Cultivating Complex Analysis: Residue theorem (5.3 part 1)

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By Cauchy, we can compute the integral over any cycle via the residues inside. That's the Residue theorem.

Theorem (Residue theorem)

Suppose $U \subset \mathbb{C}$ is open, $S \subset U$ is a finite subset, and Γ is a cycle in $U \setminus S$ homologous to zero in U. Suppose $f : U \setminus S \to \mathbb{C}$ is holomorphic (isolated singularities on S). Then

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum_{p \in S} n(\Gamma; p) \operatorname{Res}(f; p).$$

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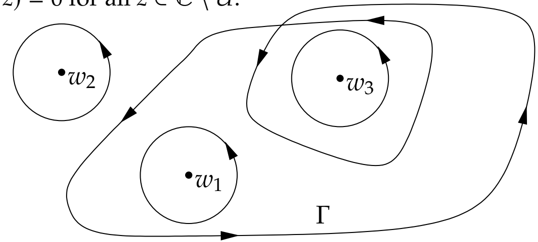

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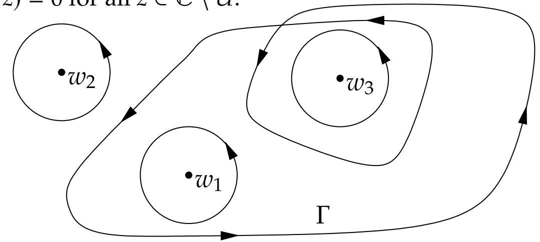

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In the picture $\quad n(\Gamma;w_1) = 1$, $n(\Gamma;w_2) = 0$, and $n(\Gamma;w_3) = 2$.

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Winding number is an integral so it is linear:

$$n(\Lambda; p) = n(\Gamma; p) - n(\Gamma; w_1)\, n(\partial\Delta_{r_1}(w_1); p) - \cdots - n(\Gamma; w_\ell)\, n(\partial\Delta_{r_\ell}(w_\ell); p).$$

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Recognize the formula for c_{-1} at w_k:

$$\frac{1}{2\pi i} \int_{\partial\Delta_{r_k}(w_k)} f(z)\, dz = \operatorname{Res}(f; w_k).$$

□