

Cultivating Complex Analysis: Laurent series (4.4 part 2)

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converging uniformly absolutely on compact subsets of $\text{ann}(p; r_1, r_2)$. The numbers c_n are given by

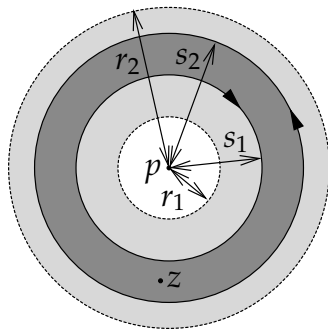
$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - p)^{n+1}} dz,$$

where γ is any circle of radius s , $r_1 < s < r_2$, centered at p oriented counterclockwise.

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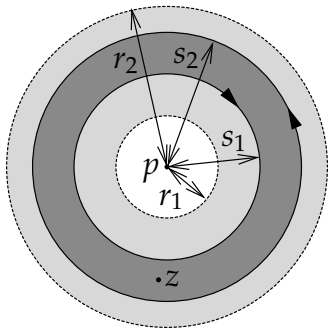
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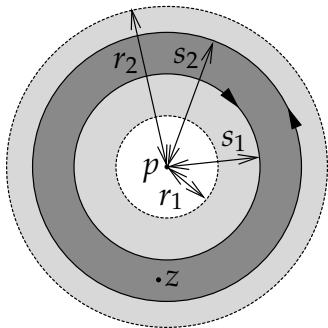


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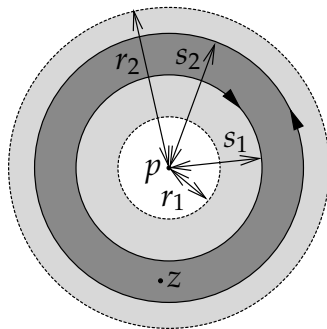
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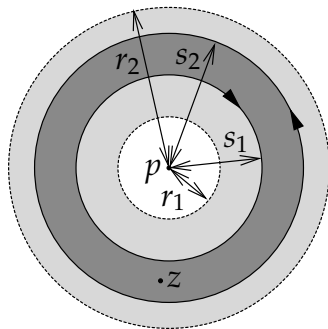
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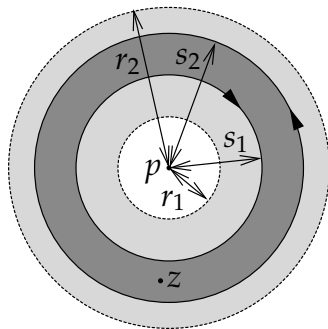
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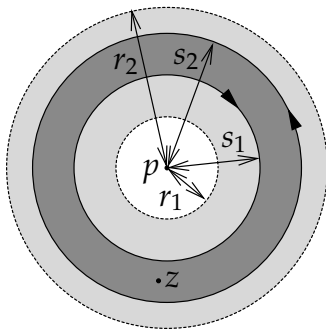
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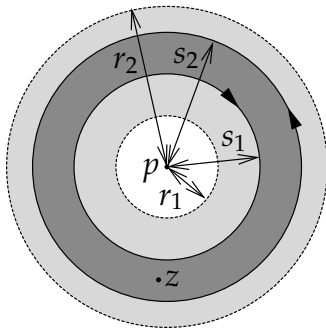
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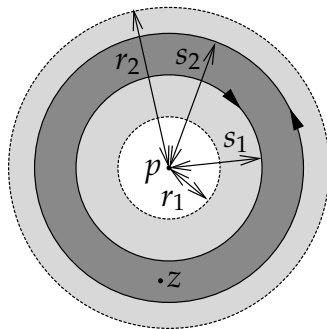
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We will expand the two integrals separately.



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We can swap the series limit with the integral as the convergence is uniform on the circle.

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Similarly for s_2 and so

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(The last equality because $\int_{\partial\Delta_s(p)} (\zeta-p)^{n-m-1} d\zeta \neq 0$ only when $n = m$.)

□

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Proposition

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$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-p)^n, \quad \text{converging uniformly on compact subsets of } \text{ann}(p; r_1, r_2).$$

Then f is holomorphic and its derivative is defined by

$$f'(z) = \sum_{n=-\infty}^{\infty} n c_n(z-p)^{n-1}, \quad \text{converging uniformly on compact subsets of } \text{ann}(p; r_1, r_2).$$

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Proof: Exercise.

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Exercise: Expand the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the sets $\text{ann}(0; 0, 1)$, $\text{ann}(0; 1, 2)$, and $\text{ann}(0; 2, \infty)$.