Cultivating Complex Analysis: The logarithm (4.1.1)

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We want to show $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$ (principal branch of the argument).

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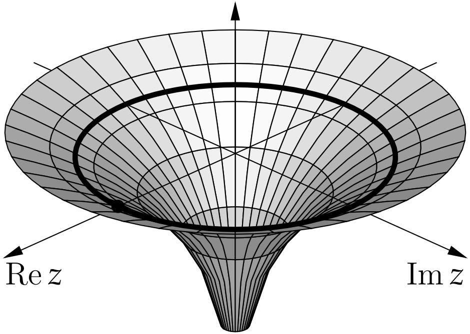
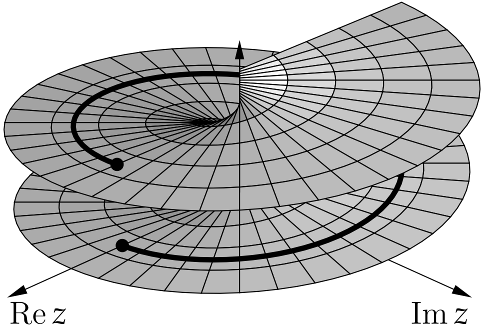
2) arg has infinitely many values.

Here are the real and imaginary parts of $\log z = \log|z| + i \arg z$:





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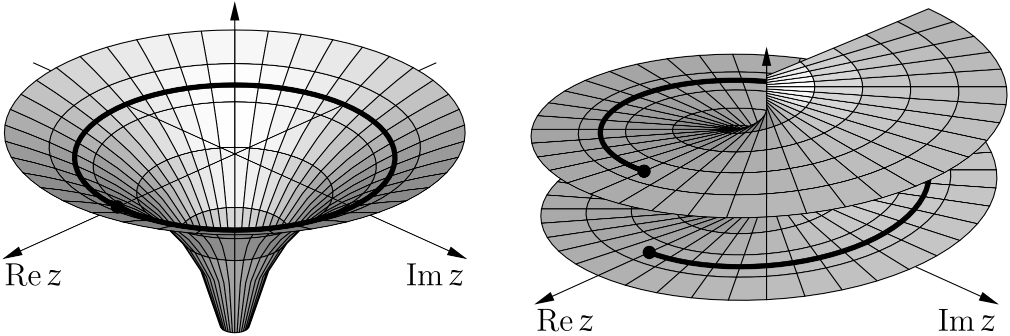

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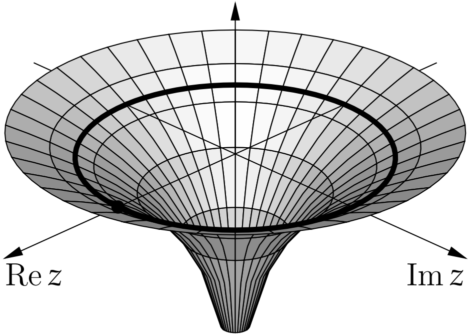

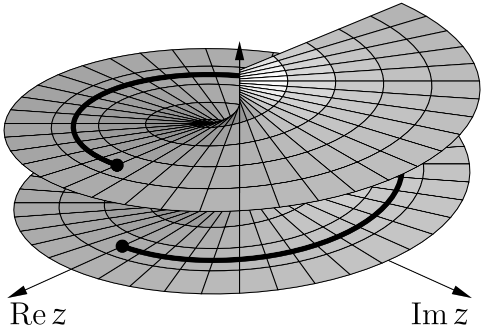

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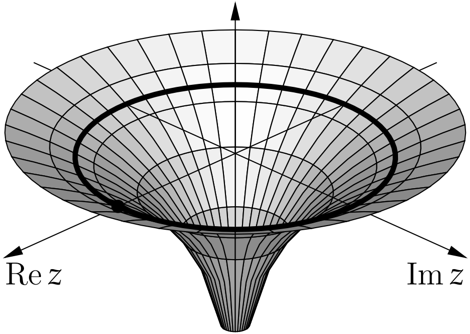

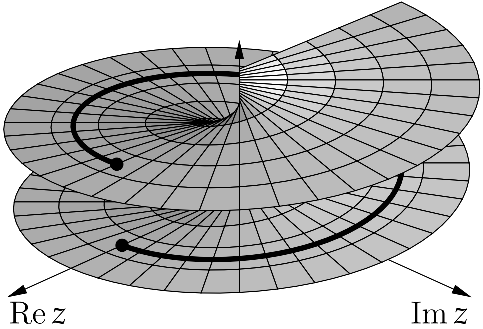

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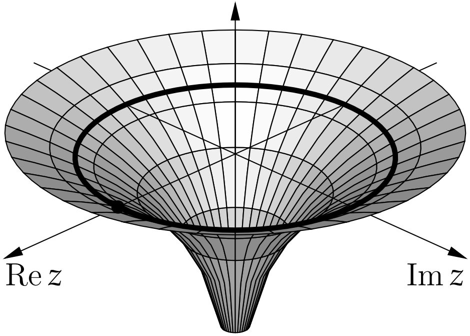

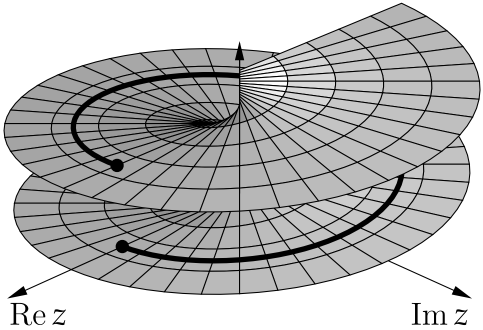

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Nevertheless, it is the correct definition. Much more useful than the principal branch.

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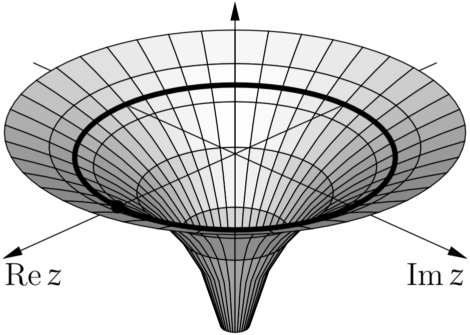
Let's see that graph again.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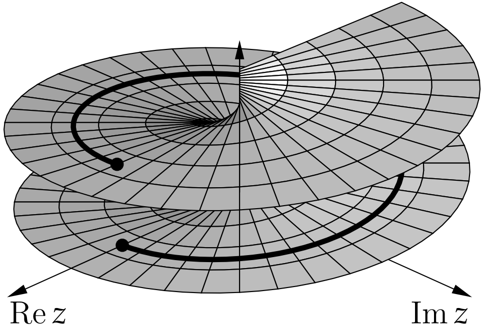
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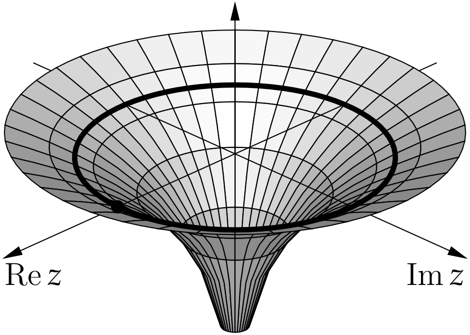

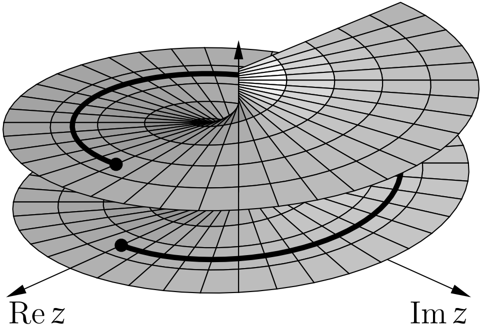

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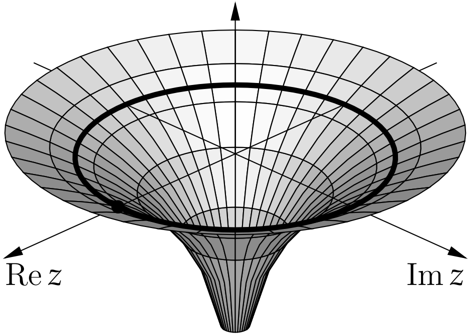

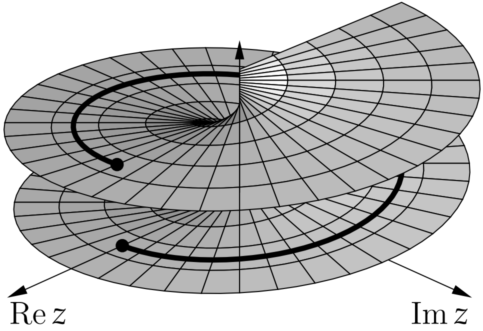

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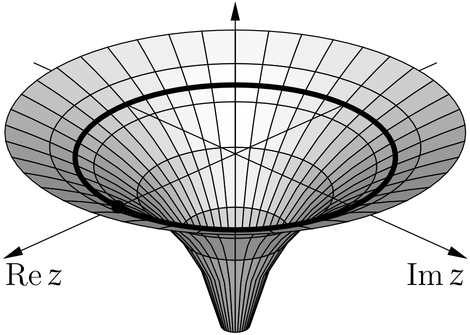

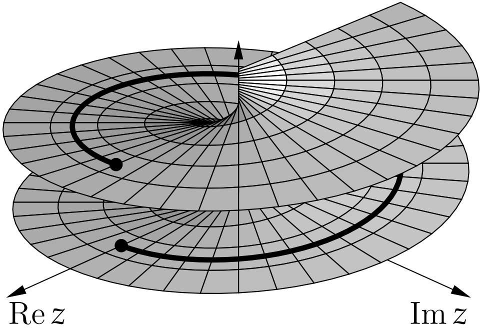

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Then we follow the graph around the circle until we end at $\log 1 = 2\pi i$.

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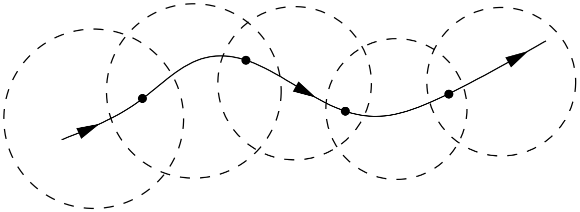

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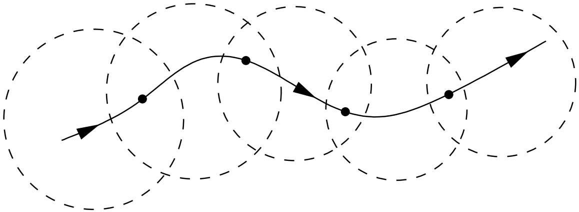

That's what we did in the computation above.