Cultivating Complex Analysis: Cauchy for star-like sets (3.2.3)

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Definition

A set $U \subset \mathbb{C}$ is called *star-like* (or *star-like with respect to* z_0) if there exists a point $z_0 \in U$ such that the segment $[z_0, z] \subset U$ for every $z \in U$.

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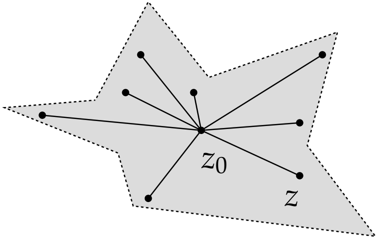

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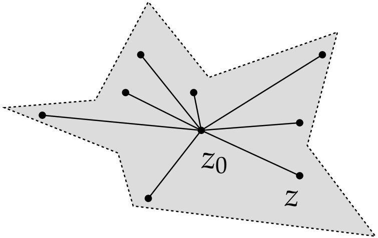

A convex set is star-like, but not vice versa.

Proposition

Suppose $U \subset \mathbb{C}$ *is open and star-like,* $f : U \longrightarrow \mathbb{C}$ *is continuous, and*

$$\int_{\partial T} f(z)\, dz = 0 \qquad \textit{for every triangle } T \subset U.$$

Then f *has a primitive: There exists a holomorphic* $F : U \longrightarrow \mathbb{C}$ *such that* $F' = f$*.*

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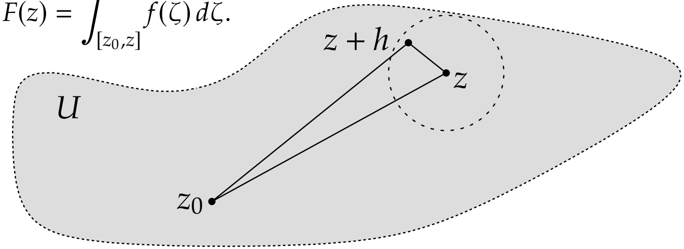
U is star-like w.r.t. $z_0 \Rightarrow$
the entire triangle with vertices
z_0, z, and $z + h$ is in U.



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By continuity of f at z,

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z). \qquad \square$$

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Remark: $\mathbb{C}$-valued function gives a vector-field on $\mathbb{R}^2$.
The corollary is a special case of a theorem from vector calculus:

In a star-like domain $U \subset \mathbb{R}^2$, if a C^1 vector field $(u, v) \colon U \to \mathbb{R}^2$ satisfies $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ (irrotational), then there exists a real-valued $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = (u, v)$ (conservative vector field).