Cultivating Complex Analysis: Laurent series (4.4 part 2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Theorem (Existence of Laurent series)

Suppose that $0 \leq r_1 < r_2 \leq \infty$ *and* $f \colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is holomorphic.*

Theorem (Existence of Laurent series)

Suppose that $0 \le r_1 < r_2 \le \infty$ *and* $f\colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is holomorphic. Then there exist unique numbers* $c_n \in \mathbb{C}$ *for* $n \in \mathbb{Z}$ *such that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n,$$

converging uniformly absolutely on compact subsets of $\operatorname{ann}(p; r_1, r_2)$.

Theorem (Existence of Laurent series)

Suppose that $0 \leq r_1 < r_2 \leq \infty$ *and* $f\colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is holomorphic. Then there exist unique numbers* $c_n \in \mathbb{C}$ *for* $n \in \mathbb{Z}$ *such that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n,$$

converging uniformly absolutely on compact subsets of $\operatorname{ann}(p; r_1, r_2)$. *The numbers* c_n *are given by*

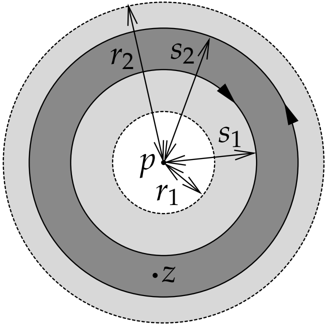
$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-p)^{n+1}}\, dz,$$

where γ *is any circle of radius* s, $r_1 < s < r_2$, *centered at* p *oriented counterclockwise.*

Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

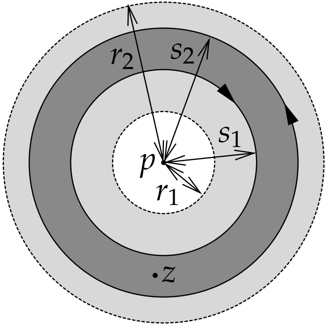
Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.



Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.



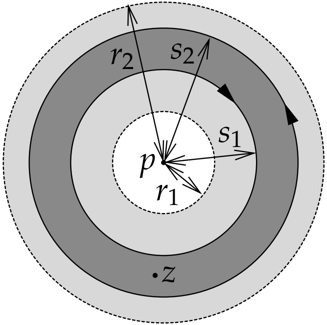
If $q \in \mathbb{C} \setminus \operatorname{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.

If $q \in \mathbb{C} \setminus \mathrm{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.



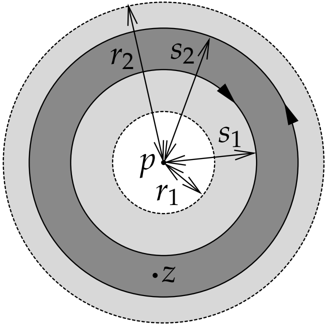
Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\qquad \Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p).$

If $q \in \mathbb{C} \setminus \operatorname{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.

If q is on the outside, $n\big(\partial\Delta_{s_j}(p); q\big) = 0$ for $j = 1, 2$.



Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

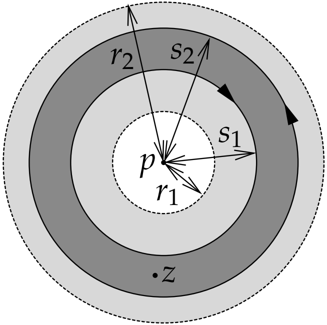
Define the cycle $\quad \Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p).$

If $q \in \mathbb{C} \setminus \text{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.

If q is on the outside, $n\big(\partial\Delta_{s_j}(p); q\big) = 0$ for $j = 1, 2$.

So Γ is homologous to zero in $\text{ann}(p; r_1, r_2)$.



Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\quad \Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.

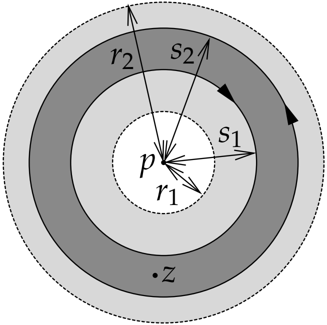
If $q \in \mathbb{C} \setminus \operatorname{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.

If q is on the outside, $n\big(\partial\Delta_{s_j}(p); q\big) = 0$ for $j = 1, 2$.

So Γ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$.

However, when $z \in \operatorname{ann}(p; s_1, s_2)$, then $n(\Gamma; z) = 1$.



Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.

If $q \in \mathbb{C} \setminus \operatorname{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.

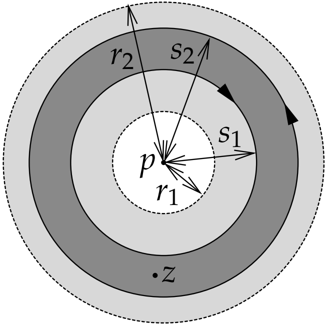
If q is on the outside, $n\big(\partial\Delta_{s_j}(p); q\big) = 0$ for $j = 1, 2$.

So Γ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$.

However, when $z \in \operatorname{ann}(p; s_1, s_2)$, then $n(\Gamma; z) = 1$.

By Cauchy's formula for $z \in \operatorname{ann}(p; s_1, s_2)$,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$



Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.

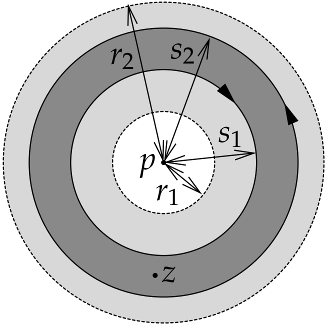
If $q \in \mathbb{C} \setminus \operatorname{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.

If q is on the outside, $n\big(\partial\Delta_{s_j}(p); q\big) = 0$ for $j = 1, 2$.

So Γ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$.

However, when $z \in \operatorname{ann}(p; s_1, s_2)$, then $n(\Gamma; z) = 1$.



By Cauchy's formula for $z \in \operatorname{ann}(p; s_1, s_2)$,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof of the theorem: Choose s_1 and s_2 such that $r_1 < s_1 < s_2 < r_2$.

Define the cycle $\quad \Gamma = \partial\Delta_{s_2}(p) - \partial\Delta_{s_1}(p)$.

If $q \in \mathbb{C} \setminus \operatorname{ann}(p; r_1, r_2)$, then $n(\Gamma; q) = 0$:

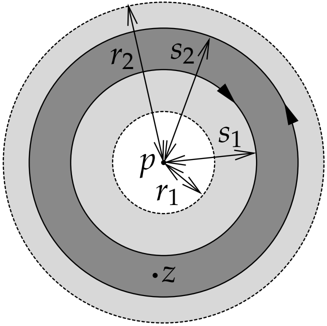
If q is in the hole, $n\big(\partial\Delta_{s_j}(p); q\big) = 1$ for $j = 1, 2$.

If q is on the outside, $n\big(\partial\Delta_{s_j}(p); q\big) = 0$ for $j = 1, 2$.

So Γ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$.

However, when $z \in \operatorname{ann}(p; s_1, s_2)$, then $n(\Gamma; z) = 1$.

By Cauchy's formula for $z \in \operatorname{ann}(p; s_1, s_2)$,



$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

We will expand the two integrals separately.

First, if $\zeta \in \partial\Delta_{s_2}$, then $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} < 1$.

First, if $\zeta \in \partial\Delta_{s_2}$, then $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} < 1$.

We follow the same reasoning as in the proof of the existence of power series.

First, if $\zeta \in \partial\Delta_{s_2}$, then $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} < 1$.

We follow the same reasoning as in the proof of the existence of power series.

$$\frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\frac{1}{1-\frac{z-p}{\zeta-p}}\, d\zeta$$

First, if $\zeta \in \partial\Delta_{s_2}$, then $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} < 1$.

We follow the same reasoning as in the proof of the existence of power series.

$$\frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\frac{1}{1-\frac{z-p}{\zeta-p}}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\sum_{n=0}^{\infty}\left(\frac{z-p}{\zeta-p}\right)^n d\zeta$$

First, if $\zeta \in \partial\Delta_{s_2}$, then $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} < 1.$

We follow the same reasoning as in the proof of the existence of power series.

$$\frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\frac{1}{1-\frac{z-p}{\zeta-p}}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\sum_{n=0}^{\infty}\left(\frac{z-p}{\zeta-p}\right)^n d\zeta$$

$$= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}}\,d\zeta\right)}_{c_n}(z-p)^n.$$

First, if $\zeta \in \partial\Delta_{s_2}$, then $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} < 1$.

We follow the same reasoning as in the proof of the existence of power series.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\frac{1}{1-\frac{z-p}{\zeta-p}}\,d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{\zeta - p}\sum_{n=0}^{\infty}\left(\frac{z-p}{\zeta-p}\right)^n d\zeta \\
&= \sum_{n=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial\Delta_{s_2}(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\,d\zeta\right)}_{c_n}(z-p)^n.
\end{aligned}$$

We can swap the series limit with the integral as the convergence is uniform on the circle.

Similarly, if $\zeta \in \partial\Delta_{s_1}$, then $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} < 1$, so

Similarly, if $\zeta \in \partial\Delta_{s_1}$, then $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} < 1$, so

$$-\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\frac{1}{1-\frac{\zeta-p}{z-p}}\, d\zeta$$

Similarly, if $\zeta \in \partial\Delta_{s_1}$, then $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} < 1$, so

$$-\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\frac{1}{1-\frac{\zeta-p}{z-p}}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\sum_{m=0}^{\infty}\left(\frac{\zeta-p}{z-p}\right)^m d\zeta$$

Similarly, if $\zeta \in \partial\Delta_{s_1}$, then $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} < 1$, so

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\,\frac{1}{1-\frac{\zeta-p}{z-p}}\, d\zeta \\
&= \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\sum_{m=0}^{\infty}\left(\frac{\zeta-p}{z-p}\right)^m d\zeta \\
&= \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} f(\zeta)(\zeta-p)^m\, d\zeta\right)(z-p)^{-m-1}
\end{aligned}
$$

Similarly, if $\zeta \in \partial\Delta_{s_1}$, then $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} < 1$, so

$$-\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\frac{1}{1-\frac{\zeta-p}{z-p}}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{z-p}\sum_{m=0}^{\infty}\left(\frac{\zeta-p}{z-p}\right)^m d\zeta$$

$$= \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} f(\zeta)(\zeta-p)^m\, d\zeta\right)(z-p)^{-m-1}$$

$$= \sum_{n=-\infty}^{-1}\underbrace{\left(\frac{1}{2\pi i}\int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta\right)}_{c_n}(z-p)^n.$$

We got the series, although perhaps not the formula for c_n for circle of any radius s.

We got the series, although perhaps not the formula for c_n for circle of any radius s.

Let s be such that $r_1 < s < r_2$.

We got the series, although perhaps not the formula for c_n for circle of any radius s.

Let s be such that $r_1 < s < r_2$.

The cycle $\partial\Delta_s(p) - \partial\Delta_{s_1}(p)$ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$, and $z \mapsto \frac{f(z)}{(z-p)^{n+1}}$ is holomorphic in $\operatorname{ann}(p; r_1, r_2)$.

We got the series, although perhaps not the formula for c_n for circle of any radius s.

Let s be such that $r_1 < s < r_2$.

The cycle $\partial\Delta_s(p) - \partial\Delta_{s_1}(p)$ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$, and $z \mapsto \frac{f(z)}{(z-p)^{n+1}}$ is holomorphic in $\operatorname{ann}(p; r_1, r_2)$.

Cauchy's theorem says

$$0 = \int_{\partial\Delta_s(p) - \partial\Delta_{s_1}(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, d\zeta$$

We got the series, although perhaps not the formula for c_n for circle of any radius s.

Let s be such that $r_1 < s < r_2$.

The cycle $\partial\Delta_s(p) - \partial\Delta_{s_1}(p)$ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$, and $z \mapsto \frac{f(z)}{(z-p)^{n+1}}$ is holomorphic in $\operatorname{ann}(p; r_1, r_2)$.

Cauchy's theorem says

$$0 = \int_{\partial\Delta_s(p)-\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta = \int_{\partial\Delta_s(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta - \int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{(\zeta-p)^{n+1}}\, d\zeta.$$

We got the series, although perhaps not the formula for c_n for circle of any radius s.

Let s be such that $r_1 < s < r_2$.

The cycle $\partial\Delta_s(p) - \partial\Delta_{s_1}(p)$ is homologous to zero in $\operatorname{ann}(p; r_1, r_2)$, and $z \mapsto \frac{f(z)}{(z-p)^{n+1}}$ is holomorphic in $\operatorname{ann}(p; r_1, r_2)$.

Cauchy's theorem says

$$0 = \int_{\partial\Delta_s(p) - \partial\Delta_{s_1}(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, d\zeta = \int_{\partial\Delta_s(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, d\zeta - \int_{\partial\Delta_{s_1}(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, d\zeta.$$

Similarly for s_2 and so

$$c_n = \frac{1}{2\pi i} \int_{\partial\Delta_s(p)} \frac{f(\zeta)}{(\zeta - p)^{n+1}} \, d\zeta.$$

Next, convergence.

Next, convergence.

For any $\epsilon > 0$, the geometric series used for the first part converges uniformly absolutely when $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\Delta_{s_2}(p)$.

Next, convergence.

For any $\epsilon > 0$, the geometric series used for the first part converges uniformly absolutely when $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\Delta_{s_2}(p)$.

The geometric series used for the second part converges uniformly absolutely when $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\mathbb{C} \setminus \overline{\Delta_{s_1}(p)}$.

Next, convergence.

For any $\epsilon > 0$, the geometric series used for the first part converges uniformly absolutely when $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\Delta_{s_2}(p)$.

The geometric series used for the second part converges uniformly absolutely when $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\mathbb{C} \setminus \overline{\Delta_{s_1}(p)}$.

Any compact subset of $\operatorname{ann}(p; r_1, r_2)$ is a compact inside both of these for some s_1, s_2.

Next, convergence.

For any $\epsilon > 0$, the geometric series used for the first part converges uniformly absolutely when $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\Delta_{s_2}(p)$.

The geometric series used for the second part converges uniformly absolutely when $\left|\frac{\zeta-p}{z-p}\right| = \frac{s_1}{|z-p|} \leq 1 - \epsilon$.

So the resulting series converges uniformly absolutely on compact subsets of $\mathbb{C} \setminus \overline{\Delta_{s_1}(p)}$.

Any compact subset of $\mathrm{ann}(p; r_1, r_2)$ is a compact inside both of these for some s_1, s_2.

So the full series converges uniformly absolutely on compact subsets of $\mathrm{ann}(p; r_1, r_2)$.

Finally, uniqueness.

Finally, uniqueness.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-p)^n,$$

converging uniformly absolutely on compact subsets of $\mathrm{ann}(p; r_1, r_2)$.

Finally, uniqueness.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} d_n(z-p)^n,$$

converging uniformly absolutely on compact subsets of $\mathrm{ann}(p; r_1, r_2)$.

Then

Finally, uniqueness.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} d_n(z-p)^n,$$

converging uniformly absolutely on compact subsets of $\mathrm{ann}(p; r_1, r_2)$.

Then

$$c_m = \frac{1}{2\pi i} \int_{\partial\Delta_s(p)} \frac{f(\zeta)}{(\zeta - p)^{m+1}} \, d\zeta$$

Finally, uniqueness.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-p)^n,$$

converging uniformly absolutely on compact subsets of $\operatorname{ann}(p; r_1, r_2)$.

Then

$$c_m = \frac{1}{2\pi i} \int_{\partial \Delta_s(p)} \frac{f(\zeta)}{(\zeta - p)^{m+1}}\, d\zeta = \frac{1}{2\pi i} \int_{\partial \Delta_s(p)} \left(\sum_{n=-\infty}^{\infty} d_n (\zeta - p)^n \right) \frac{1}{(\zeta - p)^{m+1}}\, d\zeta$$

Finally, uniqueness.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-p)^n,$$

converging uniformly absolutely on compact subsets of $\operatorname{ann}(p; r_1, r_2)$.

Then

$$c_m = \frac{1}{2\pi i} \int_{\partial \Delta_s(p)} \frac{f(\zeta)}{(\zeta - p)^{m+1}} \, d\zeta = \frac{1}{2\pi i} \int_{\partial \Delta_s(p)} \left(\sum_{n=-\infty}^{\infty} d_n (\zeta - p)^n \right) \frac{1}{(\zeta - p)^{m+1}} \, d\zeta$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} d_n \int_{\partial \Delta_s(p)} (\zeta - p)^{n-m-1} \, d\zeta$$

Finally, uniqueness.

Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-p)^n,$$

converging uniformly absolutely on compact subsets of $\operatorname{ann}(p; r_1, r_2)$.

Then

$$c_m = \frac{1}{2\pi i} \int_{\partial \Delta_s(p)} \frac{f(\zeta)}{(\zeta - p)^{m+1}}\, d\zeta = \frac{1}{2\pi i} \int_{\partial \Delta_s(p)} \left(\sum_{n=-\infty}^{\infty} d_n (\zeta - p)^n \right) \frac{1}{(\zeta - p)^{m+1}}\, d\zeta$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} d_n \int_{\partial \Delta_s(p)} (\zeta - p)^{n-m-1}\, d\zeta = d_m.$$

(The last equality because $\int_{\partial \Delta_s(p)} (\zeta - p)^{n-m-1}\, d\zeta \neq 0$ only when $n = m$.) □

We differentiate/antidifferentiate formally (except for antidifferentiating the $c_{-1}(z-p)^{-1}$).

We differentiate/antidifferentiate formally (except for antidifferentiating the $c_{-1}(z-p)^{-1}$).

Proposition

Suppose $p \in \mathbb{C}$, $0 \leq r_1 < r_2 \leq \infty$, *and* $f\colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is defined by*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

We differentiate/antidifferentiate formally (except for antidifferentiating the $c_{-1}(z-p)^{-1}$).

Proposition

Suppose $p \in \mathbb{C}$, $0 \le r_1 < r_2 \le \infty$, *and* $f\colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is defined by*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

(i) *Then* f *is holomorphic and its derivative is defined by*

$$f'(z) = \sum_{n=-\infty}^{\infty} n c_n (z-p)^{n-1}, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

We differentiate/antidifferentiate formally (except for antidifferentiating the $c_{-1}(z-p)^{-1}$).

Proposition

Suppose $p \in \mathbb{C}$, $0 \le r_1 < r_2 \le \infty$, *and* $f\colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is defined by*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

(i) *Then* f *is holomorphic and its derivative is defined by*

$$f'(z) = \sum_{n=-\infty}^{\infty} n c_n (z-p)^{n-1}, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

(ii) *If* $c_{-1} = 0$, *then* $f = F'$, *where*

$$F(z) = \sum_{n=-\infty, n\neq -1}^{\infty} \frac{c_n}{n+1} (z-p)^{n+1}, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

We differentiate/antidifferentiate formally (except for antidifferentiating the $c_{-1}(z-p)^{-1}$).

Proposition

Suppose $p \in \mathbb{C}$, $0 \leq r_1 < r_2 \leq \infty$, *and* $f\colon \operatorname{ann}(p; r_1, r_2) \to \mathbb{C}$ *is defined by*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

(i) *Then* f *is holomorphic and its derivative is defined by*

$$f'(z) = \sum_{n=-\infty}^{\infty} n c_n (z-p)^{n-1}, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

(ii) *If* $c_{-1} = 0$, *then* $f = F'$, *where*

$$F(z) = \sum_{n=-\infty, n\neq -1}^{\infty} \frac{c_n}{n+1} (z-p)^{n+1}, \quad \textit{converging uniformly on compact subsets of } \operatorname{ann}(p; r_1, r_2).$$

Proof: Exercise.

Some more useful exercises:

Some more useful exercises:

Exercise: Suppose f and g are holomorphic functions defined on $\operatorname{ann}(p; r_1, r_2)$. Let a_n be the coefficients in the Laurent series for f and b_n be the coefficients in the Laurent series for g. Suppose that $\alpha, \beta \in \mathbb{C}$. Show that the Laurent series for the function $\alpha f + \beta g$ has coefficients $\alpha a_n + \beta b_n$.

Some more useful exercises:

Exercise: Suppose f and g are holomorphic functions defined on $\operatorname{ann}(p; r_1, r_2)$. Let a_n be the coefficients in the Laurent series for f and b_n be the coefficients in the Laurent series for g. Suppose that $\alpha, \beta \in \mathbb{C}$. Show that the Laurent series for the function $\alpha f + \beta g$ has coefficients $\alpha a_n + \beta b_n$.

Exercise: Expand the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the sets $\operatorname{ann}(0; 0, 1)$, $\operatorname{ann}(0; 1, 2)$, and $\operatorname{ann}(0; 2, \infty)$.