Cultivating Complex Analysis: Hurwitz's theorem (5.4.3)

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Theorem (Hurwitz)

Let $U \subset \mathbb{C}$ *be open and* $f_n : U \to \mathbb{C}$ *a sequence of holomorphic functions converging uniformly on compact subsets to a holomorphic* $f : U \to \mathbb{C}$. *Suppose* Γ *is a cycle in* U *homologous to zero in* U, *such that* $n(\Gamma; z)$ *is* 0 *or* 1 *for all* $z \notin \Gamma$. *Suppose* f *has no zeros on* Γ *and* k *zeros (counting multiplicity) in* $V = \{z \in U : n(\Gamma; z) = 1\}$. *Then there is an* N *such that for all* $n \geq N$, f_n *has* k *zeros (counting multiplicity) in* V.

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"No zeros on Γ" is necessary: $\quad \Gamma = \partial\mathbb{D}, \quad f(z) = z - 1, \quad f_n(z) = z + (1 - \frac{1}{n})$.

Example: For every integer $k > 0$, $\exists\, N$ such that $\forall\, d \geq N$,

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Proof:

P_d are the partial sums of the power series of $\cos(z)$, which has exactly $2k$ zeros in $\Delta_{\pi k}(0)$.

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For any $\epsilon > 0$, $z^2 + \frac{1}{n}$ has two zeros in $\Delta_\epsilon(0)$, for large enough n: $\quad \pm i\sqrt{\frac{1}{n}}$.

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Corollary

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$\Rightarrow \quad f_n$ not univalent. $\square$

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