Cultivating Complex Analysis: Cycles around compacts (6.3.3)

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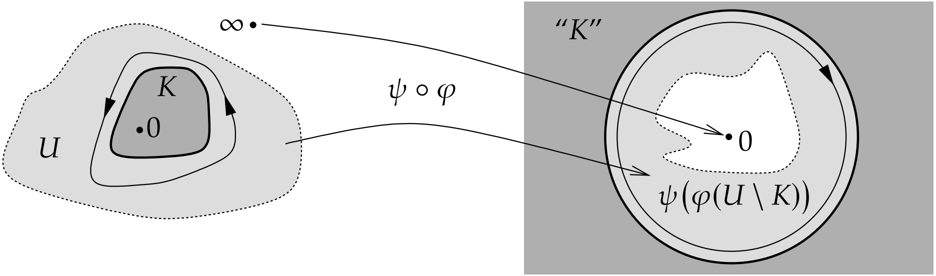
We will map to the disk, but with a twist. We'll take $\mathbb{C}_\infty \setminus K$ to the disc, and go around the "outside" in the opposite direction:



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So without loss of generality, assume that K is connected.

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Suppose $p \notin \Gamma$.

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1} \circ \psi^{-1} \circ \gamma_j} \frac{1}{z - p}\, dz$$

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$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

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(γ_1 traverses the circle backwards)

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In other words, Γ is not homologous to zero in U. □

Exercise: Suppose $\{f_n\}$ is a sequence of holomorphic functions on an open set $U \subset \mathbb{C}$ that converges uniformly on compact subsets to a nonconstant $f : U \to \mathbb{C}$. Let $K \subset U$ be a compact set. Prove that for every open neighborhood V of K in U (so $K \subset V \subset U$) there exists a smaller open neighborhood W (so $K \subset W \subset V$) and an $N \in \mathbb{N}$ such that f and f_n have the same number of zeros in W for all $n \geq N$.