Cultivating Complex Analysis: Power series (2.3 part 1)

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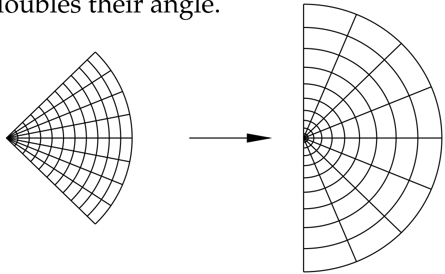
to the closed right half-plane
$\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.



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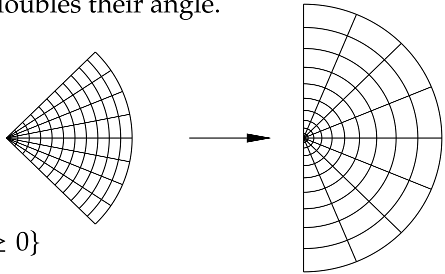

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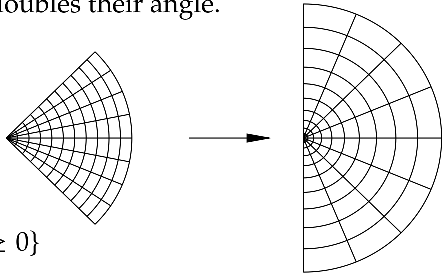

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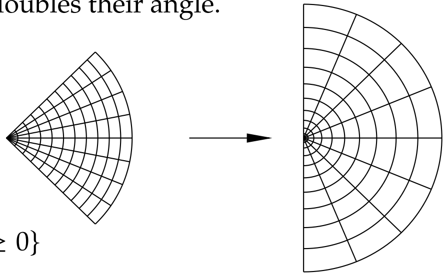

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They are equally spaced out on a circle of radius $r^{1/n}$.

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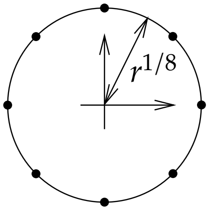
$r^{1/8}, r^{1/8}e^{i\pi/4}, r^{1/8}e^{i\pi/2}$, etc.



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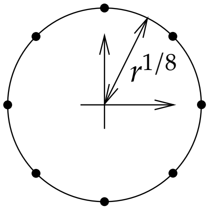
The roots of $w = 1$ are called the *roots of unity*.



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If $z \neq 1$ is such that $|z| = 1$, then z^n diverges as $n \to \infty$.

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We say a power series is *convergent* if it converges for any $z \neq p$.

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which follows by expanding $(1-z)(1+z+z^2+\cdots+z^m)$. $\square$

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Thus, an absolutely convergent series converges.