

Cultivating Complex Analysis: Simply connected domains (4.3 part 2)

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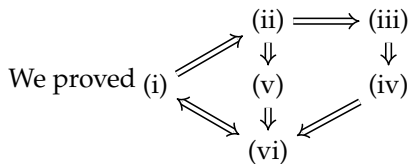
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- 2) If $U_1 \cap U_2$ is nonempty and connected, then $U_1 \cup U_2$ is simply connected.