

# Cultivating Complex Analysis: Zeros of holomorphic functions (5.1)

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This is not true for just real differentiable (not holomorphic) functions (see the exercises):

E.g., let  $f(0) = 0$  and  $f(x) = e^{-1/x^2}$  for  $x \in \mathbb{R} \setminus \{0\}$ .

$f$  is infinitely differentiable,  $f^{(k)}(0) = 0$  for all  $k$ , but  $f$  has an isolated zero.

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Plug in  $z = p$  to see  $k_2 = k_1$ .



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**Exercise:** Prove L'Hôpital's rule: If  $f$  and  $g$  are holomorphic near  $p$ , both with an isolated zero at  $p$ , and  $\lim_{z \rightarrow p} \frac{f'(z)}{g'(z)}$  exists (including possibly  $\infty$ ), then  $\lim_{z \rightarrow p} \frac{f(z)}{g(z)}$  exists and equals the same thing.