

Cultivating Complex Analysis: Arzelà–Ascoli (6.1.3)

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$\Rightarrow x \in \bar{D}$.



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Let (X, d) be a compact metric space, and let $\{f_n\}$ be pointwise bounded and equicontinuous sequence of functions $f_n: X \rightarrow \mathbb{C}$. Then $\{f_n\}$ is uniformly bounded and $\{f_n\}$ contains a uniformly convergent subsequence.

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$$|g_n(x_\ell) - g_m(x_\ell)| < \epsilon/3 \quad \text{for } \ell = 1, 2, \dots, k.$$

Let $x \in X$. $\exists \ell$ such that $x \in B(x_\ell, \delta) \Rightarrow |g_j(x) - g_j(x_\ell)| < \epsilon/3$ for all $j \in \mathbb{N}$.

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$\Rightarrow \{g_n\}$ uniformly Cauchy.

Let $D \subset X$ be dense and countable.

\exists a subsequence $\{f_{n_j}\}$ that converges pointwise on D . Write $g_j = f_{n_j}$.

$\{g_n\}$ is uniformly equicontinuous \Rightarrow Given $\epsilon > 0$, $\exists \delta > 0$ such that

$$B(x, \delta) \subset g_n^{-1}(B(g_n(x), \epsilon/3)) \quad \text{for all } x \in X, n \in \mathbb{N}.$$

Every $x \in X$ is in $B(y, \delta)$ for some $y \in D$ (density).

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$\Rightarrow \{g_n\}$ uniformly Cauchy. \mathbb{C} is a complete $\Rightarrow \{g_n\}$ uniformly convergent. □

Corollary (Arzelà–Ascoli)

Let $U \subset \mathbb{C}$ be open and let $\{f_n\}$ be pointwise bounded and equicontinuous sequence of functions $f_n: U \rightarrow \mathbb{C}$. Then $\{f_n\}$ contains a subsequence that converges uniformly on compact subsets.

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Exercise: Suppose that $f_n: [0, 1] \rightarrow \mathbb{C}$ are functions that are pointwise bounded, (real) differentiable, and for some $M > 0$, we have $|f'_n(t)| \leq M$ for all $t \in [0, 1]$ and all n . Prove that there exists a subsequence that converges uniformly on $[0, 1]$.

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Exercise: Suppose (X, d) is a compact metric space and $\{f_n\}$ an equicontinuous sequence of functions on X . If $\{f_n\}$ converges pointwise, show that it converges uniformly.