Cultivating Complex Analysis: Simply connected domains (4.3 part 2)

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We proved
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\text{(i)} \arrow[ur, Rightarrow] \arrow[dr, Leftrightarrow] & \text{(v)} \arrow[d, Rightarrow] & \text{(iv)} \arrow[dl, Rightarrow] \\
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Remark: It is important to use $\mathbb{C}_\infty$ and not $\mathbb{C}$: If $U = \mathbb{C} \setminus \{0\}$ (not simply connected), then $\mathbb{C} \setminus U = \{0\}$ is connected, but $\mathbb{C}_\infty \setminus U = \{0, \infty\}$ is not connected.

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2) If $U_1 \cap U_2$ is nonempty and connected, then $U_1 \cup U_2$ is simply connected.