# Cultivating Complex Analysis:
# The complex numbers as the plane (1.1.1)

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Start with $\mathbb{R}$ and arrive at $\mathbb{C}$ to be able to solve $z^2 + 1 = 0$.

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A graduate complex analysis course fills the student with unrealistic optimism.

Definition of the complex numbers (complex field, complex plane):

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{R}^2$$

with multiplication:

$$\begin{aligned}(a, b) + (c, d) &\stackrel{\text{def}}{=} (a + c, b + d),\\ (a, b)(c, d) &\stackrel{\text{def}}{=} (ac - bd, bc + da).\end{aligned}$$

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$i^2 = -1$ so $z^2 + 1 = 0$ has the two solutions $i$ and $-i$.

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Let's give names to the $x$ and $y$:

$$\operatorname{Re} z = \operatorname{Re}(x + iy) \stackrel{\text{def}}{=} \frac{z + \bar{z}}{2} = x \qquad \textit{real part} \text{ of } z.$$

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An expression in $x, y$ can be written in terms of $z, \bar{z}$ and vice versa:

$$x^3 + y^3 + 3ixy = \left(\frac{z + \bar{z}}{2}\right)^3 + \left(\frac{z - \bar{z}}{2i}\right)^3 + 3i\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z - \bar{z}}{2i}\right),$$

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$$z^2 - i\bar{z}^2 + z\bar{z} = (x + iy)^2 - i(x - iy)^2 + (x + iy)(x - iy).$$

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Almost looks as if $z$ and $\bar{z}$ were independent variables.