Cultivating Complex Analysis: Power series (2.3 part 3)

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So a uniformly absolutely convergent series converges uniformly.

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The result follows. $\square$

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Exercise: Suppose $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have a radius of convergence at least $r > 0$. Show that $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ has a radius of convergence at least r and converges to the sum of the two series.