Cultivating Complex Analysis: The open mapping theorem (5.5)

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For a holomorphic map, $f(V)$ is open whenever V is, unless f is constant.

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Given $p \in V$, $\quad \exists\, \overline{\Delta_r(p)} \subset V$ and a $\delta > 0$ such that $\left|f(z) - f(p)\right| > \delta$ for all $z \in \partial\Delta_r(p)$.

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Take $w \in \Delta_\delta\big(f(p)\big)$. For all $z \in \partial\Delta_r(p)$,

$$\left|\big(f(z) - w\big) - \big(f(z) - f(p)\big)\right| = \left|f(p) - w\right| < \delta < \left|f(z) - f(p)\right|.$$

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By Rouché, $z \mapsto f(z) - w$ has at least one zero in $\Delta_r(p)$. So

$$\Delta_\delta\big(f(p)\big) \subset f\big(\Delta_r(p)\big). \qquad \square$$

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The proof gives the more explicit:

$\left|f(z) - f(p)\right| > \delta$ for $z \in \partial\Delta_r(p)$, then $\Delta_\delta\big(f(p)\big) \subset f\big(\Delta_r(p)\big)$.

Exercise: Suppose $U, V \subset \mathbb{C}$ are open and $f\colon U \to \mathbb{C}$ is holomorphic and bijective. Prove that $f^{-1}\colon V \to U$ is continuous.

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Typical application/exercise of the open mapping theorem is something like:

Exercise: Let $U \subset \mathbb{C}$ be a domain and let $f\colon U \to \mathbb{C}$ be holomorphic.
Prove that if $(\operatorname{Im} f(z))^2 - (\operatorname{Re} f(z))^2 = 1$ for all $z \in U$, then f is constant.