

Cultivating Complex Analysis: Wirtinger operators (2.2.2)

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The operators are really determined by wanting

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0, \quad \frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1.$$

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Proposition

Let $U \subset \mathbb{C}$ be open. Then $f: U \rightarrow \mathbb{C}$ is holomorphic if and only if f is (real) differentiable and

$$\frac{\partial f}{\partial \bar{z}} \equiv 0.$$

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So for a holomorphic function

$$f' = \frac{\partial f}{\partial z}.$$

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For polynomials the operators work as if z and \bar{z} were separate variables. E.g. (exercise)

$$\frac{\partial}{\partial z} [z^2 \bar{z}^3 + z^{10}] = 2z \bar{z}^3 + 10z^9 \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} [z^2 \bar{z}^3 + z^{10}] = z^2 (3\bar{z}^2) + 0.$$

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Caution: Note that $\frac{d}{dz} [z^2 \bar{z}^3 + z^{10}]$ does not exist, while $\frac{\partial}{\partial z} [z^2 \bar{z}^3 + z^{10}]$ does.

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Remark 3: The chain rule for real differentiable functions can be written as

$$\frac{\partial(g \circ f)}{\partial z}\Big|_p = \frac{\partial g}{\partial z}\Big|_{f(p)} \frac{\partial f}{\partial z}\Big|_p + \frac{\partial g}{\partial \bar{z}}\Big|_{f(p)} \frac{\partial \bar{f}}{\partial z}\Big|_p \quad \text{and} \quad \frac{\partial(g \circ f)}{\partial \bar{z}}\Big|_p = \frac{\partial g}{\partial z}\Big|_{f(p)} \frac{\partial f}{\partial \bar{z}}\Big|_p + \frac{\partial g}{\partial \bar{z}}\Big|_{f(p)} \frac{\partial \bar{f}}{\partial \bar{z}}\Big|_p.$$