Cultivating Complex Analysis: Simply connected domains (4.3 part 1)

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Remark: Can a disconnected set be simply connected? We remain neutral on this.

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

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Define $$F(z) = \int_\gamma f(\zeta)\, d\zeta.$$

If α is another path from p to z, then by Cauchy

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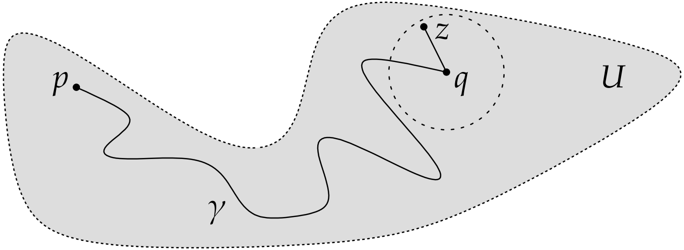
Consider $q \in U$, γ a path from p to q, and $\Delta_r(q) \subset U$. For $z \in \Delta_r(q)$

$$F(z) = \int_{\gamma+[q,z]} f(\zeta)\,d\zeta$$



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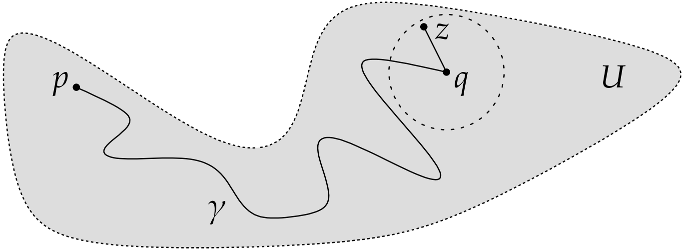

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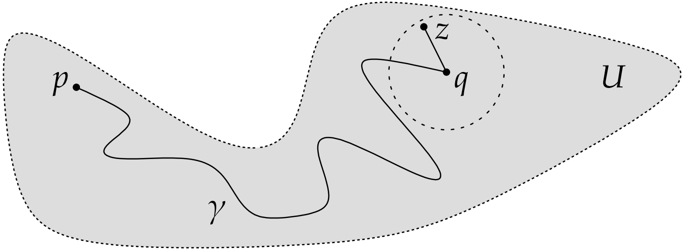

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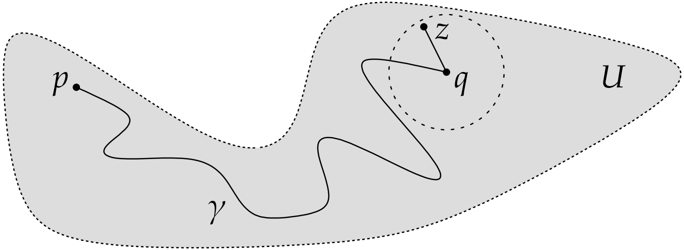

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The second term is how we defined a primitive in a star-like domain ($\Delta_r(q)$). See Proposition 3.2.11.

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$\Rightarrow \quad \exists\, C \in \mathbb{C}$ such that $e^{g(z)+C} = f(z)$. $\square$

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