

Cultivating Complex Analysis: Power series (2.3 part 2)

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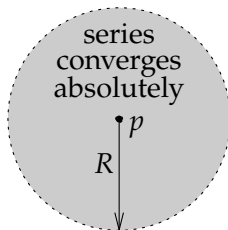
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Proposition (Cauchy–Hadamard theorem)

$\sum c_n(z-p)^n$ converges absolutely if $|z-p| < R$ and diverges if $|z-p| > R$.

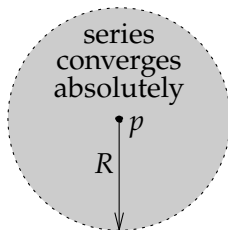
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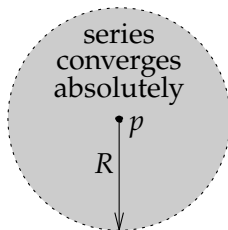


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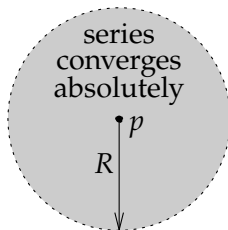
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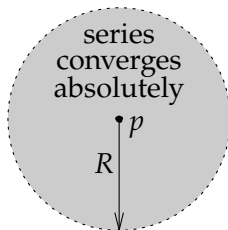
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R is called the *radius of convergence*.

Proposition

The series $\sum c_n(z - p)^n$ converges in $\Delta_R(p)$ for some $R > 0$ if and only if for every r with $0 < r < R$, there exists an $M > 0$ such that

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However, $\{nr^n\}$ is bounded for every $r < 1$.

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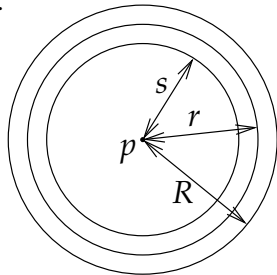
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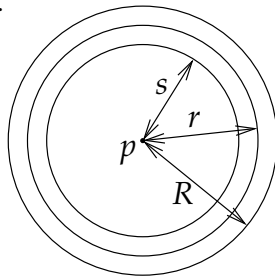
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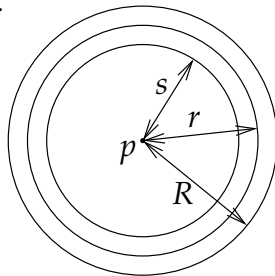
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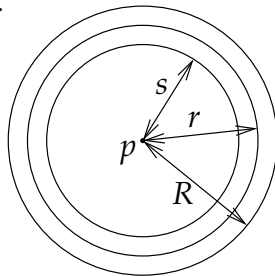
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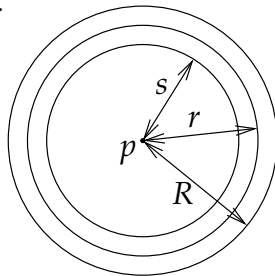
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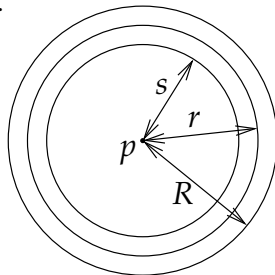
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As s and r with $0 < s < r < R$ were arbitrary, the series converges (absolutely) in $\Delta_R(p)$. □

