

Cultivating Complex Analysis:
Harmonic functions
Dirichlet problem in a disc and the Poisson kernel (7.2.1)

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On an unbounded domain, solution need not be unique.

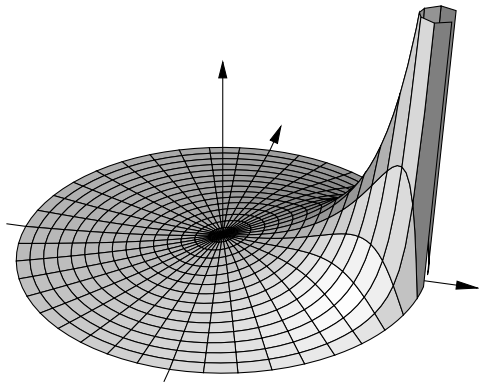
Exercise: The Dirichlet problem does not have a unique solution on the upper half plane \mathbb{H} .

The Poisson kernel for the unit disc $\mathbb{D} \subset \mathbb{C}$ is

$$P_r(\theta) = \frac{1}{2\pi} \frac{1-r^2}{1+r^2-2r\cos\theta} = \frac{1}{2\pi} \operatorname{Re} \left(\frac{1+re^{i\theta}}{1-re^{i\theta}} \right), \quad \text{for } 0 \leq r < 1.$$

As a function of $z = re^{i\theta}$:

$$\frac{1}{2\pi} \operatorname{Re} \left(\frac{1+z}{1-z} \right)$$



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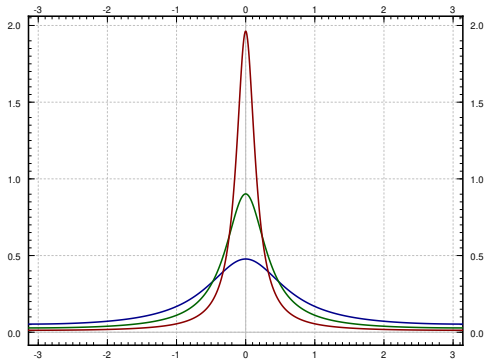
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The graph of P_r as a function of θ on $[-\pi, \pi]$ for $r = 0.5$, $r = 0.7$, and $r = 0.85$:



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Let $f: \partial\mathbb{D} \rightarrow \mathbb{R}$ be continuous. Then $Pf: \overline{\mathbb{D}} \rightarrow \mathbb{R}$, defined by

$$Pf(re^{i\theta}) = \begin{cases} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t) dt & \text{if } r < 1, \\ f(e^{i\theta}) & \text{if } r = 1, \end{cases}$$

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Let $f: \partial\mathbb{D} \rightarrow \mathbb{R}$ be continuous. Then $Pf: \overline{\mathbb{D}} \rightarrow \mathbb{R}$, defined by

$$Pf(re^{i\theta}) = \begin{cases} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t) dt & \text{if } r < 1, \\ f(e^{i\theta}) & \text{if } r = 1, \end{cases}$$

is harmonic in \mathbb{D} and continuous on $\overline{\mathbb{D}}$.

Proof: Let $z = re^{i\theta}$. For fixed t ,

$$P_r(\theta - t) = \frac{1}{2\pi} \operatorname{Re} \left(\frac{1 + re^{i(\theta-t)}}{1 - re^{i(\theta-t)}} \right) = \frac{1}{2\pi} \operatorname{Re} \left(\frac{1 + ze^{-it}}{1 - ze^{-it}} \right)$$

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The integral from δ to π is exactly the same.

$$\left|\int_{-\delta}^{\delta}\left(f\left(e^{i(\theta-t)}\right)-f\left(e^{i \theta}\right)\right) P_r(t) d t\right|$$

$$\left| \int_{-\delta}^{\delta} \left(f(e^{i(\theta-t)}) - f(e^{i\theta}) \right) P_r(t) dt \right| \leq \int_{-\delta}^{\delta} |f(e^{i(\theta-t)}) - f(e^{i\theta})| P_r(t) dt$$

$$\begin{aligned}
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So $1 - \delta' < r < 1 \quad \Rightarrow \quad |Pf(re^{i\theta}) - f(e^{i\theta})|$

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So $1 - \delta' < r < 1 \quad \Rightarrow \quad |Pf(re^{i\theta}) - f(e^{i\theta})| < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.$

$$\left| \int_{-\delta}^{\delta} \left(f(e^{i(\theta-t)}) - f(e^{i\theta}) \right) P_r(t) dt \right| \leq \int_{-\delta}^{\delta} |f(e^{i(\theta-t)}) - f(e^{i\theta})| P_r(t) dt \leq \int_{-\delta}^{\delta} \frac{\epsilon}{2} P_r(t) dt$$

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$\implies Pf(re^{i\theta}) \rightarrow f(e^{i\theta})$ uniformly in θ as $r \uparrow 1$.

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Fix $z_0 = e^{i\theta_0} \in \partial\mathbb{D}$ and $\epsilon > 0$.

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For possibly smaller δ

$$1 - \delta < r \leq 1 \Rightarrow |Pf(re^{i\theta}) - Pf(e^{i\theta})| < \epsilon/2 \quad \text{for all } \theta.$$

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So if $z = re^{i\theta}$ satisfies $1 - \delta < r \leq 1$ and $|\theta - \theta_0| < \delta$,

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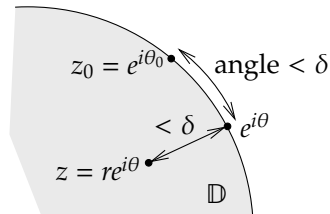
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For possibly smaller δ

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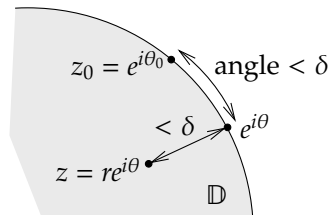
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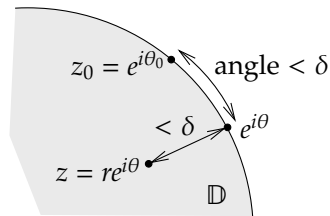
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Remark: We used polar coordinates (homeomorphism near z_0).



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Translation and scaling gives (proof an exercise):

Corollary

Let $f: \partial\Delta_R(p) \rightarrow \mathbb{R}$ be continuous. Then $Pf: \overline{\Delta_R(p)} \rightarrow \mathbb{R}$, defined by

$$Pf(p + re^{i\theta}) = \begin{cases} \int_{-\pi}^{\pi} f(p + Re^{it}) P_{r/R}(\theta - t) dt & \text{if } r < R, \\ f(p + Re^{i\theta}) & \text{if } r = R, \end{cases}$$

is harmonic in $\Delta_R(p)$ and continuous on $\overline{\Delta_R(p)}$.

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A quick consequence: If f is harmonic on a neighborhood of $\overline{\Delta_R(p)}$, then

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + Re^{it}) dt.$$

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For general functions Cf and Pf are different in the disc:

Consider $f(z) = z + \bar{z} = 2 \operatorname{Re}(z)$ on the unit circle.

$$Pf(z) = z + \bar{z} \quad Cf(z) = z$$

Exercise: Given a bounded continuous $f: \mathbb{R} \rightarrow \mathbb{R}$, prove that $Pf: \overline{\mathbb{H}} \rightarrow \mathbb{R}$,

$$Pf(x + iy) = \begin{cases} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{y}{(x-t)^2 + y^2} dt & \text{if } y > 0, \\ f(x) & \text{if } y = 0, \end{cases}$$

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Exercise: Derive the *Schwarz integral formula*, which recovers a holomorphic function out of the real parts of the boundary values and the value of the imaginary part at one point.

If $f: \overline{\mathbb{D}} \rightarrow \mathbb{C}$ is continuous and holomorphic on \mathbb{D} , then for all $z \in \mathbb{D}$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} \frac{\operatorname{Re} f(\zeta)}{\zeta} d\zeta + i \operatorname{Im} f(0).$$