Cultivating Complex Analysis: Riemann mapping theorem (6.3.1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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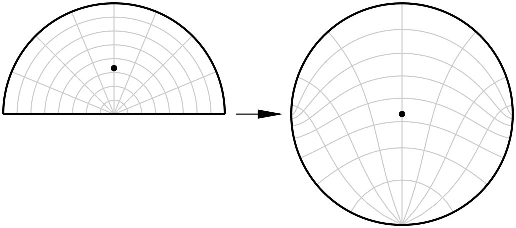

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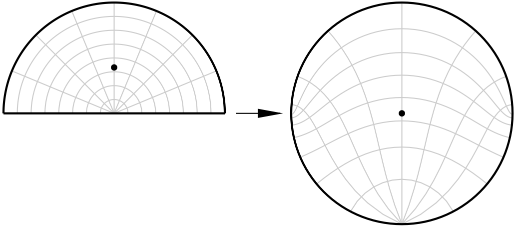

Proof is to "maximize" $|f'(p)|$ among all maps into the disc and $f(p) = 0$.

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Uniqueness left as an exercise. $\square$

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The nicer the boundary, the nicer the map will be.

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Exercise: Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire holomorphic and injective, prove that f is onto.