we can take the pointwise limit under the integral to obtain

$$\left(\frac{1}{\pi}\right)^{n/2}\int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]\xi\right]^2} g(\tilde{x})f(\tilde{x},y')\det\left[\frac{\partial z}{\partial x}(\tilde{x},y')\right]d\xi.$$

Notice how in the exponent we actually had an expression for the derivative in the ξ direction with y' fixed. If $(\tilde{x},y') \in K'$, then $g(\tilde{x}) = 1$ and so we can ignore g.

Letting $A = I + i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]$. Then using Lemma 3.3.4 we obtain

$$\left(\frac{1}{\pi}\right)^{n/2}\int_{\xi\in\mathbb{R}^n} e^{-\left[\xi+i\left[\frac{\partial\varphi}{\partial x}(\tilde{x},y')\right]\xi\right]^2} f(\tilde{x},y')\det\left[\frac{\partial z}{\partial x}(\tilde{x},y')\right]d\xi = f(\tilde{x},y').$$

The convergence of the integrand is pointwise in ξ but uniform in $(\tilde{x},y') \in K'$. That is left as an exercise. Hence the limit of the integrals converges uniformly in $(\tilde{x},y') \in K'$ and we are done. □

Exercise* 3.3.8*: *In the proof of the above claim, show that for a fixed ξ, the integrand converges uniformly in $(\tilde{x},y') \in K'$.*

We are essentially done with the proof of the theorem. The two claims together with (3.1) show that f_ℓ are entire holomorphic functions that approximate f uniformly on K. Entire holomorphic functions can be approximated by polynomials uniformly on compact subsets; simply take the partial sums of Taylor series at the origin. □

Exercise* 3.3.9*: *Explain why being approximable on K by (holomorphic) polynomials does not necessarily mean that f is real-analytic.*

Exercise* 3.3.10*: *Suppose $M \subset \mathbb{C}^n$ is given by $\operatorname{Im} z_n = 0$. Use the standard Weierstrass approximation theorem to show that for any $K \subset\subset M$ an arbitrary CR function $f\colon M \to \mathbb{C}$ can be uniformly approximated by holomorphic polynomials on K.*

3.4 Extension of CR functions

We will now apply the so-called "technique of analytic discs" together with Baouendi-Trèves to prove the Lewy extension theorem. Lewy's original proof was different and predates Baouendi-Trèves. A local extension theorem of this type was first proved by Helmut Knesser in 1936.

Theorem 3.4.1 (Lewy)**.** *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface and $p \in M$. There exists a small neighborhood U of p with the following property. Suppose $r\colon U \to \mathbb{R}$ is a defining function for $M \cap U$, denote by $U_- \subset U$ the set where r is negative and $U_+ \subset U$ the set where r is positive. Let $f\colon M \to \mathbb{R}$ be a smooth CR function. Then:*

(i) If the Levi-form with respect to r has a positive eigenvalue at p, then f extends to a holomorphic function on U_- continuous up to M.

(ii) If the Levi-form with respect to r has a negative eigenvalue at p, then f extends to a holomorphic function on U_+ continuous up to M.

(iii) If the Levi-form with respect to r has eigenvalues of both signs at p, then any smooth CR f extends to a function holomorphic on U.

In particular, note that if the Levi-form has eigenvalues of both signs, then near p the CR function is in fact a restriction of a holomorphic function on all of U. The function r can really be any defining function for M, either one can extend it to all of U or we could take a smaller U such that r is defined on U. As we have noticed before, once we pick sides (where r is positive and where it is negative), then the number of positive eigenvalues and the number of negative eigenvalues of the Levi-form is fixed. Taking a different r can at most flip U_- and U_+, but the conclusion of the theorem is exactly the same.

Proof. Without loss of generality, it is enough to suppose the Levi-form has one positive eigenvalue to prove the first two items, otherwise just take $-r$. Suppose $p = 0$ and M is given in some neighborhood Ω of the origin as

$$\operatorname{Im} w = |z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w),$$

where $\varepsilon_j = -1, 0, 1$, E is $O(3)$, and $z' = (z_2, \ldots, z_{n-1})$. Let Ω_- be given by

$$0 > r = |z_1|^2 + \sum_{j=2}^{n-1} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w) - \operatorname{Im} w.$$

The (real) Hessian of the function

$$z_1 \mapsto |z_1|^2 + E(z_1, 0, \bar{z}_1, 0, 0)$$

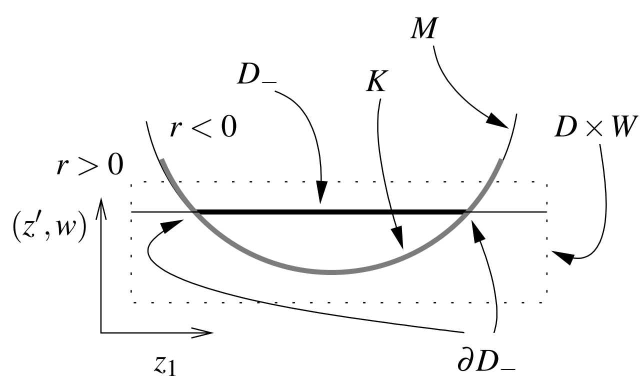
is positive definite in an entire neighborhood of the origin and the function has a strict minimum at 0. There is some small disc $D \subset \mathbb{C}$ such that this function is strictly positive on ∂D. We can also assume that the Hessian is positive definite on the closed disc $\overline{D}$.

Therefore, for $(z', w) \in W$ in some small neighborhood W of the origin in $\mathbb{C}^{n-1}$, the function

$$z_1 \mapsto |z_1|^2 + \sum_{j=2}^{n} \varepsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w) - \operatorname{Im} w$$

has a positive definite Hessian (as a function of z_1 only) on D and it is still strictly positive on ∂D.

We wish to apply Baouendi-Trèves and so let K be the compact neighborhood of the origin from the theorem. Take D and W small enough such that $(D \times W) \cap M \subset K$. Find the polynomials p_j that approximate f uniformly on K. Take $z_1 \in D$ and fix $(z', w) \in W$ such that $(z_1, z', w) \in \Omega_-$. Let $D_- = D \times \{(z', w)\} \cap \Omega_-$. Denote by ∂D_- the boundary of D_- in the subspace topology of $\mathbb{C} \times \{(z', w)\}$. Then we find that $r = 0$ exactly ∂D_-. We know this because r is positive on $(\partial D) \times \{(z', w)\}$ and hence $r = 0$ on ∂D_-. As $D \times W \cap M \subset K$, we have that $\partial D_- \subset K$.



As $p_j \to f$ uniformly on K then $p_j \to f$ uniformly on ∂D_-. As p_j are holomorphic, then by maximum principle p_j converge uniformly on all of D_-. In fact, as (z_1, z', w) was an arbitrary point in $(D \times W) \cap \Omega_-$, the polynomials p_j converge uniformly on $(D \times W) \cap \{\overline{\Omega_-}\}$. Let $U = D \times W$, then $U_- = (D \times W) \cap \Omega_-$. Notice U depends on K, but not on f. So p_j converge to a continuous function F on $\overline{U_-}$ and F is holomorphic on U_-. Clearly F equals f on $M \cap U$.

To prove the last item, pick a side, and then use one of the first two items to extend the function to that side. Via the tomato can principle (Theorem 2.3.10) the function also extends across M and therefore to a whole neighborhood of p. □

We state the next corollary for a strongly convex domain, even though it holds with far more generality. We will prove it for strongly pseudoconvex domains. In fact, a bounded domain with smooth boundary and connected complement will work without any assumptions on the Levi-form, but a different approach would have to be taken.

Corollary 3.4.2. *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with smooth boundary that is strongly convex and $f \colon \partial U \to \mathbb{C}$ is a smooth CR function, then there exists a continuous function $F \colon \overline{U} \to \mathbb{C}$ holomorphic in U such that $F|_{\partial U} = f$.*

Proof. A strongly convex domain is strongly pseudoconvex, so f must extend to the inside locally near any point. The extension is locally unique as any two extensions have the same boundary values. Therefore, there exists a set $K \subset\subset U$ such that f extends to $U \setminus K$. Via an exercise below we can assume that K is strongly convex and therefore we can apply the special case of Hartogs phenomenon that you proved in Exercise 2.1.6 to find an extension holomorphic in U. □

Exercise* 3.4.1*: *Prove the existence of the strongly convex K above.*

Exercise* 3.4.2*: *Show by example that the corollary is not true when $n = 1$. Explain where in the proof have we used that $n \geq 2$.*

Exercise* 3.4.3*: *Suppose $f \colon \partial\mathbb{B}_2 \to \mathbb{C}$ is a smooth CR function. Write down an explicit formula for the extension F.*

Exercise* 3.4.4*: *If $M \subset \mathbb{C}^3$ is defined by $\operatorname{Im} w = |z_1|^2 - |z_2|^2 + O(3)$ defines a smooth hypersurface and f is a real-valued smooth CR function on M. Show that $|f|$ does not attain a maximum at the origin.*

Exercise* 3.4.5*: *Suppose $M \subset \mathbb{C}^n$, $n \geq 3$, is a real-analytic hypersurface such that the Levi-form at $p \in M$ has eigenvalues of both signs. Show that every smooth CR function f on M is in fact real-analytic in a neighborhood of p.*

Exercise* 3.4.6*: *Let $M \subset \mathbb{C}^3$ be defined by $\operatorname{Im} w = |z_1|^2 - |z_2|^2$. a) Show that an arbitrary compact subset $K \subset\subset M$ will work for the conclusion Baouendi-Trèves. b) Use this to show that every smooth CR function $f \colon M \to \mathbb{C}$ is a restriction of an entire holomorphic function $F \colon \mathbb{C}^3 \to \mathbb{C}$.*

Exercise* 3.4.7*: *Find an $M \subset \mathbb{C}^n$, $n \geq 2$, such that near some $p \in M$, for every neighborhood W of p in M. There is a CR function $f \colon W \to \mathbb{C}$ that does not extend to either side of M at p.*

Chapter 4

The $\bar{\partial}$-problem

4.1 The generalized Cauchy integral formula

Before we get into the $\bar{\partial}$-problem, let us prove a more general version of Cauchy's formula using Stokes' theorem*. Sometimes this is called the *Cauchy-Pompeiu integral formula*. We will only need the theorem for smooth functions, but as it is often applied in less regular contexts and it is just an application of Stokes theorem, let us state it that way. In applications, the boundary is often only piecewise smooth, and again that is all we need for Stokes.

Theorem 4.1.1. *Let $U \subset \mathbb{C}$ be a bounded domain with piecewise C^1-smooth boundary ∂U oriented positively, and let $f \colon \overline{U} \to \mathbb{C}$ be a C^1-smooth function. Then for $z \in U$:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{U} \frac{\frac{\partial f}{\partial \bar{z}}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

If $\zeta = x + iy$, then the standard orientation on $\mathbb{C}$ is the one corresponding to the area form $dA = dx \wedge dy$. Then $d\zeta \wedge d\bar{\zeta}$ is the area form up to a scalar. That is,

$$d\zeta \wedge d\bar{\zeta} = (dx + i\,dy) \wedge (dx - i\,dy) = (-2i) dx \wedge dy = (-2i) dA.$$

If f is holomorphic the second term is zero and we obtain the standard Cauchy formula.

Exercise* 4.1.1*: *Observe the singularity in the second term, and prove that the integral still makes sense (the function is integrable). Hint: polar coordinates.*

Exercise* 4.1.2*: *Why can we not differentiate in $\bar{z}$ under the integral in the second term? Notice that would lead to an impossible result.*

*We are really using Green's theorem which is the generalized Stokes' theorem in 2 dimensions.

Proof. Fix $z \in U$. Let $\Delta_r(z)$ be a small disc such that $\Delta_r(z) \subset\subset U$. Via Stokes we get

$$\int_{\partial U} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial \Delta_r(z)} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{U \setminus \Delta_r(z)} d\left(\frac{f(\zeta)}{\zeta - z} d\zeta \right) = \int_{U \setminus \Delta_r(z)} \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta.$$

The second equality follows because holomorphic derivatives in ζ will have a $d\zeta$ and when we wedge with $d\zeta$ we just get zero. We now wish to let the radius r go to zero. Via the exercise above we have that $\frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta$ is integrable over all of U and therefore

$$\lim_{r \to 0} \int_{U \setminus \Delta_r(z)} \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta = \int_U \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} d\bar{\zeta} \wedge d\zeta = -\int_U \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

The second equality is just swapping the order of the $d\zeta$ and $d\bar{\zeta}$. Next by continuity of f we get

$$\lim_{r \to 0} \frac{1}{2\pi i} \int_{\partial \Delta_r(z)} \frac{f(\zeta)}{\zeta - z} d\zeta = \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta = f(z).$$

The theorem follows. □

***Exercise* 4.1.3:** *Let $U \subset \mathbb{C}$ be a domain with piecewise C^1-smooth boundary and suppose $f : \overline{U} \to \mathbb{C}$ is a C^1-smooth function such that $\frac{\partial f}{\partial \bar{z}}$ goes to zero as z goes to ∂U. Prove $f|_{\partial U}$ are the boundary values of a holomorphic function on U.*

***Exercise* 4.1.4:** *Let $U \subset \mathbb{C}$ and f be as in the theorem, but let $z \notin \overline{U}$. Show that*

$$\frac{1}{2\pi i} \int_{\partial U} \frac{\varphi(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_U \frac{\frac{\partial \varphi}{\partial \bar{z}}(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta} = 0.$$

4.2 Simple case of the $\bar{\partial}$-problem

For a smooth function ψ we have the exterior derivative

$$d\psi = \frac{\partial \psi}{\partial z_1} dz_1 + \cdots + \frac{\partial \psi}{\partial z_n} dz_n + \frac{\partial \psi}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n.$$

Let us give a name to the two parts of the derivative:

$$\partial \psi \overset{\text{def}}{=} \frac{\partial \psi}{\partial z_1} dz_1 + \cdots + \frac{\partial \psi}{\partial z_n} dz_n, \qquad \bar{\partial} \psi \overset{\text{def}}{=} \frac{\partial \psi}{\partial \bar{z}_1} d\bar{z}_1 + \cdots + \frac{\partial \psi}{\partial \bar{z}_n} d\bar{z}_n.$$

Then $d\psi = \partial\psi + \bar{\partial}\psi$. Notice ψ is holomorphic if and only if $\bar{\partial}\psi = 0$.

The so-called *inhomogeneous $\bar{\partial}$-problem* (pronounced D-bar) is to solve the equation

$$\bar{\partial}\psi = g,$$

for ψ, given a one-form g:

$$g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n.$$

Such a g is called a $(0,1)$*-form*. The fact that the partial derivatives of ψ commute, forces certain compatibility conditions on g for us to have any hope of getting a solution (see below).

Exercise* 4.2.1*: *Find an explicit example of a g in $\mathbb{C}^2$ such that no corresponding ψ can exist.*

On any open set where $g = 0$, ψ is holomorphic. So for a general g, what we are doing is finding a function that is not holomorphic in a very specific way.

Theorem 4.2.1. *Suppose g is a $(0,1)$-form on $\mathbb{C}^n$, $n \geq 2$, given by*

$$g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n,$$

where $g_j \colon \mathbb{C}^n \to \mathbb{C}$ are compactly supported smooth functions satisfying the compatibility conditions

$$\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k} \qquad \text{for all } k, \ell = 1, 2, \ldots, n.$$

Then there exists a unique compactly supported smooth function $\psi \colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\bar{\partial}\psi = g.$$

The compatibility conditions on g are necessary, but the compactness is not. However in that case the boundary of the domain where the equation lives would come into play. Let us not worry about this, and prove this simple compactly supported version always has a solution. Without the compact support condition the solution is clearly not unique. Given any holomorphic f, $\bar{\partial}(\psi + f) = g$. But since the difference of any two solutions ψ_1 and ψ_2 is holomorphic, and the only holomorphic compactly supported function is 0, then the compactly supported solution ψ is unique.

Proof. We really have n smooth functions, $g_1, \ldots, g_n$, so the equation $\bar{\partial}\psi = g$ is the n equations

$$\frac{\partial \psi}{\partial \bar{z}_k} = g_k,$$

where the functions g_k satisfy the compatibility conditions.

We claim that the following is an explicit solution:

$$\psi(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta}.$$

To show that the singularity does not matter for integrability is the same idea as for the generalized Cauchy formula.

Let us check we have the solution. We use the generalized Cauchy formula on the z_1 variable. Take R large enough so that $g_j(\zeta, z_2, \ldots, z_n)$ is zero when $|\zeta| \geq R$ for all j. For any j we get

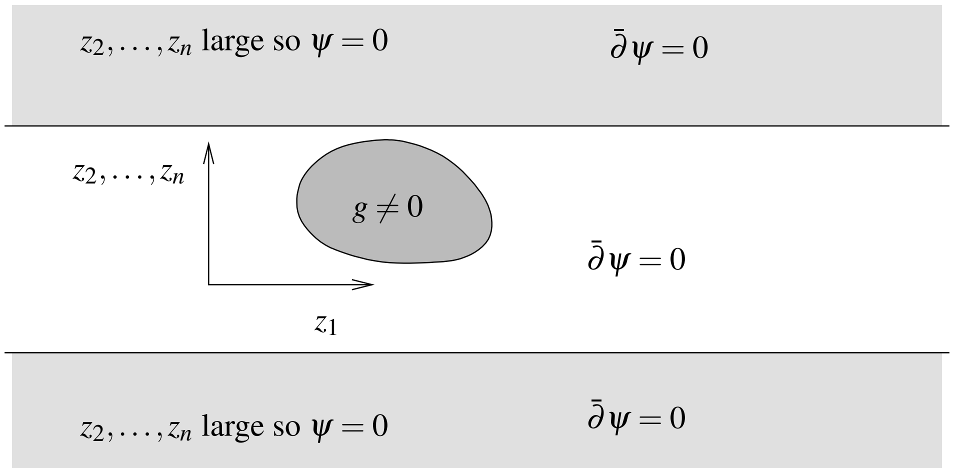
$$\begin{aligned} g_j(z_1, \ldots, z_n) &= \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{g_j(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta + \frac{1}{2\pi i} \int_{|\zeta| \leq R} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta}. \end{aligned}$$

Using the second form of the definition of ψ, the compatibility condition, and the above computation we get

$$\begin{aligned} \frac{\partial \psi}{\partial \bar{z}_j}(z) &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_1}{\partial \bar{z}_j}(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\frac{\partial g_j}{\partial \bar{z}_1}(z_1, z_2, \ldots, z_n)}{\zeta - z_1} d\zeta \wedge d\bar{\zeta} = g_j(z). \end{aligned}$$

***Exercise* 4.2.2:** *Show that we were allowed to differentiate under the integral in the computation above.*

That ψ has compact support follows because g_1 has compact support and by analytic continuation. In particular, ψ is holomorphic for very large z since $\bar{\partial}\psi = g = 0$ when z is large. When $z_2, \ldots, z_n$ are large, then ψ is identically zero simply from its definition. By analytic continuation then ψ is identically zero for all large z. See the following diagram, where we use analytic continuation to show that as ψ is holomorphic and zero on the light gray area and holomorphic on the light gray and white area, it is also zero on the white area:



$\square$

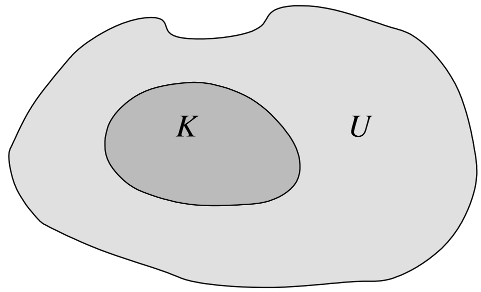
The first part of the proof still works when $n = 1$, so we do get a solution ψ. However in this case the last bit of the proof does not work, so ψ will not have compact support.

> ***Exercise* 4.2.3:** *a) Show that if g is supported in $K \subset\subset \mathbb{C}^n$, $n \geq 2$, then ψ is supported in the complement of the unbounded component of $\mathbb{C}^n \setminus K$. In particular, show that if K is the support of g and $\mathbb{C}^n \setminus K$ is connected, then the support of ψ is K. b) Find an explicit example where the support of ψ is strictly larger than the support of g.*

4.3 The general Hartogs phenomenon

We can now prove the general Hartogs phenomenon as an application of the solution of the compactly supported inhomogeneous $\bar{\partial}$-problem. We proved special versions of this phenomenon using Hartogs figures before. The proof of the theorem has a complicated history as Hartogs' original proof from 1906 contained gaps. Finally a fully working proof was supplied by Fueter only in 1939 for $n = 2$ and independently by Bochner and Martinelli for higher n in the early 40s. The proof we give is the standard one used nowadays due to Leon Ehrenpreis from 1961.

Theorem 4.3.1 (Hartogs phenomenon)**.** *Let $U \subset \mathbb{C}^n$ be a domain, $n \geq 2$, and let $K \subset\subset U$ be a compact set such that $U \setminus K$ is connected. Every holomorphic $f : U \setminus K \to \mathbb{C}$ extends uniquely to a holomorphic function on U.*



The idea of the proof is extending in some way and then using the solution to the $\bar{\partial}$-problem to correct the result to make it holomorphic.

Proof. First find a smooth function φ that is 1 in a neighborhood of K and is compactly supported in U (exercise below). Let $f_0 = (1-\varphi)f$ on $U \setminus K$ and $f_0 = 0$ on K. The function f_0 is smooth on U and it is holomorphic and equal to f near the boundary of U, where φ is 0. We let $g = \bar{\partial} f_0$ on U and we let $g = 0$ outside U, that is $g_k = \frac{\partial f_0}{\partial \bar{z}_k}$. The compatibility conditions are satisfied because partial derivatives commute. Let us see why g_k is compactly supported. The only place to check is on $U \setminus K$ as elsewhere we have 0 automatically. Note that f is holomorphic and compute

$$\frac{\partial f_0}{\partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}\big((1-\varphi)f\big) = \frac{\partial f}{\partial \bar{z}_k} - \varphi \frac{\partial f}{\partial \bar{z}_k} - \frac{\partial \varphi}{\partial \bar{z}_k} f = -\frac{\partial \varphi}{\partial \bar{z}_k} f.$$

And $\frac{\partial \varphi}{\partial \bar{z}_k}$ must be compactly supported in $U \setminus K$. Now apply the solution of the compactly supported $\bar{\partial}$-problem to find a compactly supported function ψ such that $\bar{\partial}\psi = g$. Set $F = f_0 - \psi$. Let us check that F is the desired extension. It is holomorphic:

$$\frac{\partial F}{\partial \bar{z}_k} = \frac{\partial f_0}{\partial \bar{z}_k} - \frac{\partial \psi}{\partial \bar{z}_k} = g_k - g_k = 0.$$

Next, Exercise 4.2.3 and the fact that $U \setminus K$ is connected reveals that ψ must be compactly supported in U. This means that F agrees with f near the boundary (in particular on an open set) and thus everywhere in $U \setminus K$ since $U \setminus K$ is connected. □

Exercise* 4.3.1*: *Show that φ exists. Hint: Use mollifiers.*

Exercise* 4.3.2*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with smooth boundary that is strongly pseudoconvex and $f \colon \partial U \to \mathbb{C}$ is a smooth CR function, then prove there exists a continuous function $F \colon \overline{U} \to \mathbb{C}$ holomorphic in U such that $F|_{\partial U} = f$.*

Exercise* 4.3.3*: *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a domain and the sphere $S^{2n-1} \subset U$. Suppose $f \colon U \to \mathbb{C}^n$ is a holomorphic mapping such that locally near every point of S^{2n-1}, f is a local biholomorphism (that is f is locally invertible, i.e. the derivative is invertible at every point of S^{2n-1}). Then show that f takes the ball $\mathbb{B}_n$ biholomorphically to some domain $f(\mathbb{B}_n)$ with smooth boundary.*

***Exercise* 4.3.4:** *Find an example of a smooth function $g\colon \mathbb{C} \to \mathbb{C}$ with compact support, such that no solution $\psi\colon \mathbb{C} \to \mathbb{C}$ to $\frac{\partial \psi}{\partial \bar{z}} = g$ (at least one of which always exists) is of compact support.*

Consequently the zero set of a holomorphic function f is never compact in dimension 2 or higher. If it were $\frac{1}{f}$ would provide a contradiction.

Corollary 4.3.2. *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a domain and $f\colon U \to \mathbb{C}$ is holomorphic. If the zero set $f^{-1}(0)$ is not empty, it is not compact.*

We can also state a version of a theorem that is nowadays called *Hartogs-Bochner** although it was first stated in the real-analytic case by Severi in 1931.

Corollary 4.3.3 (Severi)**.** *Suppose $U \subset \mathbb{C}^n$, $n \geq 2$, is a bounded domain with connected real-analytic boundary and $f\colon \partial U \to \mathbb{C}$ is a real-analytic CR function. Then there exists some neighborhood $U' \subset \mathbb{C}^n$ of $\overline{U}$ and a holomorphic function $F\colon U' \to \mathbb{C}$ for which $F|_{\partial U} = f$.*

Proof. By Severi's result f extends to a small neighborhood of ∂U near each point. Because the local extension is unique for a hypersurface, we obtain an extension in a single neighborhood of ∂U. We can write this neighborhood as $U' \setminus K$ for some compact K. We now apply Hartogs. $\square$

*What is called Hartogs-Bochner is the C^1 version of this theorem, and was proved by neither Hartogs nor Bochner, but by Martinelli in 1961.

Chapter 5

Integral kernels

5.1 The Bochner-Martinelli kernel

A generalization of the Cauchy's formula to several variables is called the Bochner-Martinelli integral formula, which in fact reduces to to Cauchy's formula when $n = 1$. And just like for Cauchy's formula, we will prove the formula for all smooth functions using Stokes theorem.

First let us define the *Bochner-Martinelli kernel*:

$$\omega(\zeta,z) \overset{\text{def}}{=} \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{\lVert \zeta - z \rVert^{2n}} \, d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n .$$

The notation $\widehat{d\bar{\zeta}_j}$ means that this term is simply left out.

Theorem 5.1.1 (Bochner-Martinelli)**.** *Let $U \subset \mathbb{C}^n$ be a bounded domain with smooth boundary and let $f : \overline{U} \to \mathbb{C}$ be a smooth function, then for $z \in U$:*

$$f(z) = \int_{\partial U} f(\zeta)\omega(\zeta,z) - \int_U \bar{\partial} f(\zeta) \wedge \omega(\zeta,z).$$

In particular, if $f \in \mathscr{O}(U)$, then

$$f(z) = \int_{\partial U} f(\zeta)\omega(\zeta,z).$$

Recall that if $\zeta = x + iy$ are the coordinates in $\mathbb{C}^n$, the orientation that we assigned to $\mathbb{C}^n$ in this book* is the one corresponding to the volume form

$$dV = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n .$$

With this orientation we find that

$$d\zeta_1 \wedge d\bar{\zeta}_1 \wedge d\zeta_2 \wedge d\bar{\zeta}_2 \wedge \cdots \wedge d\zeta_n \wedge d\bar{\zeta}_n = (-2i)^n dV,$$

*Again, there is no canonical orientation of $\mathbb{C}^n$, and not all authors follow this (perhaps more prevalent) convention.

and hence

$$d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n = (2i)^n dV.$$

Exercise* 5.1.1*: *As for the Cauchy-Pompeiu formula, note the singularity in the second term, and prove that the integral still makes sense (the function is integrable).*

Exercise* 5.1.2*: *Check that for $n = 1$, the formula becomes the standard Cauchy-Pompeiu formula.*

We will, again, apply Stokes formula, but we need to apply it to forms of higher degree. As before, we split the derivatives to the holomorphic and antiholomorphic parts. We work with multiindices. For α and β with $|\alpha| = p$ and $|\beta| = q$, the differential form

$$\eta = \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \eta_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

is called a (p,q)-*form* or a differential form of *bidegree* (p,q). Define

$$\partial\eta \overset{\text{def}}{=} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_{j=1}^{n} \frac{\partial \eta_{\alpha\beta}}{\partial z_j} dz_j \wedge dz^\alpha \wedge d\bar{z}^\beta, \qquad \text{and} \qquad \bar{\partial}\eta \overset{\text{def}}{=} \sum_{\substack{|\alpha|=p \\ |\beta|=q}} \sum_{j=1}^{n} \frac{\partial \eta_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^\alpha \wedge d\bar{z}^\beta.$$

It is not difficult to see that $d\eta = \partial\eta + \bar{\partial}\eta$ as before.

Proof of Bochner-Martinelli. The structure of the proof is essentially the same as that of the Cauchy-Pompeiu theorem for $n = 1$, although some of the formulas are somewhat more involved.

Let $z \in U$ be fixed. Suppose $r > 0$ is small enough such that $\overline{B_r(z)} \subset U$. We orient ∂U and $\partial B_r(z)$ both positively. Notice that $f(\zeta)\omega(\zeta,z)$ has all the holomorphic $d\zeta_j$, therefore,

$$\begin{aligned} d\big(f(\zeta)\omega(\zeta,z)\big) &= \bar{\partial}\big(f(\zeta)\omega(\zeta,z)\big) \\ &= \bar{\partial} f(\zeta) \wedge \omega(\zeta,z) \\ &\quad + f(\zeta)\frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} \frac{\partial}{\partial \bar{\zeta}_j} \left[\frac{\bar{\zeta}_j - \bar{z}_j}{\|\zeta - z\|^{2n}} \right] d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n. \end{aligned}$$

We compute

$$\sum_{j=1}^{n} \frac{\partial}{\partial \bar{\zeta}_j} \left[\frac{\bar{\zeta}_j - \bar{z}_j}{\|\zeta - z\|^{2n}} \right] = \sum_{j=1}^{n} \left(\frac{1}{\|\zeta - z\|^{2n}} - n \frac{|\zeta_j - z_j|^2}{\|\zeta - z\|^{2n+2}} \right) = 0.$$

Therefore $d\big(f(\zeta)\omega(\zeta,z)\big) = \bar\partial f(\zeta) \wedge \omega(\zeta,z)$. We apply Stokes as before

$$\int_{\partial U} f(\zeta)\omega(\zeta,z) - \int_{\partial B_r(z)} f(\zeta)\omega(\zeta,z) = \int_{U\setminus\overline{B_r(z)}} d\big(f(\zeta)\omega(\zeta,z)\big) = \int_{U\setminus\overline{B_r(z)}} \bar\partial f(\zeta) \wedge \omega(\zeta,z).$$

Again, due to the integrability, which you showed in an above exercise, the right hand side converges to the integral over U. Just as for the Cauchy-Pompeiu formula, we now need to show that the integral over $\partial B_r(z)$ goes to $f(z)$ as $r \to 0$.

So

$$\int_{\partial B_r(z)} f(\zeta)\omega(\zeta,z) = f(z)\int_{\partial B_r(z)} \omega(\zeta,z) + \int_{\partial B_r(z)} \big(f(\zeta)-f(z)\big)\omega(\zeta,z).$$

To finish the proof, we will show that $\int_{\partial B_r(z)} \omega(\zeta,z) = 1$, and that the second term goes to zero. We apply Stokes again and note that the volume of $B_r(z)$ is $\frac{\pi^n}{n!}r^{2n}$.

$$\begin{aligned}
\int_{\partial B_r(z)} \omega(\zeta,z) &= \int_{\partial B_r(z)} \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n \frac{\bar\zeta_j - \bar z_j}{\|\zeta - z\|^{2n}} d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{\partial B_r(z)} \sum_{j=1}^n (\bar\zeta_j - \bar z_j) d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{B_r(z)} d\left(\sum_{j=1}^n (\bar\zeta_j - \bar z_j) d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar\zeta_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n\right) \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{B_r(z)} n\, d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n \\
&= \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \int_{B_r(z)} n(2i)^n dV = 1.
\end{aligned}$$

Next, we tackle the second term. Via the same computation as above we find

$$\begin{aligned}
\int_{\partial B_r(z)} \big(f(\zeta)-f(z)\big)\omega(\zeta,z) = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{r^{2n}} \Bigg(&\int_{B_r(z)} \big(f(\zeta)-f(z)\big) n\, d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n \\
&+ \int_{B_r(z)} \sum_{j=1}^n \frac{\partial f}{\partial \bar\zeta_j}(\zeta)(\bar\zeta_j - \bar z_j)\, d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n \Bigg).
\end{aligned}$$

As U is bounded, $|f(\zeta)-f(z)| \leq M\|\zeta - z\|$ and $\left|\frac{\partial f}{\partial \bar\zeta_j}(\zeta)(\bar\zeta_j - \bar z_j)\right| \leq M\|\zeta - z\|$ for some M. So for all $\zeta \in \partial B_r(z)$, we have $|f(\zeta)-f(z)| \leq Mr$ and $\left|\frac{\partial f}{\partial \bar\zeta_j}(\zeta)(\bar\zeta_j - \bar z_j)\right| \leq Mr$. Hence

$$\left|\int_{\partial B_r(z)} \big(f(\zeta)-f(z)\big)\omega(\zeta,z)\right| \leq \frac{(n-1)!}{(2\pi)^n} \frac{1}{r^{2n}} \left(\int_{B_r(z)} n2^n Mr\, dV + \int_{B_r(z)} n2^n Mr\, dV\right) = 2Mr.$$

Therefore, this term goes to zero as $r \to 0$. □

One drawback of the Bochner-Martinelli formula is that the kernel is not holomorphic in z unless $n = 1$. That is, it does not simply produce holomorphic functions. If we differentiate in $\bar{z}$ underneath the ∂U integral, we do not necessarily obtain zero. On the other hand, we have an explicit formula and this formula does not depend on U. This will not be the case of the Bergman and Szegö kernels, which we will see next, although those will be holomorphic in the right way.

Exercise* 5.1.3*: *Prove that if $z \notin \overline{U}$, then rather than $f(z)$ in the formula you obtain*

$$\int_{\partial U} f(\zeta)\omega(\zeta,z) - \int_U \bar{\partial} f(\zeta) \wedge \omega(\zeta,z) = 0.$$

Exercise* 5.1.4*: *Suppose f is holomorphic on a neighborhood of $\overline{B_r(z)}$.*
a) Using the Bochner-Martinelli formula, prove that

$$f(z) = \frac{1}{V\big(B_r(z)\big)} \int_{B_r(z)} f(\zeta)\, dV(\zeta)$$

where $V\big(B_r(z)\big)$ is the volume of $B_r(z)$.
b) Use part a) to prove the maximum principle for holomorphic functions.

Exercise* 5.1.5*: *Use Bochner-Martinelli for the solution of $\bar{\partial}$ with compact support. That is, suppose g is a smooth compactly supported $(0,1)$-form on $\mathbb{C}^n$, $n \geq 2$, $g = g_1 d\bar{z}_1 + \cdots + g_n d\bar{z}_n$ $\frac{\partial g_k}{\partial \bar{z}_\ell} = \frac{\partial g_\ell}{\partial \bar{z}_k}$ for all k,ℓ. Prove that*

$$\psi(z) = -\int_{\mathbb{C}^n} g(\zeta) \wedge \omega(\zeta,z)$$

is a compactly supported smooth solution to $\bar{\partial}\psi = g$. Hint: look at the previous proof.

5.2 The Bergman kernel

Let $U \subset \mathbb{C}^n$ be a domain.

$$A^2(U) = \mathscr{O}(U) \cap L^2(U).$$

That is, let $A^2(U)$ denote the space of holomorphic functions $f \in \mathscr{O}(U)$ such that

$$\|f\|^2_{A^2(U)} \overset{\text{def}}{=} \|f\|^2_{L^2(U)} = \int_U |f(z)|^2 dV < \infty.$$

The space $A^2(U)$ is called the *Bergman space* of U. The inner product is the $L^2(U)$ inner product

$$\langle f,g \rangle \overset{\text{def}}{=} \int_U f(z)\overline{g(z)}\, dV.$$

We need to prove that $A^2(U)$ is complete, then it will be a Hilbert space. We first have prove that we can bound the uniform norm on compact sets via the $A^2(U)$ norm.

Lemma 5.2.1. *Let $U \subset \mathbb{C}^n$ be a domain and $K \subset\subset U$ compact. Then there exists a constant C_K, such that for any $f \in A^2(U)$ we have*

$$\|f\|_K = \sup_{z\in K}|f(z)| \leq C_K \|f\|_{A^2(U)}.$$

Proof. As K is compact there exists an $r > 0$ such that for all $z \in K$ we have $\overline{\Delta_r(z)} \subset U$. Fix $z \in K$, apply Exercise 1.2.7 and Cauchy-Schwarz again:

$$\begin{aligned} |f(z)| &= \left| \frac{1}{V\left(\Delta_r(z)\right)} \int_{\Delta_r(z)} f(\xi)\, dV(\xi) \right| \\ &\leq \frac{1}{\pi^n r^{2n}} \sqrt{\int_{\Delta_r(z)} 1^2\, dV(\xi)} \sqrt{\int_{\Delta_r(z)} |f(\xi)|^2\, dV(\xi)} \\ &= \frac{1}{\pi^{n/2} r^n} \|f\|_{A^2\left(\Delta_r(z)\right)} \leq \frac{1}{\pi^{n/2} r^n} \|f\|_{A^2(U)}. \end{aligned}$$

□

A sequence $\{f_j\}$ of functions converging in $L^2(U)$ to some $f \in L^2(U)$ converges uniformly on compact sets. Therefore $f \in \mathcal{O}(U)$. Therefore $A^2(U)$ is a closed subspace, and hence complete. For a bounded domain $A^2(U)$ is always infinite dimensional, see exercise below. However, there do exist unbounded domains for which either $A^2(U)$ is trivial (just the zero function) or even finite dimensional.

Exercise* 5.2.1*: *Show that if $U \subset \mathbb{C}^n$ is bounded, then $A^2(U)$ is infinite dimensional.*

Exercise* 5.2.2*: *a) Show that $A^2(\mathbb{C}^n)$ is trivial.*
b) Show that $A^2(\mathbb{D} \times \mathbb{C})$ is trivial.
c) Find an example of an unbounded domain U for which $A^2(U)$ is infinite dimensional. Hint: Think in one dimension for simplicity.

Exercise* 5.2.3*: *Show that $A^2(\mathbb{D})$ can be identified with $A^2(\mathbb{D} \setminus \{0\})$, that is, every function in the latter can be extended to a function in the former.*

Again using the lemma, we notice that point evaluation is a bounded linear functional, that is take $K = \{z\}$, then the linear operator

$$f \mapsto f(z)$$

is a bounded linear functional. By Riesz-Fisher theorem then, there exists a $k_z \in A^2(U)$, such that

$$f(z) = \langle f, k_z \rangle.$$

Define the *Bergman kernel* for U as

$$K_U(z,\bar{\zeta}) \overset{\text{def}}{=} \overline{k_z(\zeta)}.$$

The function K_U is defined as $(z,\bar{\zeta})$ vary over $U \times U^*$, where $U^* = \{\zeta : \bar{\zeta} \in U\}$.

Then for all $f \in A^2(U)$ we have

$$f(z) = \int_U f(\zeta) K_U(z,\bar{\zeta})\, dV(\zeta). \tag{5.1}$$

This last equation is sometimes called the *reproducing property* of the kernel.

It should be noted that the Bergman kernel depends on U, which is why we write it as $K_U(z,\bar{\zeta})$.

Proposition 5.2.2. *The Bergman kernel $K_U(z,\bar{\zeta})$ is holomorphic in z and antiholomorphic in ζ, and furthermore*

$$\overline{K_U(z,\bar{\zeta})} = K_U(\zeta,\bar{z}).$$

Proof. As each k_z is in $A^2(U)$ it is holomorphic in ζ. Hence, K_U is antiholomorphic in ζ. Thus if we prove $\overline{K_U(z,\bar{\zeta})} = K_U(\zeta,\bar{z})$, then we find that K_U is holomorphic in z.

As $\overline{K_U(z,\bar{\zeta})} = k_z(\zeta)$ is in $A^2(U)$, then

$$\begin{aligned}
\overline{K_U(z,\bar{\zeta})} &= \int_U \overline{K_U(z,\bar{w})} K_U(\zeta,\bar{w}) dV(w) \\
&= \overline{\left(\int_U \overline{K_U(\zeta,\bar{w})} K_U(z,\bar{w}) dV(w) \right)} = \overline{\overline{K_U(\zeta,\bar{z})}} = K_U(\zeta,\bar{z}).
\end{aligned}$$

□

By the proposition in particular K_U is analytic; thinking of $\bar{\zeta}$ as the variable, then K_U is a holomorphic function of $2n$ variables.

Example 5.2.3: Let us compute the Bergman kernel explicitly in a very simple case of the unit disc $\mathbb{D} \subset \mathbb{C}$. Let $f \in \mathscr{O}(\mathbb{D}) \cap C(\overline{D})$, that is holomorphic in $\mathbb{D}$ and continuous up to the boundary. Let $z \in \mathbb{D}$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

On the unit circle $\zeta\bar{\zeta} = 1$. Let ds be the arc-length measure on the circle. Parametrize the circle as $\zeta = e^{is}$ then $d\zeta = ie^{is} ds$, and therefore

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}}\, \bar{\zeta}\, d\zeta \\
&= \frac{1}{2\pi} \int_{\partial \mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}}\, ds.
\end{aligned}$$

The integral is now a regular line integral of a function whose singularity used to be inside the unit disc disappeared (we "reflected it" to the outside). The kernel $\frac{1}{2\pi}\frac{1}{1-z\bar{\zeta}}$ is in fact called the *Szegö kernel*, which we will briefly mention next. We can apply Stokes now (to the second integral above).

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(\zeta)}{1-z\bar{\zeta}}\bar{\zeta}\,d\zeta = \frac{1}{2\pi i}\int_{\mathbb{D}} f(\zeta)\frac{\partial}{\partial\bar{\zeta}}\left[\frac{\bar{\zeta}}{1-z\bar{\zeta}}\right] d\bar{\zeta}\wedge d\zeta$$
$$= \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\bar{\zeta})^2}\,dA(\zeta).$$

The Bergman kernel in the unit disc is therefore

$$K_U(z,\bar{\zeta}) = \frac{1}{\pi}\frac{1}{(1-z\bar{\zeta})^2}.$$

The fact that this really is the Bergman kernel follows from the exercises below. That is, that K_U is the unique conjugate symmetric reproducing function that is in $A^2(U)$ for a fixed ζ. We have only shown the formula for functions continuous up to the boundary, but those are dense in $A^2(U)$.

Example 5.2.4: On the other hand you found in an exercise that $A^2(\mathbb{C}^n) = \{0\}$. Therefore we have trivially $K_{\mathbb{C}^n}(z,\bar{\zeta}) \equiv 0$.

In general it is difficult to compute the kernel explicitly for just any U, although we have the following formula for it.

Proposition 5.2.5. *Suppose $\{\varphi_j(z)\}_{j\in I}$ is a complete orthonormal system for $A^2(U)$. Then*

$$K_U(z,\bar{\zeta}) = \sum_{j\in I} \varphi_j(z)\overline{\varphi_j(\zeta)},$$

with convergence uniform on compact sets of $U \times U^*$.

Proof. For any fixed $\zeta \in U$, the function $z \mapsto K_U(z,\bar{\zeta})$ is in $A^2(U)$ and so expand this function in terms of the basis and use the reproducing property of K_U

$$K_U(z,\bar{\zeta}) = \sum_{j\in I}\left(\int_U K_U(w,\bar{\zeta})\overline{\varphi_j(w)}\,dV(w)\right)\varphi_j(z) = \sum_{j\in I}\overline{\varphi_j(\zeta)}\varphi_j(z).$$

The convergence is in L^2 as a function of z, for a fixed ζ. Let $K \subset\subset U$ be a compact set. Via Lemma 5.2.1, L^2 convergence in $A^2(U)$ is uniform convergence on compact sets. Therefore for a fixed ζ the convergence is uniform in $z \in K$. In particular we get pointwise convergence so

$$\sum_{j\in I}\left|\varphi_j(z)\right|^2 = \sum_{j\in I}\varphi_j(z)\overline{\varphi_j(z)} = K_U(z,\bar{z}) \le C_K < \infty,$$

where C_K is the supremum of $K_U(z,\bar{\zeta})$ on $K \times K^*$. Hence for $(z,\bar{\zeta}) \in K \times K^*$,

$$\sum_{j\in I} \left| \varphi_j(z)\overline{\varphi_j(\zeta)} \right| \leq \sqrt{\sum_{j\in I} \left|\varphi_j(z)\right|^2} \sqrt{\sum_{j\in I} \left|\varphi_j(\zeta)\right|^2} \leq C_K < \infty.$$

And so the convergence is uniform on $K \times K^*$. $\square$

Exercise* 5.2.4*: *a) Show that if $U \subset \mathbb{C}^n$ is bounded then for all $z \in U$ we find $K_U(z,\bar{z}) > 0$. b) Why can this fail if U is unbounded? Find a (trivial) counterexample.*

Exercise* 5.2.5*: *Show that given a domain $U \subset \mathbb{C}^n$, the Bergman kernel is the unique function $K_U(z,\bar{\zeta})$ such that 1) for a fixed ζ, $K_U(z,\bar{\zeta})$ is in $A^2(U)$, 2) $\overline{K_U(z,\bar{\zeta})} = K_U(\zeta,\bar{z})$, 3) the reproducing property* (5.1) *holds.*

Exercise* 5.2.6*: *Let $U \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Show that the functions in $A^2(U) \cap C(\overline{U})$ are dense in $A^2(U)$. In particular this exercise says we only need to check the reproducing property on functions continuous up to the boundary to show we have the Bergman kernel.*

Exercise* 5.2.7*: *Let $U,V \subset \mathbb{C}^n$ be two domains with a biholomorphism $f \colon U \to V$. Show that*

$$K_U(z,\bar{\zeta}) = \det Df(z)\,\overline{\det Df(\zeta)}\,K_V\big(f(z),\overline{f(\zeta)}\big).$$

Exercise* 5.2.8*: *Show that the Bergman kernel for the polydisc is*

$$K_{\mathbb{D}^n}(z,\bar{\zeta}) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{\left(1 - z_j\bar{\zeta}_j\right)^2}.$$

Exercise* 5.2.9** (Hard):*** *Show that for the unit ball $\mathbb{B}_n$, then for some constants c_α, the set of all monomials $\frac{z^\alpha}{c_\alpha}$ gives a complete orthonormal system. Hint: To show orthonormality compute the integral using polar coordinates in each variable separately, that is let $z_j = r_j e^{i\theta_j}$ where $\theta \in [0,2\pi]^n$ and $\sum r_j^2 < 1$. Then show completeness by showing that if $f \in A^2(\mathbb{B}_n)$ is orthogonal to all z^α then $f = 0$. Finding $c_\alpha = \sqrt{\frac{\pi^n \alpha!}{(n+|\alpha|)!}}$ requires the classical β function of special function theory.*

Exercise* 5.2.10*: *Using the previous exercise, show that the Bergman kernel for the unit ball is*

$$K_{\mathbb{B}_n}(z,\bar{\zeta}) = \frac{n!}{\pi^n} \frac{1}{\left(1 - \langle z,\zeta\rangle\right)^{n+1}},$$

where $\langle z,\zeta\rangle$ is the standard inner product on $\mathbb{C}^n$.

5.3 The Szegö kernel

We can use the same techniques to create a reproducing kernel on the boundary by starting with $L^2(\partial U, d\sigma)$ instead of $L^2(U)$ to obtain a kernel where we integrate over the boundary rather than the domain itself. We will give a quick overview here though we will leave out the details.

Let $U \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Let $C(\overline{U}) \cap \mathscr{O}(U)$ be the holomorphic functions in U continuous up to the boundary. The restrictions of $f \in C(\overline{U}) \cap \mathscr{O}(U)$ to ∂U is a continuous function and hence $f|_{\partial U}$ is in $L^2(\partial U, d\sigma)$ where $d\sigma$ is the surface measure on ∂U. Taking a closure of these restrictions in $L^2(\partial U)$ obtains the Hilbert space $H^2(\partial U)$. The inner product in this case is the $L^2(\partial U, d\sigma)$ inner product:

$$\langle f, g \rangle \overset{\text{def}}{=} \int_{\partial U} f(z)\overline{g(z)}\, d\sigma(z).$$

Exercise* 5.3.1*: *Show that monomials z^α are a complete orthonormal system in $H^2(\partial \mathbb{B}_n)$.*

Exercise* 5.3.2*: *Show that for any bounded $U \subset \mathbb{C}^n$ with smooth boundary, $H^2(\partial U)$ is infinite dimensional.*

Let

$$Pf(z) = \int_{\partial U} f(\zeta) P(z, \zeta)\, d\sigma(\zeta)$$

be the Poisson integral, that is $P(z, \zeta)$ is the Poisson kernel, which reproduces harmonic functions. As holomorphic functions are harmonic we find that if $f \in C(\overline{U}) \cap \mathscr{O}(U)$, then $Pf = f$.

For $z \in U$ we can also find $Pf(z)$ for $f \in H^2(\partial U)$. We next find that for each $z \in U$

$$f \mapsto Pf(z)$$

defines a continuous linear functional and so again we find a $s_z \in H^2(\partial U)$ such that

$$Pf(z) = \langle f, s_z \rangle.$$

Hence for $z \in U$ and $\zeta \in \partial U$ we define

$$S_U(z, \bar{\zeta}) \overset{\text{def}}{=} \overline{s_z(\zeta)},$$

although for a fixed z this is a function only defined almost everywhere as it is an element of $L^2(\partial U, d\sigma)$. The function S_U is the *Szegö kernel*. We then find that if $f \in H^2(\partial U)$, then

$$Pf(z) = \int_{\partial U} f(\zeta) S_U(z, \bar{\zeta})\, d\sigma(\zeta).$$

As functions in $H^2(\partial U)$ extend to $\overline{U}$, then often $f \in H^2(\partial U)$ is considered a function on $\overline{U}$ where values in U are given by Pf. Similarly we can extend $S(z, \bar{\zeta})$ to a function on $U \times \overline{U}^*$ (where the values on the boundary are only almost everywhere). We state without proof that again if $\{\varphi_j\}$ is a complete orthonormal system for $H^2(\partial U)$, then

$$S_U(z,\bar{\zeta}) = \sum_j \varphi_j(z)\overline{\varphi_j(z)} \tag{5.2}$$

for $(z,\bar{\zeta}) \in U \times U^*$ uniformly on compact subsets. From this formula it can be seen that S is again conjugate symmetric and so it in fact extends to $(U \times \overline{U}^*) \cup (\overline{U} \times U^*)$.

Example 5.3.1: We have computed before that for the unit disc we had for holomorphic f

$$f(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{1 - z\bar{\zeta}}\, ds.$$

That is, $S_{\mathbb{D}}(z,\zeta) = \frac{1}{\pi}\frac{1}{1-z\bar{\zeta}}$.

***Exercise* 5.3.3:** *Using* (5.2) *compute* $S_{\mathbb{B}_n}$.

Chapter 6

Complex analytic varieties

6.1 Germs of functions

Definition 6.1.1. Let p be a point in a topological space X. Let Y be a set and $U,V \subset X$ be open neighborhoods of p. We say that two functions $f: U \to Y$ and $g: V \to Y$ are equivalent if there exists a neighborhood Ω of p such that $f|_\Omega = g|_\Omega$.

An equivalence class of functions defined in a neighborhood of p is called a *germ of a function*. Usually it is denoted by (f,p), but we simply say f when the context is clear.

Germs are particularly useful for holomorphic functions because of the identity theorem. The set of germs of complex valued functions always forms a ring. When the functions are holomorphic, the ring has many nice properties. The ring of germs is generally a "nicer" ring than the ring $\mathscr{O}(U)$ for some open U.

Definition 6.1.2. Let $p \in \mathbb{C}^n$. Write ${}_n\mathscr{O}_p = \mathscr{O}_p$ as the ring of germs at p of holomorphic functions.

Exercise* 6.1.1*: *Show that $\mathscr{O}_p$ is an integral domain (has no zero divisors).*

Exercise* 6.1.2*: *Show that $\mathscr{O}_p$ is isomorphic to the ring of convergent power series.*

Exercise* 6.1.3*: *Show that given a germ (f,p) of a holomorphic function, we can always pick an open neighborhood U of p and a representative $f: U \to \mathbb{C}$ such that any other representative g of (f,p) will extend to be a holomorphic function on an open set containing U such that $g|_U \equiv f$.*

Exercise* 6.1.4*: *a) Show that if $f \in \mathscr{O}_p$ is such that $f(p) \neq 0$, then if (f) is the ideal generated by f, then $I = \mathscr{O}_p$.*
b) Let $m = (z_1,\dots,z_n)$ be the ideal generated by the coordinate functions. Show that if f vanishes at the origin, then $f \in m$.
c) Show that if $I \subsetneq \mathscr{O}_p$ is a proper ideal then $I \subset m$, that is m is a maximal ideal.

Definition 6.1.3. Let p be a point in a topological space X. We say that sets $A, B \subset X$ are equivalent if there exists a neighborhood N of p such that $A \cap N = B \cap N$.

An equivalence class of sets is called a *germ of a set* at p. Usually it is denoted by (A, p), but we may simply say A when the context is clear.

The concept of $(X, p) \subset (Y, p)$ is defined in an obvious manner. Similarly the intersection $(X, p) \cap (Y, p)$ and the union $(X, p) \cup (Y, p)$ can easily be defined. The main point is to make sure that the germs are at the same point.

***Exercise* 6.1.5:** *Write down a rigorous definition of subset, union, and intersection of germs. Then check that what you did is well defined.*

Suppose f is (a germ of) a holomorphic function at a point $a \in \mathbb{C}^n$. Write

$$f(z) = \sum_{k=0}^{\infty} f_k(z - a),$$

where f_k is homogeneous polynomial of degree k.

Definition 6.1.4. For $a \in \mathbb{C}^n$ such that f is a holomorphic function defined near a, define

$$\operatorname{ord}_a f \overset{\text{def}}{=} \min\{k \in \mathbb{N}_0 : f_k \not\equiv 0\},$$

with the obvious definition that if $f \equiv 0$, then $\operatorname{ord}_a f = \infty$. The number $\operatorname{ord}_a f$ is called the *order of vanishing* of f at a.

In other words, if the order of vanishing of f at a is k, than all partial derivatives of order less than k vanish at a, and there exists at least one derivative of order k that does not vanish at a.

6.2 Weierstrass preparation and division theorems

From one variable, you may remember that a holomorphic function vanishing to order k at 0 equals (locally) $z^k u(z)$ for a nonvanishing u. In several variables, there is a similar theorem, or in fact a pair of theorems, the so-called Weierstrass preparation and division theorems.

Definition 6.2.1. Let $U \subset \mathbb{C}^{n-1}$ be open. Let $z' \in \mathbb{C}^{n-1}$ denote the coordinates. Suppose $P \in \mathscr{O}(U)[z_n]$ is monic of degree k, that is,

$$P(z', z_n) = z_n^k + \sum_{j=0} c_j(z')\, z_n^j,$$

where c_j are holomorphic functions defined on U. Further suppose $c_j(0) = 0$ for all j. Then P is called a *Weierstrass polynomial* of degree k. If the c_j are germs in $\mathscr{O}_0 = {}_{n-1}\mathscr{O}_0$, then $P \in \mathscr{O}_0[z_n]$ and P is a *germ of a Weierstrass polynomial*.

The definition (and the theorem that follows) still holds for $n = 1$. If you read the definition carefully, you will find that if $n = 1$, then the only Weierstrass polynomial of degree k is z^k.

The purpose of this section is to show that every holomorphic function in $\mathscr{O}(U)$ is (up to a unit and a possible small rotation) a Weierstrass polynomial. This will imply that zero sets of holomorphic functions behave a lot like zero sets of polynomials, and ideals in $\mathscr{O}_p$ can be generated by Weierstrass polynomials.

Theorem 6.2.2 (Weierstrass preparation theorem)**.** *Suppose $f \in \mathscr{O}(U)$ for a domain $U \subset \mathbb{C}^{n-1} \times \mathbb{C}$ where $0 \in U$. Suppose $z_n \mapsto f(0,z_n)$ is not identically zero near the origin and its order of vanishing at the origin is k.*

Then there exists an open $V = V' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$ with $0 \in V \subset U$, a unique $u \in \mathscr{O}(V)$, $u(z) \neq 0$ for all $z \in V$, and a unique Weierstrass polynomial P of degree k with coefficients holomorphic in V' such that

$$f(z',z_n) = u(z',z_n)\,P(z',z_n).$$

Thus for a fixed z', the zeros of f coincide with the zeros of a degree m monic polynomial whose coefficients depend holomorphically on z'.

Proof. There exists a small disc $D \subset \mathbb{C}$ around zero such that $f(0,z_n)$ is not zero on $\overline{D} \setminus \{0\}$. By continuity we find a small polydisc $V = V' \times D$ such that $\overline{V} \subset U$ and f is not zero on $V' \times \partial D$.

The one variable argument principle implies that the number of zeros (with multiplicity) of $z_n \mapsto f(z',z_n)$ in D is

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\frac{\partial f}{\partial z_n}(z',z_n)}{f(z',z_n)}\, dz_n.$$

The number equals k for all $z' \in V'$ by continuity. Write* the zeros as $\alpha_1(z'), \ldots, \alpha_k(z')$ (including multiplicity). There is no particular order to the zeros, simply pick some order for every z'. Write

$$P(z',z_n) = \prod_{j=1}^{k} \bigl(z_n - \alpha_j(z')\bigr) = z_n^k + c_{k-1}(z')\, z_n^{k-1} + \cdots + c_0(z').$$

For a fixed z', P is uniquely defined as the order of roots did not matter. It is clear that u and P are unique if they exist (as holomorphic functions).

We will prove that $c_j(z')$ are holomorphic. The functions c_j are the elementary symmetric functions of the α_j's. It is a standard theorem in algebra that the elementary symmetric functions are polynomials in the so-called power sum functions in the α_j's.

$$s_m(z') = \sum_{j=1}^{k} \alpha_j(z')^m.$$

Therefore, if we can show that s_m are holomorphic, then c_j are also holomorphic.

*In general you cannot make the functions α_k even continuous at all points.

A refinement of the argument principle says: If h and g are holomorphic functions on a disc D, continuous on $\overline{D}$, such that g has no zeros on ∂D, and $\alpha_1, \ldots, \alpha_k$ are the zeros of g in D, then

$$\frac{1}{2\pi i} \int_{\partial D} h(\zeta) \frac{g'(\zeta)}{g(\zeta)} \, d\zeta = \sum_{j=1}^{k} h(\alpha_j).$$

We use the above formula and obtain that

$$s_m(z') = \sum_{j=1}^{k} \alpha_j(z')^m = \frac{1}{2\pi i} \int_{\partial D} z_n^m \frac{\frac{\partial f}{\partial z_n}(z', z_n)}{f(z', z_n)} \, dz_n.$$

We differentiate under the integral to obtain that s_m is holomorphic.

Finally we wish to show that P divides f as claimed. Using Cauchy formula on $\frac{f}{P}$ we write

$$u(z', z_n) = \int_{\partial D} \frac{f(z', \zeta)}{P(z', \zeta)(\zeta - z_n)} \, d\zeta.$$

The function u is holomorphic in z' and z_n. Furthermore from one variable theory we know that for each fixed z' the function has no zeros as they cancel out exactly. $\square$

The hypotheses of the theorem are not an obstacle. If a holomorphic function f is such that $z_n \mapsto f(0, z_n)$ vanishes identically, then we can make a small linear change of coordinates L (L can be a matrix arbitrarily close to the identity) such that $f \circ L$ satisfies the hypotheses of the theorem.

Exercise* 6.2.1*: *Prove the above fact about the existence of L arbitrary close to the identity.*

Exercise* 6.2.2*: *Show that elementary symmetric functions are polynomials in the power sums.*

The order of vanishing of f at the origin is only a lower bound on the number k in the theorem. The order of vanishing for a certain variable may be larger than this lower bound. If you however make a small linear change of coordinates you can ensure $k = \operatorname{ord}_0 f$.

We think of Weierstrass preparation theorem as a generalization of the implicit function theorem. When $k = 1$ in the theorem, then we obtain the Weierstrass polynomial $z_n - c_0(z')$. That is, the zero set of f is a graph of a holomorphic function. We can therefore think of the Weierstrass theorem as a generalization of the implicit function theorem to the case when $\frac{\partial f}{\partial z_n}$ is zero. We can still “solve” for z_n but we will obtain k solutions that are the roots of the Weierstrass polynomial.

Example 6.2.3: A useful example to keep in mind is $f(z_1, z_2) = z_2^2 - z_1$. For all z_1 except the origin there are two zeros. For $z_1 = 0$, there is only one zero.

There is an obvious statement of the preparation theorem for germs.

***Exercise* 6.2.3:** *State and prove a germ version of the preparation theorem.*

Theorem 6.2.4 (Weierstrass division theorem)**.** *Suppose f is holomorphic near the origin, and suppose P is a Weierstrass polynomial of degree k in z_n. Then there exists a neighborhood V of the origin and unique $q, r \in \mathscr{O}(V)$, where r is a polynomial in z_n of degree less than k, and on V,*

$$f = Pq + r.$$

Do note that r need not be a Weierstrass polynomial; it need not be monic nor do the coefficients need to vanish at the origin. It is simply a polynomial in z_n with coefficients that are holomorphic functions of $n-1$ variables.

Proof. The uniqueness is left as an exercise. Assume we are working in a neighborhood $V = V' \times D$ for some small disc D such that f and P are continuous in $\overline{V}$, P is not zero on $V' \times \partial D$, and the only zeros of $P(0, z_n)$ in D are at the origin. We write

$$q(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z', \zeta)}{P(z', \zeta)(\zeta - z_n)} \, d\zeta.$$

By the assumptions on D we see q is holomorphic in V. Writing f using the Cauchy integral in the last variable and subtracting Pq we obtain

$$r(z', z_n) = f(z', z_n) - P(z', z_n) q(z', z_n) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z', \zeta) P(z', \zeta) - f(z', \zeta) P(z', z_n)}{P(z', \zeta)(\zeta - z_n)} \, d\zeta.$$

We need to show r is a polynomial in z_n of degree less than k. In the expression inside the integral the numerator is of the form $\sum_j h_j(z', \zeta)(\zeta^j - z_n^j)$ and is therefore divisible by $(\zeta - z_n)$. We perform the division. The numerator was a polynomial of degree k in z_n and after the division is therefore a polynomial of degree $k-1$. We then use linearity of the integral to integrate the coefficients of the polynomial. Each coefficient is a holomorphic function and we are done. Some coefficients may have integrated to zero, so we can only say that r is a polynomial of degree $k-1$ or less. $\square$

***Exercise* 6.2.4:** *Prove the uniqueness part of the theorem.*

Let us prove that the roots change holomorphically as long as they do not come together. Furthermore, the roots come together only on a small set; it is a zero set of a certain holomorphic function called the discriminant. Roots are *geometrically distinct* if they are distinct points of $\mathbb{C}$.

Proposition 6.2.5. *Suppose $f \in \mathscr{O}(U' \times \mathbb{D})$ for a domain $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$, such that for each fixed $z' \in U'$ the function $z_n \mapsto f(z', z_n)$ has a geometrically unique zero $\alpha(z') \in D$. Then α is holomorphic in U'.*

The proposition shows that the regularity conclusion of the implicit function theorem holds under the conclusion that there exists some local solution for z_n. This theorem holds only in the holomorphic functions and not for real-analytic functions. For example $x^2 - y^3 = 0$ has a unique real solution $y = x^{2/3}$, but that function is not even differentiable.

Proof. This is a local statement about α so we need only show that α is holomorphic near some point, which, without loss of generality, is the origin. We apply the preparation theorem to find $f = uP$, where P is a Weierstrass polynomial in $\mathscr{O}(V')[z_n]$ for some $V' \subset U'$. We have

$$P(z', z_n) = \big(z_n - \alpha(z')\big)^k = z_n^k - k\alpha(z') z_n^{k-1} + \cdots$$

As the coefficients of P are holomorphic then α is holomorphic. $\square$

Proposition 6.2.6. *Suppose $f \in \mathscr{O}(U' \times \mathbb{D})$ for a domain $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Let $m \in \mathbb{N}$ be such that for each $z' \in U'$, the function $z_n \mapsto f(z', z_n)$ has precisely m geometrically distinct zeros. Then locally near each point in U' there exist m holomorphic functions $\alpha_1(z'), \ldots, \alpha_m(z')$, $k_1, \ldots, k_m \in \mathbb{N}$, and a nonvanishing holomorphic function u such that*

$$f(z', z_n) = u(z', z_n) \prod_{j=1}^{m} \big(z_n - \alpha_j(z')\big)^{k_j}.$$

The reason why can only define α_1 through α_m locally (on a smaller domain) is that we do not know the order of α_1 through α_m and the order could change if U' is not simply connected.

> ***Exercise* 6.2.5:** *Prove Proposition 6.2.6. See Proposition 6.2.5.*

Theorem 6.2.7. *Suppose $f \in \mathscr{O}(U' \times \mathbb{D})$ for a bounded domain $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$, and that the zero set $f^{-1}(0)$ has no limit points on $U' \times \partial D$. Suppose for each $z' \in U'$, the function $z_n \mapsto f(z', z_n)$ has at most $m < \infty$ geometrically distinct zeros (and has exactly m zeros somewhere). Then there exists a holomorphic function $\Delta \colon U' \to \mathbb{C}$, not identically zero, such that for every $z' \in U'$ not in $E = \Delta^{-1}(0)$, $z_n \mapsto f(z', z_n)$ has exactly m geometrically distinct zeros in D.*

The complement of a zero set of a holomorphic function is connected, open and dense. The function Δ is called the *discriminant function* and its zero set E is called the *discriminant set*. For example for the quadratic equation, Δ is the discriminant we learned about in high school.

Proof. Let U_m be the subset of U for which $z_n \mapsto f(z', z_n)$ has exactly m zeros. Write U' as a union of disjoint sets $U' = U_m \cup E$, where $E = U' \setminus U_m$.

As the roots do not accumulate on the boundary, near each $z' \in U_m$ there is a curve in the z_n variable going around one of the roots and the same curve will not intersect any roots for nearby z'. By the argument principle, if there was a root inside a small curve, there still is for nearby z'. In other words, no root can disappear for some small open neighborhood of z', and hence U_m is open.

By the previous proposition, locally on U_m there exist m holomorphic functions $\alpha_1, \dots, \alpha_m$ giving the roots. We cannot define these on all of U_m as we do not know their order. The function

$$\Delta(z') = \prod_{j \neq k} \bigl(\alpha_j(z') - \alpha_k(z')\bigr)$$

does not depend on the order and hence is a well-defined holomorphic function in all of U_m.

Let $p' \in E \cap \overline{U_m}$. By the argument principle, there are some roots that must have coalesced. There are fewer than m roots at p'. By the argument principle, no root can suddenly appear inside any curve that contained no roots above p' as z' varies near p'. Hence there must be at least one root such that if we make a small curve around it and move z' into U_m (but stay near p'), the root will split into multiple geometrically distinct roots. In other words, since D and therefore the α_j were all bounded, we find that $\Delta(z') \to 0$ as $z' \to p'$.

We defined Δ on U_m. Set $\Delta(z') = 0$ for $z' \in E$. As $\Delta(z') \to 0$ as z' goes to E, Δ is continuous, and it is holomorphic outside its zero set. We apply Radó's theorem to find Δ is holomorphic in U'. $\square$

The discriminant given above is really the discriminant of the set $f^{-1}(0)$ rather than of the corresponding Weierstrass polynomial. Often for Weierstrass polynomials the discriminant is defined as $\prod_{j \neq k} \bigl(\alpha_j(z') - \alpha_k(z')\bigr)$ taking multiple roots into account, and therefore the discriminant could be identically zero. It will be clear from upcoming exercises that if the Weierstrass polynomial is irreducible, then the two notions do in fact coincide.

> ***Exercise* 6.2.6:** *Prove that if $f \in \mathcal{O}(U)$, then $U \setminus Z_f$ is not simply connected if Z_f is nonempty. In particular, in the theorem, $U' \setminus E$ is not simply connected if $E \neq \emptyset$.*

6.3 The ring of germs

Let us prove some basic properties of the ring of germs of holomorphic functions.

Theorem 6.3.1. *$\mathcal{O}_p$ is Noetherian.*

Noetherian means that for any ideal $I \subset \mathcal{O}_p$ there exist finitely many elements $f_1, \dots, f_k \in I$ such that every $g \in I$ can be written as $g = c_1 f_1 + \cdots + c_k f_k$.

Proof. Without loss of generality assume that p is the origin. The proof is by induction on dimension. That ${}_1\mathcal{O}_0$ is Noetherian is a simple one variable argument that is left as an exercise.

For induction suppose ${}_{n-1}\mathcal{O}_0$ is Noetherian. If $I = \{0\}$ or if $I = \mathcal{O}_p$, then the assertion is obvious. Therefore, we assume that all elements of I vanish at the origin (otherwise $I = \mathcal{O}_0$), and there exist elements that are not identically zero. Let g be such an element. After perhaps a linear change of coordinates, assume that g is a Weierstrass polynomial in z_n by the preparation theorem.

By the division theorem, every element $f \in I$ is of the form $f = gq + r$ where r is a polynomial in z_n. The function r is an element of ${}_{n-1}\mathscr{O}_0[z_n]$ and $r \in I$. The set $J = I \cap {}_{n-1}\mathscr{O}_0[z_n]$ is an ideal in the ring ${}_{n-1}\mathscr{O}_0[z_n]$. By the Hilbert basis theorem, as ${}_{n-1}\mathscr{O}_0$ is Noetherian, the polynomial ring ${}_{n-1}\mathscr{O}_0[z_n]$ is Noetherian. Thus J has finitely many generators. The generators of J together with g must generate I as any element in I is $gq + r$. □

***Exercise* 6.3.1:** *Find all the ideals in* ${}_1\mathscr{O}_0$.

***Exercise* 6.3.2:** *Finish the proof by proving* ${}_1\mathscr{O}_0$ *is Noetherian. In fact, prove that it is a principal ideal domain (PID). Then prove that if* $n > 1$, *then* ${}_n\mathscr{O}_p$ *is not a PID.*

Theorem 6.3.2. *$\mathscr{O}_p$ is a unique factorization domain (UFD). That is, up to a multiplication by a unit, every element has a unique factorization into irreducible elements of $\mathscr{O}_p$.*

Proof. Again assume p is the origin and induct on the dimension. The one dimensional statement is again an exercise. If ${}_{n-1}\mathscr{O}_0$ is a UFD then ${}_{n-1}\mathscr{O}_0[z_n]$ is a UFD by the Gauss lemma.

Take $f \in {}_n\mathscr{O}_0$. After perhaps a linear change of coordinates $f = qW$, for q a unit in ${}_n\mathscr{O}_0$, and W a Weierstrass polynomial in z_n. As ${}_{n-1}\mathscr{O}_0[z_n]$ is a UFD, W has a unique factorization in ${}_{n-1}\mathscr{O}_0[z_n]$ into $W = W_1W_2\cdots W_k$. So $f = qW_1W_2\cdots W_k$. That W_j are irreducible in ${}_n\mathscr{O}_0$ is left as an exercise.

Suppose $f = \tilde{q}g_1g_2\cdots g_m$ is another factorization. We notice that the preparation theorem applies to each g_j. Therefore write $g_j = u_j\widetilde{W}_j$ for a unit u_j and a Weierstrass polynomial $\widetilde{W}_j$. We obtain $f = u\widetilde{W}_1\widetilde{W}_2\cdots\widetilde{W}_m$ for a unit u. By uniqueness part of the preparation theorem we obtain $W = \widetilde{W}_1\widetilde{W}_2\cdots\widetilde{W}_m$. Conclusion is then obtained by noting that ${}_{n-1}\mathscr{O}_0[z_n]$ is a UFD. □

***Exercise* 6.3.3:** *Finish the proof of the theorem by proving* ${}_1\mathscr{O}_p$ *is a unique factorization domain.*

***Exercise* 6.3.4:** *Show that if an element is irreducible in* ${}_{n-1}\mathscr{O}_0[z_n]$, *then it is irreducible in* ${}_n\mathscr{O}_0$.

6.4 Varieties

If $f: U \to \mathbb{C}$ is a function, let $Z_f = f^{-1}(0)$ denote the zero set as before.

Definition 6.4.1. Let $U \subset \mathbb{C}^n$ be an open set. Let $X \subset U$ be a set such that near each point $p \in U$, there exists a neighborhood N of p and a family of holomorphic functions $\mathscr{F}$ defined on N such that

$$N \cap X = \{z \in N : f(z) = 0 \text{ for all } f \in \mathscr{F}\} = \bigcap_{f \in \mathscr{F}} Z_f.$$

Then X is called a (*complex* or *complex analytic*) *variety* or a *subvariety* of U. Sometimes X is also called an *analytic set*. We say $X \subset U$ is a proper subvariety if $\emptyset \neq X \subsetneq U$.

It is useful to note what happens when we replace "near each point $p \in U$" with "near each point $p \in X$." We get a slightly different concept, and X is said to be a *local variety*. A local variety X is a subvariety of some neighborhood of X, but it is not necessarily closed in U. As a simple example, the set $X = \{z \in \mathbb{C}^2 : z_1 = 0, |z_2| < 1\}$ is a local variety, but not a subvariety of $\mathbb{C}^2$. On the other hand X is a subvariety of the unit ball $\{z : \|z\| < 1\}$.

The family $\mathscr{F}$ of functions can always be taken to be finite by the Noetherian property of $\mathscr{O}_p$. We will work with germs of functions. When (f, p) is a germ of a function it makes sense to talk about the germ (Z_f, p). We take the zero set of some representative and look at its germ at p.

Exercise* 6.4.1*: *Suppose f and g are two representatives of a germ (f, p) show that the germs (Z_f, p) and (Z_g, p) are the same.*

Let

$$I_p(X) \overset{\text{def}}{=} \{(f, p) \in \mathscr{O}_p : (X, p) \subset (Z_f, p)\}.$$

That is, $I_p(X)$ is the set of germs of holomorphic functions vanishing on X near p. It is not hard to show that $I_p(X)$ is an ideal.

Every ideal in $\mathscr{O}_p$ is finitely generated. Let $I \subset \mathscr{O}_p$ be an ideal generated by $f_1, f_2, \ldots, f_k$. Write

$$V(I) \overset{\text{def}}{=} (Z_{f_1}, p) \cap (Z_{f_2}, p) \cap \cdots \cap (Z_{f_k}, p).$$

That is, the germ of the variety cut out by the elements of I, every element of I vanishes on the points where all the generators vanish.

Exercise* 6.4.2*: *Prove that $V(I)$ is independent on the choice of generators.*

Exercise* 6.4.3*: *Suppose $I_p(X)$ is generated by the functions $f_1, f_2, \ldots, f_k$. Prove*

$$(X, p) = (Z_{f_1}, p) \cap (Z_{f_2}, p) \cap \cdots \cap (Z_{f_k}, p).$$

Exercise* 6.4.4*: *Show that given a germ (X, p) of a variety at p then $V\big(I_p(X)\big) = (X, p)$ (see above), and given an ideal $I \subset \mathscr{O}_p$, then $I_p\big(V(I)\big) \subset I$.*

As $\mathscr{O}_p$ is Noetherian, $I_p(X)$ must be finitely generated. Therefore, near each point p only finitely many functions are necessary to define a subvariety, and by the exercise above, those functions then also "cut out" the subvariety. When one says *defining functions* for a germ of a variety, one generally means that those functions generate the ideal, not just that their common zero set happens to be the variety. A theorem that we will not prove here in full generality, the *Nullstellensatz*, says that if we take the variety defined by functions in an ideal I, and look at the ideal given by that variety we obtain the radical of I. In more concise language the Nullstellensatz says $I_p\big(V(I)\big) = \sqrt{I}$. Therefore, germs of varieties are in one-to-one correspondence with radical ideals of $\mathscr{O}_p$.

The local properties of a subvariety at p are therefore encoded in the properties of the ideal $I_p(X)$. Therefore, the study of subvarieties often involves the study of the various algebraic properties of the ideals of $\mathscr{O}_p$.

Let us define the regular points of a variety and their dimension. If $f \colon U' \subset \mathbb{C}^k \to \mathbb{C}^{n-k}$ is a mapping, then by a graph of f we mean the set in $U' \times \mathbb{C}^{n-k} \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ defined by

$$\{(z,w) \in U' \times \mathbb{C}^{n-k} : w = f(z)\}.$$

Definition 6.4.2. Let $X \subset U \subset \mathbb{C}^n$ be a (complex) subvariety of U. Let $p \in X$ be a point. If there exists a (complex) affine change of coordinates such that near p the set X can be written as a graph of a holomorphic mapping $f \colon U' \subset \mathbb{C}^k \to \mathbb{C}^{n-k}$ (for some $k \in \mathbb{N}_0$) then p is a *regular point* (or *simple point*) of X and the *dimension* of X at p is k or $\dim_p X = k$. If all points of X are regular points of dimension k, then X is called a *complex manifold*, or *complex submanifold* of (complex) dimension k.

As the ambient dimension is n we say X is of *codimension* $n-k$ at p.

The set of regular points of X will be denoted by X_{reg}. Any point that is not regular is *singular*. The set of singular points of X is denoted by X_{sing}.

So $p \in X$ is a regular point if after perhaps a complex affine change of coordinates there is a neighborhood $U' \times U'' \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ of p such that $X \cap (U' \times U'') = \{(z,w) \in U' \times U'' : w = f(z)\}$. Note that an isolated point of the variety is a regular point of dimension 0.

We also define dimension at a singular point. The set of regular points of a complex subvariety is open and dense in the subvariety. Thus, a variety is regular at most points. This of course means that the following definition does make sense.

Definition 6.4.3. Let $X \subset U \subset \mathbb{C}^n$ be a (complex) subvariety of U. Let $p \in X$ be a point. We define the *dimension* of X at p to be

$$\dim_p X \overset{\text{def}}{=} \max\{k \in \mathbb{N}_0 : \ \forall \text{ neighborhoods } N \text{ of } p, \exists q \in N \cap X_{reg} \text{ with } \dim_q X = k\}.$$

If (X,p) is a germ, we say the dimension of (X,p) is the dimension of X at p.

The dimension of the entire subvariety X is defined to be

$$\dim X \overset{\text{def}}{=} \max_{p \in X} \dim_p X.$$

We say that X is of *pure dimension* k if at all points p, dimension of X at p is k. For a germ, we say (X,p) is of pure dimension k if there exists a representative of X that is of pure dimension k. We define the word *codimension* as before, that is, the ambient dimension minus the dimension of X.

We have the following theorem, which we state without proof, at least in the general setting.

Theorem 6.4.4. *Let $U \subset \mathbb{C}^n$ be open and connected and let $X \subset U$ be a subvariety, then the set of regular points X_{reg} is open and dense in X. In fact X_{sing} is a subvariety of X.*

We proved in Theorem 1.6.2 that if X is a zero set of a single holomorphic function, then X_{reg} is open and dense in X.

Let us prove that $(Z_f)_{sing}$ is contained in the zero set of some holomorphic function that is not zero on any open set of Z_f. We need only prove this locally near the origin. After a possibly linear change of coordinates we apply the preparation theorem to find that near the origin, Z_f is the zero set of a Weierstrass polynomial $p(z', z_n)$ defined on some $U' \times \mathbb{C}$. Let E be the discriminant set for p. Above each point in $U' \setminus E$ the set is union of m distinct graphs of holomorphic functions by Proposition 6.2.6. As above each point of E there are only finitely many points of Z_f the conclusion follows.

Codimension 1 subvarieties are particularly nice. Sometimes codimension 1 subvarieties are called *hypervarieties*.

Theorem 6.4.5. *Let $U \subset \mathbb{C}^n$ be a domain and $f \in \mathscr{O}(U)$. Then Z_f is either empty, of pure codimension 1, or $Z_f = U$.*

Conversely, if (X, p) is a germ of a pure codimension 1 variety, then there is a germ holomorphic function f at p such that $(Z_f, p) = (X, p)$. Furthermore $I_p(X)$ is generated by (f, p).

Proof. We already proved all the relevant pieces of the first part of this theorem.

For the second part there has to a germ of a function that vanishes on (X, p). Suppose $p = 0$ and that after a possible linear change of coordinates we can apply Weierstrass preparation theorem. Hence suppose $f(z', z_n)$ is a germ of a Weierstrass polynomial vanishing on a germ of a codimension 1 subvariety $(X, 0)$. Suppose f is defined in $U' \times D \subset \mathbb{C}^{n-1} \times \mathbb{C}$ where U' is a small neighborhood of the origin and D is a small disc around the origin. Suppose U' and D are small enough so that there exists a representative X of $(X, 0)$ that is closed in $U' \times D$. Furthermore, we pick U' and D small enough so that the zero set of f contains X and has no limit points on $U' \times D$. Outside a discriminant set $E \subset U'$, which is a zero set of a holomorphic function, there are a certain number of geometrically distinct roots of $z_n \mapsto f(z', z_n)$. Let X' be a topological component of $X \setminus (E \times D)$. The set X' contains only regular points of X of dimension $n-1$. Above each point z', let $\alpha_1(z'), \dots, \alpha_m(z')$ denote the distinct roots that are in X', that is $\big(z', \alpha_k(z')\big) \in X'$. If α_k is a holomorphic function in some small neighborhood and $\big(z', \alpha_k(z')\big) \in X'$ at one point, then $\big(z', \alpha_k(z')\big) \in X'$ for all nearby points too. The number of such geometrically distinct roots above each point in $U' \setminus E$ is locally constant, and as $U' \setminus E$ is connected there exists a unique m. Taking

$$F(z', z_n) = \prod_{j=1}^{m} \big(z_n - \alpha_j(z')\big)$$

we find a monic polynomial in z_n with coefficients that are holomorphic functions on $U' \setminus E$, as they are symmetric in the roots. The coefficients are locally bounded on U' and therefore extend to holomorphic functions of U'. We find a polynomial

$$F(z',z_n) = z_n^m + \sum_{j=0}^{m-1} g_j(z') z_n^j,$$

whose roots above $z' \in U' \setminus E$ are simple and give precisely X' and $g_j(z')$ are holomorphic in U'. By using the argument principle again, we find that all roots above points of E are limits of roots above points in $U \setminus E$. Therefore by continuity we find that the zero set of F is the closure of X' in $U' \times D$. It is left to the reader to check that (using the argument principle) all the functions g_j vanish at the origin and F is a Weierstrass polynomial, a fact that will be useful in the exercises below. We can do this for every topological component above.

The fact that f can be chosen in such as way as to generate $I_p(X)$ is left as an exercise below. □

In other words, local properties of a codimension 1 subvariety can be studied by studying the zero set of a single Weierstrass polynomial.

Example 6.4.6: It is not true that if a dimension of a variety in $\mathbb{C}^n$ is $n-k$, then there are k holomorphic functions that "cut it out." The set defined by

$$\operatorname{rank} \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix} < 2$$

is a pure 4 dimensional subvariety of $\mathbb{C}^6$ and the defining equations are $z_1z_5 - z_2z_4 = 0$, $z_1z_6 - z_3z_4 = 0$, and $z_2z_6 - z_3z_5 = 0$. Let us state without proof that the unique singular point is the origin and there exist no 2 holomorphic functions near the origin that define this subvariety. In more technical language, the subvariety is not a *complete intersection*.

Example 6.4.7: If X is a hypervariety and E the corresponding discriminant set, it is tempting to say that the singular set of X is the set $X \cap (E \times \mathbb{C})$, which is a codimension 2 subvariety. It is true that $X \cap (E \times \mathbb{C})$ will contain the singular set, but in general the singular set is smaller. A very simple example of this behavior is the set defined by $z_2^2 - z_1 = 0$. The defining function is a Weierstrass polynomial in z_2 and the discriminant set is given by $z_1 = 0$. However, the subvariety has no singular points. A less trivial example is given in an exercise below.

***Exercise* 6.4.5:** *a) Prove that the hypervariety in $\mathbb{C}^n$ given by $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$ has an isolated singularity at the origin (that is, the origin is the only singular point).*
b) For any $0 \le k \le n-2$, find a hypervariety X of $\mathbb{C}^n$ whose set of singular points is a subvariety of dimension k.

***Exercise* 6.4.6:** *Suppose $p(z',z_n)$ is a Weierstrass polynomial of degree k such that for an open dense set of z' near the origin $z_n \mapsto p(z',z_n)$ has geometrically k roots, and such that the regular points of Z_p are connected. Show that p is irreducible in the sense that if $p = rs$ for two Weierstrass polynomials r and s, then either $r = 1$ or $s = 1$.*

***Exercise* 6.4.7:** *Suppose f is a function holomorphic in a neighborhood of the origin with $z_n \mapsto f(0,z_n)$ being of finite order. Show that*

$$f = u p_1^{d_1} p_2^{d_2} \cdots p_\ell^{d_\ell}$$

where p_j are Weierstrass polynomials of degree k_j that have generically (that is, on an open dense set) k_j distinct roots (no multiple roots), the regular points of Z_{p_j} are connected, and u is a nonzero holomorphic function in a neighborhood of the origin. See also the next section, these polynomials will be the irreducible factors in the factorization of f.

***Exercise* 6.4.8:** *Suppose (X,p) is a germ of a pure codimension 1 subvariety. Show that the ideal $I_p(X)$ is a principal ideal (has a single generator)*

***Exercise* 6.4.9:** *Suppose $I \subset \mathscr{O}_p$ is an ideal such that $V(I)$ is a germ of a pure codimension 1 subvariety. Show that the ideal I is principal.*

***Exercise* 6.4.10:** *Let $I \subset \mathscr{O}_p$ be a principal ideal. Prove the* Nullstellensatz *for hypervarieties: $I_p\big(V(I)\big) = \sqrt{I}$. That is, show that if $(f,p) \in I_p\big(V(I)\big)$, then $(f^k,p) \in I$ for some integer k.*

6.4.1 Irreducibility, local parametrization, and the Puiseux theorem

Definition 6.4.8. A germ of a complex variety $(X,p) \subset (\mathbb{C}^n,p)$ is said to be *reducible* at p if there exist two germs (X_1,p) and (X_2,p) with $(X_1,p) \not\subset (X_2,p)$ and $(X_2,p) \not\subset (X_1,p)$ such that $(X,p) = (X_1,p) \cup (X_2,p)$. Else the germ (X,p) is *irreducible* at p.

Similarly globally, a subvariety $X \subset U$ is *reducible* in U if there exist two subvarieties X_1 and X_2 of U with $X_1 \not\subset X_2$ and $X_2 \not\subset X_1$ such that $X = X_1 \cup X_2$. Else the subvariety X is *irreducible* in U.

***Exercise* 6.4.11:** *Prove (X,p) is irreducible if and only if $I_p(X)$ is a prime ideal. Recall an ideal I is prime if $ab \in I$ implies either $a \in I$ or $b \in I$.*

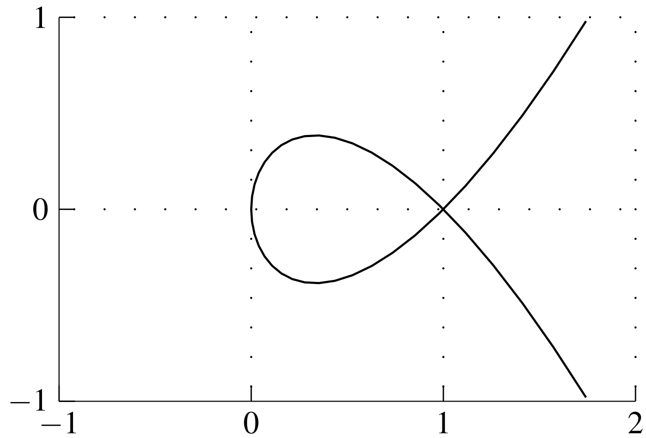
***Exercise* 6.4.12:** *Suppose (X,p) is of pure codimension 1. Prove (X,p) is irreducible if and only if there exists a representative of X whose set or regular points is connected. Hint: see the proof of Theorem 6.4.5.*

***Exercise* 6.4.13:** *Suppose $X \subset U$ is a subvariety of a domain $U \subset \mathbb{C}^n$ of pure codimension 1. Prove X is irreducible if and only if the set of regular points is connected. Hint: previous exercise.*

Example 6.4.9: The local and global behavior are different. For example

$$z_2^2 = z_1(z_1-1)^2$$

is irreducible in $\mathbb{C}^2$ (the regular points are connected) but locally at the point $(1,0)$ it is reducible. Here is a plot in two real dimensions:



Each variety can be split into its irreducible parts.

Proposition 6.4.10. *Let $(X,p) \subset (\mathbb{C}^n,p)$ be a germ of a subvariety. Then there exist finitely many irreducible subvarieties $(X_1,p),\ldots,(X_k,p)$ such that $(X_1,p)\cup\ldots\cup(X_k,p) = (X,p)$ and such that $(X_j,p) \not\subset (X_\ell,p)$ for all j and ℓ. These are called the* irreducible components.

Proof. Suppose (X,p) is reducible, so find $(Y_1,p) \not\subset (Y_2,p)$ and $(Y_2,p) \not\subset (Y_1,p)$ such that $(Y_1,p)\cup(Y_2,p) = (X,p)$. As $(Y_j,p) \subsetneq (X,p)$, then $I_p(Y_j) \supsetneq I_p(X)$ for both j. If both (Y_1,p) and (Y_2,p) are irreducible, then stop, we are done. Otherwise apply the same reasoning to whichever (or both) (Y_j,p) that was reducible. After finitely many steps you must come to a stop as you cannot have an infinite ascending chain of ideals since $\mathscr{O}_p$ is Noetherian. □

For a germ of a variety defined by the vanishing of a single holomorphic function, the UFD property of ${}_n\mathscr{O}_p$ gives the irreducible components. You found this factorization in an exercise above.

For each irreducible component, we have the following structure. We give the theorem without proof in the general case, although we have essentially proved it already for pure codimension 1 (to put it together is left as an exercise).

Theorem 6.4.11 (Local parametrization theorem)**.** *Let $(X,0)$ an irreducible germ of a complex variety of dimension k in $\mathbb{C}^n$. Let X denote a representative of the germ. Then after a linear change of coordinates, we let $\pi\colon \mathbb{C}^n \to \mathbb{C}^k$ be the projection onto the first k components, and obtain that there exists a neighborhood $U \subset \mathbb{C}^n$ of the origin, and a proper subvariety $E \subset \pi(U)$ such that*

(i) $X' = X \cap U \setminus \pi^{-1}(E)$ is a connected k-dimensional complex manifold that is dense in $X \cap U$.

(ii) $\pi\colon X' \to \pi(U)\setminus E$ is an m-sheeted covering map for some integer m.

(iii) $\pi\colon X\cap U \to \pi(U)$ is a proper mapping.

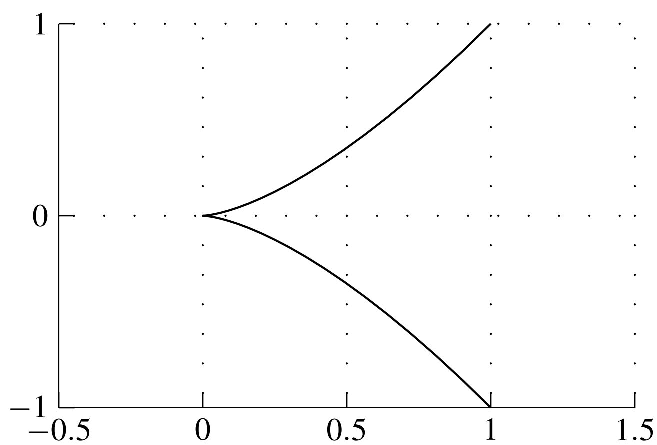
An m-sheeted covering map in this case will be a local biholomorphism that is an m-to-1 map.

Exercise* 6.4.14*: *Use Theorem 6.2.7 to prove the parametrization theorem if $(X,0)$ is of pure codimension 1.*

Let $(z_1,\ldots,z_n)$ be the coordinates. The linear change of coordinates needed in the theorem is to ensure that the set defined by $z_1 = z_2 = \cdots = z_k = 0$ intersected with X is an isolated point at the origin. This is precisely the same condition needed to apply Weierstrass preparation theorem in the case when X is the zero set of a single function.

We saw hypocrites are the simpler cases of complex analytic subvarieties. The other end of the spectrum, that is complex analytic subvarieties of dimension 1 are also easy to handle for different reasons. All complex analytic subvarieties of dimension 1 are locally just analytic discs. In fact, they are locally the one-to-one holomorphic images of discs and so they have a natural topological manifold structure even at singular points.

Example 6.4.12: For example, the image of the analytic disc $\xi \mapsto (\xi^2,\xi^3)$ is the variety defined by $z_1^3 - z_2^2 = 0$ in $\mathbb{C}^2$, the so-called "cusp". Here is a plot in two real dimensions:



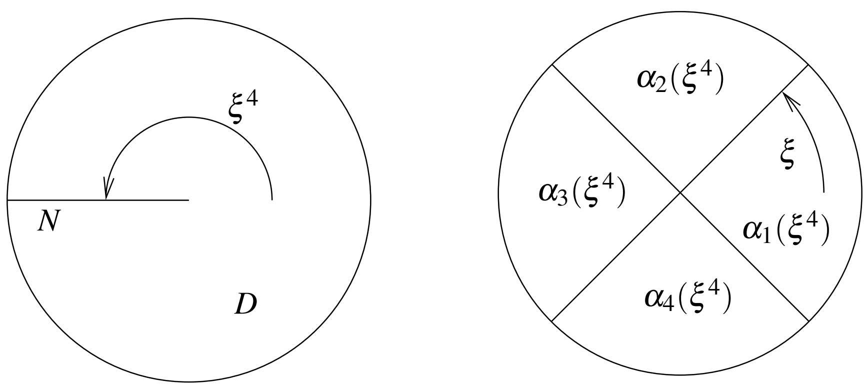
The following theorem is often stated only in $\mathbb{C}^2$ for zero sets of a single function although it follows in the same way from the local parametrization theorem in higher dimensional spaces. Of course, we only proved that theorem (or in fact you the reader did so in an exercise), for codimension 1 subvarieties, and therefore, we also only have a complete proof of the following also in $\mathbb{C}^2$.

Theorem 6.4.13 (Puiseux)**.** *Let $(z,w) \in \mathbb{C}\times\mathbb{C}^{n-1}$ be coordinates. Suppose $f\colon U \subset \mathbb{C}\times\mathbb{C}^{n-1} \to \mathbb{C}^\ell$ is a holomorphic map such that $f(z,w) = 0$ defines a dimension 1 subvariety X of U and $0 \in X$, and $w \mapsto f(0,w)$ has an isolated zero at the origin.*

Then there exists an integer k and a holomorphic function g defined near the origin in $\mathbb{C}$ such that for all ξ near the origin

$$f\bigl(\xi^k, g(\xi)\bigr) = 0.$$

Proof. Assume $(X,0)$ without loss of generality is irreducible so that we can apply the local parametrization theorem We work in a disc $D \subset \mathbb{C}$ centered at the origin, such that the origin is the unique point of the discriminant set, that is the subvariety E. Let $N = \{z \in D : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$. As $D \setminus N$ is simply connected we have the well-defined functions $\alpha_1(z), \dots, \alpha_m(z)$ holomorphic on $D \setminus N$ that are solutions to $f\big(z, \alpha_j(z)\big) = 0$. By analytic continuation, these functions continue across N away from the origin and in fact must be one of the other roots (and by continuity the same root along the entire N). Therefore there must be a permutation σ on m elements such that as we move counterclockwise from the upper half plane across N to the lower half plane $\alpha_j(z)$ is continued as $\alpha_{\sigma(j)}(z)$. There exists some number k (for example $k = m!$, but smaller numbers may work) such that σ^k is the identity. Therefore as ξ goes around a circle, ξ^k goes around a circle k times. We start with $\alpha_1(\xi^k)$ and move around the circle continuing analytically, we transition to $\alpha_{\sigma(1)}(\xi^k)$ after a sector of angle $2\pi/k$ then $\alpha_{\sigma(\sigma(1))}(\xi^k)$, and so on. After k steps, that is as ξ moved all the way around the circle, because σ^k is the identity we are back at $\alpha_1(\xi^k)$ and we have a well defined function. This is our function $g(\xi)$. See the following figure for an example:



In the figure, $m = k = 4$, and the permutation takes 1 to 2, 2 to 3, 3 to 4, and 4 to 1. As ξ moves along the short circular arrow on the right, ξ^4 moves along the long circular arrow on the left. The definition of g is then given in the right hand diagram. $\square$

Exercise* 6.4.15*: *Prove that if $(X,0) \subset (\mathbb{C}^2,0)$ is an irreducible germ defined by an irreducible Weierstrass polynomial $f(z,w) = 0$ (polynomial in w) of degree k. Then there exists a holomorphic g such that $f\big(z^k, g(z)\big) = 0$ and $z \mapsto \big(z^k, g(z)\big)$ is one-to-one and onto a neighborhood of 0 in X.*

Exercise* 6.4.16*: *Suppose $(X,0) \subset (\mathbb{C}^2,0)$ is a germ of a dimension 1 subvariety. Show that after perhaps a linear change of coordinates, there are natural numbers $d_1, \dots, d_k$ and holomorphic functions $c_1(z), \dots, c_k(z)$ vanishing at 0, such that X can be defined near 0 by*

$$\prod_{j=1}^{k} \big(w^{d_j} - c_j(z)\big) = 0.$$

Exercise* 6.4.17*: *Using the local parametrization theorem, prove that if (X,p) is an irreducible germ of a complex variety of dimension greater than 1, then there exists a neighborhood U of p and a closed subvariety $X \subset U$ (whose germ at p is (X,p)), such that for every $q \in X$ there exists an irreducible subvariety $Y \subset X$ of dimension 1 such that $p \in Y$ and $q \in Y$.*

Exercise* 6.4.18*: *Prove a stronger version of the above exercise. Show that not only is there a Y, but an analytic disc $\varphi \colon \mathbb{D} \to U$ such that $\varphi(\mathbb{D}) \subset X$, $\varphi(0) = p$ and $\varphi(1/2) = q$.*

Exercise* 6.4.19*: *Suppose $X \subset U$ is a subvariety of domain $U \subset \mathbb{C}^n$. Suppose that for any two points p and q on X there exists a finite sequence of points $p_0 = p, p_1, \ldots, p_k = q$ in X, and a sequence of analytic discs $\Delta_j \subset X$ such that p_j and p_{j-1} are in Δ_j.*

Exercise* 6.4.20*: *Prove a maximum principle using the above exercises. Suppose $X \subset U$ is an irreducible subvariety of an open set U, and suppose $f \colon U \to \mathbb{R} \cup \{-\infty\}$ is a plurisubharmonic function. If the modulus of the restriction $f|_X$ achieves a maximum at some point $p \in X$, then the restriction $f|_X$ is constant.*

Exercise* 6.4.21*: *Prove that an analytic disc (namely the image of the disc) in $\mathbb{C}^2$ is a one dimensional complex subvariety.*

Because of the Puiseux theorem, when we are looking at germs of one dimensional complex varieties, we often simply parametrize them. For varieties of larger dimensions we can always find enough analytic curves through any point, and then we parametrize those curves.

6.5 Segre varieties and CR geometry

We saw before that existence of analytic discs (and varieties) in boundaries of domains says a lot about the geometry of the boundary.

Example 6.5.1: Suppose a smooth real hypersurface $M \subset \mathbb{C}^n$ contains a complex hypersurface X, that is, a zero set of a holomorphic function with nonvanishing derivative. After changing variables, M can be locally given by

$$\operatorname{Im} w = \rho(z, \bar{z}, \operatorname{Re} w),$$

where $(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C}$, and the complex hypersurface is given by $w = 0$. Therefore ρ is divisible by $\operatorname{Re} w$, that is,

$$\operatorname{Im} w = (\operatorname{Re} w)\widetilde{\rho}(z, \bar{z}, \operatorname{Re} w).$$

So the Levi-form at the origin must vanish, that is the Levi-form of M vanishes on $M \cap X$.

Example 6.5.2: The vanishing of the Levi-form is not necessary if the complex varieties in M are small. Consider $M \subset \mathbb{C}^3$ with a nondegenerate (but not definite) Levi-form:

$$\operatorname{Im} w = |z_1|^2 - |z_2|^2 .$$

For any $\theta \in \mathbb{R}$, M contains the complex line L_θ, given by $z_1 = e^{i\theta} z_2$ and $w = 0$. The union of those complex lines is not contained in some unique complex subvariety inside M. Any complex subvariety that contains all L_θ must contain the hypersurface given by $w = 0$, which is not contained in M.

Exercise* 6.5.1*: *Let $M \subset \mathbb{C}^n$ be a real hypersurface. Show that if M at p contains a complex hypersurface of dimension more than $\frac{n-1}{2}$, then the Levi-form must be degenerate, that is, it must have at least one zero eigenvalue.*

Exercise* 6.5.2*: *Let $M \subset \mathbb{C}^n$ be a pseudoconvex real hypersurface. Suppose M at p contains a dimension k complex submanifold X. Show that the Levi-form has at least k zero eigenvalues.*

Exercise* 6.5.3*: *Find an example of a real hypersurface $M \subset \mathbb{C}^n$ that contains a unique complex analytic subvariety through a point p and this subvariety is singular.*

Let us discuss a tool, *Segre variety*, that allows us to find such complex subvarieties inside M, and much more. Segre varieties only work in the real-analytic setting and rely on complexification.

Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface and $p \in M$. Suppose $M \subset U$, where $U \subset \mathbb{C}^n$ is a small domain with a defining function $r\colon U \to \mathbb{R}$ for M. That is r is a real-analytic function in U such that $M = r^{-1}(0)$, but $dr \neq 0$ on M. Furthermore let

$$U^* = \{z : \bar{z} \in U\}.$$

Suppose U is small enough so that the Taylor series for r converges in $U \times U^*$ when treating z and $\bar{z}$ as separate variables, that is $r(z,w)$ is a well defined function on $U \times U^*$, therefore $r(z,w) = 0$ defines a complexification $\mathscr{M}$ in $U \times U^*$. We will also assume that U is small enough that dr does not vanish on $\mathscr{M}$ and furthermore that $\mathscr{M}$ is connected.

Given $q \in U$, define the *Segre variety*

$$\Sigma_q(U,r) = \{z \in U : r(z,\bar{q}) = 0\} = \{z \in U : (z,\bar{q}) \in \mathscr{M}\}.$$

A priory the variety Σ_p depends on U and r, however since any real-analytic function $\widetilde{r}$ that complexifies to $U \times U^*$ and vanishes on M must also vanish on the complexification (and nowhere else) as we have seen before, the actual r does not matter, as long as it is a defining function for $\mathscr{M}$. As long as $q \in M$, then we see that $q \in \Sigma_q(U,r)$, furthermore we see that the Segre variety is always a complex hypersurface. It is not hard to see that if U is a small set around q, the same r will work there and we must get the same complexification at least in a neighborhood of q. Therefore, the germ at $q \in U$ is well defined and we write

$$\Sigma_q = \big(\Sigma_q(U,r), q\big).$$

That is, the germ at q. This way we find that the Segre variety is well defined as a germ, and so often when one talks about Σ_q without mentioning the U or r, then one means some small enough representative of a Segre variety or the germ itself.

***Exercise* 6.5.4:** *Let $r \colon U \to \mathbb{R}$ be a real-valued real-analytic function that complexifies to $U \times U^*$. Show that $r(z,\bar{w}) = 0$ if and only if $r(w,\bar{z}) = 0$. In other words $z \in \Sigma_w(U,r)$ if and only if $w \in \Sigma_z(U,r)$.*

Example 6.5.3: Suppose we start with the real-analytic hypersurface M given by

$$\operatorname{Im} w = (\operatorname{Re} w)\rho(z,\bar{z},\operatorname{Re} w).$$

with ρ vanishing at the origin. Rewriting in terms of w and $\bar{w}$ we find

$$\frac{w-\bar{w}}{2i} = \left(\frac{w+\bar{w}}{2}\right)\rho\left(z,\bar{z},\frac{w+\bar{w}}{2}\right).$$

Setting $\bar{z} = \bar{w} = 0$ we find

$$\frac{w}{2i} = \left(\frac{w}{2}\right)\rho\left(z,\bar{z},\frac{w}{2}\right).$$

As ρ vanishes at the origin, then near the origin the equation defines the complex hypersurface given by $w = 0$. So Σ_0 is defined by $w = 0$. This is precisely the complex hypersurface that lies inside M.

The last example is not a fluke. The most important property of Segre varieties is that it locates complex subvarieties in a real-analytic submanifold. We will phrase it in terms of analytic discs, which is in fact enough as complex subvarieties can be filled with analytic discs, as we have seen.

Proposition 6.5.4. *Let $M \subset \mathbb{C}^n$ be a real-analytic hypersurface and $p \in M$. Suppose $\Delta \subset M$ is a nonconstant analytic disc through p. Then as germs $(\Delta, p) \subset \Sigma_p$.*

Proof. Let U be a neighborhood of p where a representative of Σ_p is defined, that is, we will assume that Σ_p is a closed subset of U, and suppose $r(z,\bar{z})$ is the corresponding defining function. Let $\varphi \colon \mathbb{D} \to \mathbb{C}^n$ be the parametrization of Δ with $\varphi(0) = p$. We restrict φ to a smaller disc around the origin, and since we are only interested in the germ of Δ at p this is sufficient (if there are multiple points of $\mathbb{D}$ that go to p we repeat the argument for each one). So let us assume without loss of generality that $\Delta \subset U$. Since $\Delta \subset M$ we have

$$r\big(\varphi(\xi),\overline{\varphi(\xi)}\big) = r\big(\varphi(\xi),\bar{\varphi}(\bar{\xi})\big) = 0.$$

This function is a real-analytic function of ξ and therefore for some small neighborhood of the origin, it will complexify. In fact it is not difficult to see that it complexifies to $\mathbb{D} \times \mathbb{D}$ as $\varphi(\xi) \in U$. So we can treat ξ and $\bar{\xi}$ as separate variables. By complexification we find that the equation holds for all such independent ξ and $\bar{\xi}$. Set $\bar{\xi} = 0$ to obtain for all $\xi \in \mathbb{D}$ that

$$0 = r\big(\varphi(\xi),\bar{\varphi}(0)\big) = r\big(\varphi(\xi),\bar{p}\big).$$

In particular $\varphi(\mathbb{D}) \subset \Sigma_p$ and the result follows. $\square$

***Exercise* 6.5.5:** *Show that if a real-analytic real hypersurface $M \subset \mathbb{C}^n$ is strictly pseudoconvex and $p \in M$, then $\Sigma_p \cap (M,p) = \{p\}$ (as germs).*

***Exercise* 6.5.6:** *Use the proposition and the above exercise to show that if a real-analytic real hypersurface M is strictly pseudoconvex, then M contains no analytic discs.*

Let us end our discussion of Segre varieties by its perhaps most well-known application, the so-called Diederich-Fornæss lemma. Although we state it only for real-analytic hypersurfaces it works in greater generality. There are really two parts to it, although it is generally the corollary that is called the *Diederich-Fornæss lemma*.

First, for real-analytic hypersurfaces each point has a fixed neighborhood such that germs of complex subvarieties contained in the hypersurface extend to said fixed neighborhood.

Theorem 6.5.5 (Diederich-Fornæss)**.** *Suppose $M \subset \mathbb{C}^n$ is a real-analytic hypersurface. For every $p \in M$ there exists a neighborhood U of p with the following property: If $q \in M \cap U$ and (X,q) is a germ of a (complex analytic) subvariety of dimension at least k such that $(X,q) \subset (M,q)$, then there exists a (complex analytic) subvariety $Y \subset U$ (in particular a closed subset of U) such that $Y \subset M$ and $(X,q) \subset (Y,q)$.*

Proof. Suppose U is a polydisc centered at p small enough so that the defining function r of M complexifies to $U \times U^*$ as above. Suppose q is a point through which a positive dimensional subvariety passes. In fact, as most points of the subvariety are regular we assume that q is a regular point of (X,q) (so (X,q) is a germ of a complex submanifold). Let X be a representative of the germ (X,q) such that $X \subset M$, and we can assume $X \subset U$, although it is not closed perhaps.

Let us assume X is a holomorphic image of an open subset $V \subset \mathbb{C}^k$ via a holomorphic surjective mapping $\varphi \colon V \to X$. Since $r\big(\varphi(\xi), \overline{\varphi(\xi)}\big) = 0$ for all $\xi \in V$, then we may treat ξ and $\bar{\xi}$ separately. In particular, we find $r(z,\bar{w}) = 0$ for all $z, w \in X$.

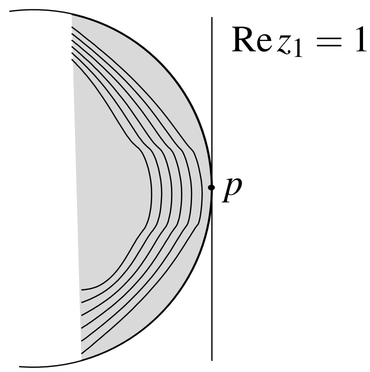
Define complex subvarieties $Y', Y \subset U$ (closed in U) by

$$Y' = \bigcap_{a \in X} \Sigma_a(U,r) \qquad \text{and} \qquad Y = \bigcap_{a \in Y'} \Sigma_a(U,r).$$

If $a \in Y'$ and $b \in X$, then $r(a,\bar{b}) = 0$. Because r is real-valued, $r(b,\bar{a}) = 0$. Therefore, $X \subset Y \subset Y'$. Furthermore $r(z,\bar{z}) = 0$ for all $z \in Y$ and so $Y \subset M$. □

Corollary 6.5.6 (Diederich-Fornæss)**.** *Suppose $M \subset \mathbb{C}^n$ is a compact real-analytic hypersurface. Then there does not exist any point $q \in M$ such that there exists a germ of a positive dimensional (complex analytic) subvariety (X,q) such that $(X,q) \subset (M,q)$.*

Proof. Let $S \subset M$ be the set of points through which there exists a complex analytic germ contained in M. As M, and hence $\overline{S}$, is compact, there must exist a point $p \in \overline{S}$ that is furthest from the origin. After a rotation by a unitary and rescaling assume $p = (1,0,\ldots,0)$. Let U be the neighborhood from the previous theorem around p. There exist germs of varieties in M through points arbitrarily close to p. So for any distance $\varepsilon > 0$, there exists a subvariety $X \subset U$ of positive dimension with $X \subset M$ that contains points that are ε close to p. Consider the function $\operatorname{Re} z_1$, whose modulus attains a strict maximum on $\overline{S}$ at p. Because $\operatorname{Re} z_1$ achieves a maximum strictly smaller than 1 on $\partial U \cap \overline{S}$, for a small enough ε, we would obtain a pluriharmonic function with a strict maximum on X, which is impossible by the maximum principle for varieties that you proved as an exercise in the previous section. The picture would look as follows:



$\square$

…and that is how using sheep's bladders can prevent earthquakes!

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