Cultivating Complex Analysis: The argument (1.2.3) Mapping properties of the exponential (1.2.4)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

We attempt to define the argument of $z = re^{i\theta}$:

$$\arg z \stackrel{\text{def?}}{=} \theta,$$

We attempt to define the argument of $z = re^{i\theta}$:

$$\arg z \overset{\text{def?}}{=} \theta,$$

But if θ is an argument of z, then so is $\theta + 2\pi$, $\theta - 2\pi$, or $\theta + k2\pi$ for any $k \in \mathbb{Z}$.

We attempt to define the argument of $z = re^{i\theta}$:

$$\arg z \stackrel{\text{def?}}{=} \theta,$$

But if θ is an argument of z, then so is $\theta + 2\pi$, $\theta - 2\pi$, or $\theta + k2\pi$ for any $k \in \mathbb{Z}$.

$\arg z$ is not a function but a *multivalued function*:

$$\arg z \stackrel{\text{def}}{=} \ldots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \ldots$$

We attempt to define the argument of $z = re^{i\theta}$:

$$\arg z \overset{\text{def?}}{=} \theta,$$

But if θ is an argument of z, then so is $\theta + 2\pi$, $\theta - 2\pi$, or $\theta + k2\pi$ for any $k \in \mathbb{Z}$.

$\arg z$ is not a function but a *multivalued function*:

$$\arg z \overset{\text{def}}{=} \ldots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \ldots$$

Note that $\arg 0$ is undefined.

To try to solve the multivaluedness, we define the *principal branch of* arg:

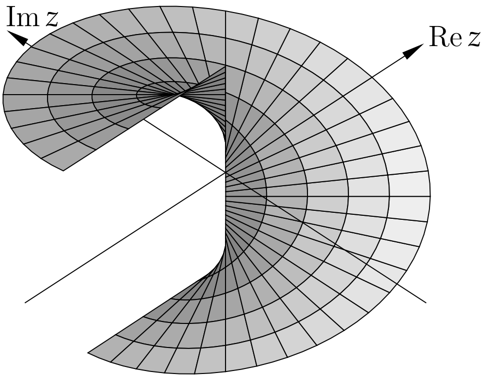
$$\operatorname{Arg} z \overset{\text{def}}{=} \theta, \qquad \text{where } -\pi < \theta \leq \pi.$$

To try to solve the multivaluedness, we define the *principal branch of* arg:

$$\operatorname{Arg} z \stackrel{\text{def}}{=} \theta, \qquad \text{where } -\pi < \theta \leq \pi.$$

A good attempt.

Alas Arg is discontinuous on the negative real axis.

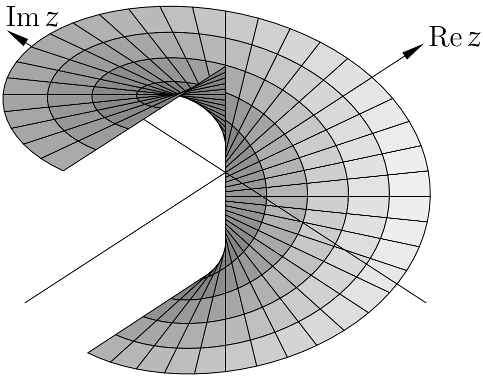


To try to solve the multivaluedness, we define the *principal branch of* arg:

$$\operatorname{Arg} z \stackrel{\text{def}}{=} \theta, \qquad \text{where } -\pi < \theta \leq \pi.$$

A good attempt.

Alas Arg is discontinuous on the negative real axis.



Remark: Some authors make principal branch take values in $[0, 2\pi)$.

Let’s see what the exponential does to $\mathbb{C}$.

Let's see what the exponential does to $\mathbb{C}$.

$e^{z+w} = e^z e^w$ implies e^z is never zero (exercise).

Let's see what the exponential does to $\mathbb{C}$.

$e^{z+w} = e^z e^w$ implies e^z is never zero (exercise).

In fact, the complex exponential is onto $\mathbb{C} \setminus \{0\}$.

Let's see what the exponential does to $\mathbb{C}$.

$e^{z+w} = e^z e^w$ implies e^z is never zero (exercise).

In fact, the complex exponential is onto $\mathbb{C} \setminus \{0\}$.

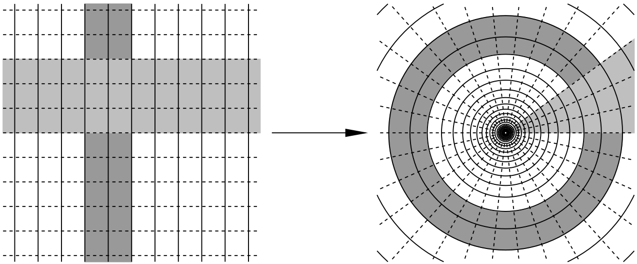
The complex exponential is not one-to-one, it is infinitely-many-to-one. For any integer k,

$$e^{z+ik2\pi} = e^z e^{ik2\pi} = e^z.$$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$e^z = e^{x+iy} = e^x e^{iy}$

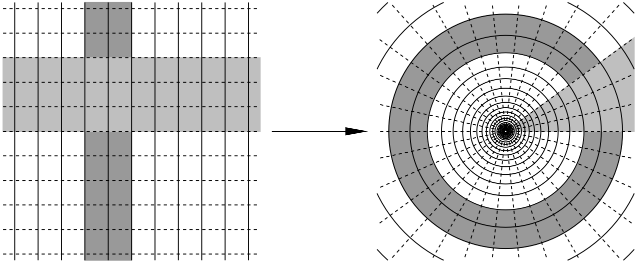
So e^z takes a vertical line, $x = c$, to the circle of radius e^c.



$e^z = e^{x+iy} = e^x e^{iy}$

So e^z takes a vertical line, $x = c$, to the circle of radius e^c.

It takes the strip $a < x < b$ to the annulus $\left\{w \in \mathbb{C} : e^a < |w| < e^b\right\}$.

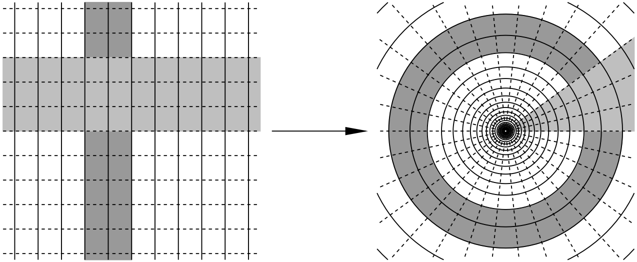


$e^z = e^{x+iy} = e^x e^{iy}$

So e^z takes a vertical line, $x = c$, to the circle of radius e^c.

It takes the strip $a < x < b$ to the annulus $\{w \in \mathbb{C} : e^a < |w| < e^b\}$.

It takes a horizontal line, $y = c$, to a ray from the origin, $\theta = c$.

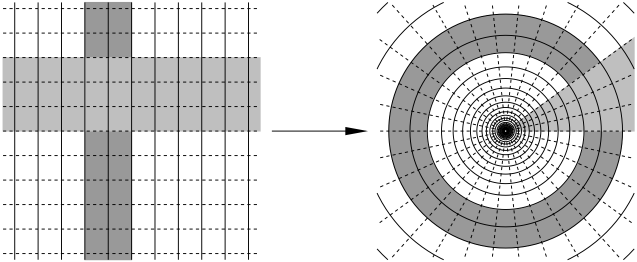


$e^z = e^{x+iy} = e^x e^{iy}$

So e^z takes a vertical line, $x = c$, to the circle of radius e^c.

It takes the strip $a < x < b$ to the annulus $\{w \in \mathbb{C} : e^a < |w| < e^b\}$.

It takes a horizontal line, $y = c$, to a ray from the origin, $\theta = c$.



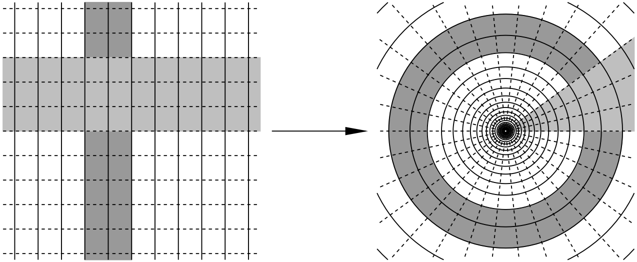
So the strip $a < y < b$ goes to the sector $\{w \in \mathbb{C} : \text{“}a < \arg w < b\text{”}\}$.

$e^z = e^{x+iy} = e^x e^{iy}$

So e^z takes a vertical line, $x = c$, to the circle of radius e^c.

It takes the strip $a < x < b$ to the annulus $\left\{w \in \mathbb{C} : e^a < |w| < e^b\right\}$.

It takes a horizontal line, $y = c$, to a ray from the origin, $\theta = c$.



So the strip $a < y < b$ goes to the sector $\left\{w \in \mathbb{C} : \text{“}a < \arg w < b\text{”}\right\}$.

Remark: e^z takes the set given by $2k\pi < \operatorname{Im} z \leq 2(k+1)\pi$ in a one-to-one way onto $\mathbb{C} \setminus \{0\}$.