Cultivating Complex Analysis: Holomorphic functions (2.1.1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

We can't even solve a simple equation like: $f' = f$.

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

We can't even solve a simple equation like: $f' = f$.

Fix $z_0 \in \mathbb{C}$ and expand $P(z)$:

$$P(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n,$$

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

We can't even solve a simple equation like: $f' = f$.

Fix $z_0 \in \mathbb{C}$ and expand $P(z)$:

$$P(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n,$$

or

$$P(z_0 + h) = c_0 + c_1 h + c_2 h^2 + \cdots + c_n h^n.$$

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

We can't even solve a simple equation like: $f' = f$.

Fix $z_0 \in \mathbb{C}$ and expand $P(z)$:

$$P(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n,$$

or

$$P(z_0 + h) = c_0 + c_1 h + c_2 h^2 + \cdots + c_n h^n.$$

Then

$$\lim_{h \to 0} \frac{P(z_0 + h) - P(z_0)}{h} = \lim_{h \to 0} \frac{P(z_0 + h) - c_0}{h} = c_1.$$

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

We can't even solve a simple equation like: $f' = f$.

Fix $z_0 \in \mathbb{C}$ and expand $P(z)$:

$$P(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n,$$

or

$$P(z_0 + h) = c_0 + c_1 h + c_2 h^2 + \cdots + c_n h^n.$$

Then

$$\lim_{h \to 0} \frac{P(z_0 + h) - P(z_0)}{h} = \lim_{h \to 0} \frac{P(z_0 + h) - c_0}{h} = c_1.$$

$P(z_0 + h)$ is approximated (locally) by $c_0 + c_1 h$ up to an error that vanishes faster than h.

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

Polynomials are nice, but there aren't many of them.

We can't even solve a simple equation like: $f' = f$.

Fix $z_0 \in \mathbb{C}$ and expand $P(z)$:

$$P(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n,$$

or

$$P(z_0 + h) = c_0 + c_1 h + c_2 h^2 + \cdots + c_n h^n.$$

Then

$$\lim_{h \to 0} \frac{P(z_0 + h) - P(z_0)}{h} = \lim_{h \to 0} \frac{P(z_0 + h) - c_0}{h} = c_1.$$

$P(z_0 + h)$ is approximated (locally) by $c_0 + c_1 h$ up to an error that vanishes faster than h.

Key point: The limits are "as a *complex* h goes to 0."

We want functions with such a derivative. That is, functions approximated by $c_0 + c_1 h$:

$$f(z_0 + h) = \underbrace{f(z_0)}_{c_0} + \underbrace{\xi h}_{c_1 h} + o(|h|) \qquad \text{for some } \xi \in \mathbb{C}.$$

We want functions with such a derivative. That is, functions approximated by $c_0 + c_1 h$:

$$f(z_0+h) = \underbrace{f(z_0)}_{c_0} + \underbrace{\xi h}_{c_1 h} + o(|h|) \qquad \text{for some } \xi \in \mathbb{C}.$$

Definition

Let $U \subset \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *complex differentiable* at $z_0 \in U$ if the limit

$$f'(z_0) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \qquad \text{exists.}$$

We want functions with such a derivative. That is, functions approximated by $c_0 + c_1 h$:

$$f(z_0+h) = \underbrace{f(z_0)}_{c_0} + \underbrace{\xi h}_{c_1 h} + o(|h|) \qquad \text{for some } \xi \in \mathbb{C}.$$

Definition

Let $U \subset \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *complex differentiable* at $z_0 \in U$ if the limit

$$f'(z_0) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \qquad \text{exists.}$$

We call $f'(z_0)$ the *complex derivative* of f at z_0. Sometimes $\frac{df}{dz}$ is used.

We want functions with such a derivative. That is, functions approximated by $c_0 + c_1 h$:

$$f(z_0 + h) = \underbrace{f(z_0)}_{c_0} + \underbrace{\xi h}_{c_1 h} + o(|h|) \qquad \text{for some } \xi \in \mathbb{C}.$$

Definition

Let $U \subset \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *complex differentiable* at $z_0 \in U$ if the limit

$$f'(z_0) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \qquad \text{exists.}$$

We call $f'(z_0)$ the *complex derivative* of f at z_0. Sometimes $\frac{df}{dz}$ is used.

$f : U \to \mathbb{C}$ is *holomorphic* if it is complex differentiable at every point.

We proved above:

Proposition

If $P(z)$ is a polynomial, then $P\colon \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic.

We proved above:

Proposition

If $P(z)$ is a polynomial, then $P \colon \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic.

An exercise:

Proposition

If $U \subset \mathbb{C}$ is open and $f \colon U \longrightarrow \mathbb{C}$ is holomorphic, then f is continuous.