

Cultivating Complex Analysis:
The geometry and topology of the plane (1.1.2)
Complex-valued functions (1.1.3)

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Proposition

Complex addition, multiplication, division, and conjugation are continuous: Suppose $\{a_n\}$ and $\{b_n\}$ are two convergent sequences of complex numbers. Then,

$$(i) \quad \lim_{n \rightarrow \infty} (a_n + b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) + \left(\lim_{n \rightarrow \infty} b_n \right),$$

$$(ii) \quad \lim_{n \rightarrow \infty} a_n b_n = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right),$$

$$(iii) \quad \lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{\lim_{n \rightarrow \infty} a_n}, \text{ as long as } \lim_{n \rightarrow \infty} a_n \neq 0,$$

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All these operations are defined in terms of operations on the real and imaginary parts which are continuous. Details left as exercise.

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Definition

An open and connected set $U \subset \mathbb{C}$ is called a *domain*.

If $f: X \rightarrow \mathbb{C}$, write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$: $f = u + iv$.

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If $f: [a, b] \rightarrow \mathbb{C}$, f is (Riemann) integrable if u and v are, and

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Proposition

Suppose $f: [a, b] \rightarrow \mathbb{C}$ is (Riemann) integrable. Then $|f|$ is (Riemann) integrable and

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt.$$