

Cultivating Complex Analysis: Convergence of sequences of holomorphic functions (3.4.2)

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$|x|^{1+1/n}$ is C^1 on \mathbb{R} and converges uniformly on compact subsets to $|x|$.

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We can take the limit $n \rightarrow \infty$ underneath the integral.

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We still need to prove the “Moreover” bit:

That $\{f_n^{(\ell)}\}$ also converges uniformly on compact subsets.

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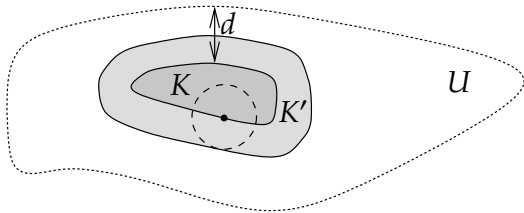
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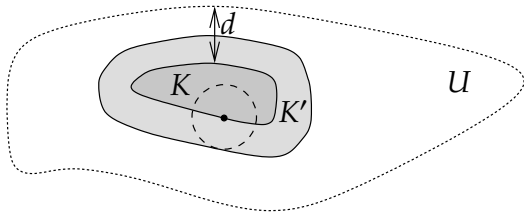
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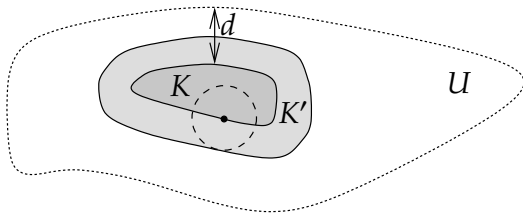
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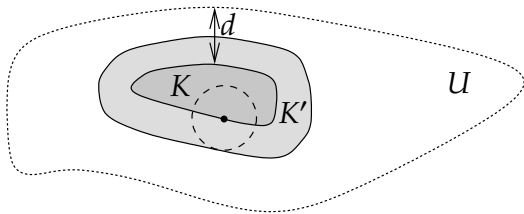
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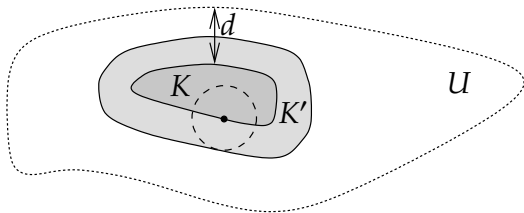
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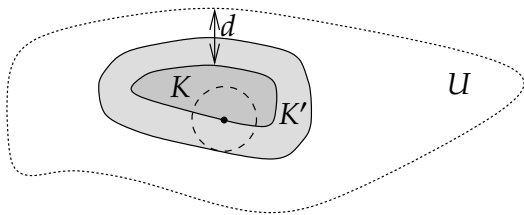
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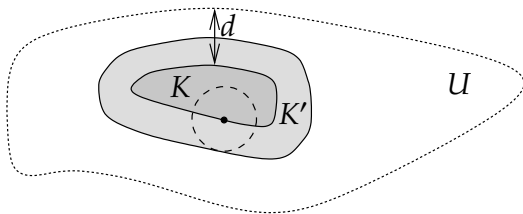
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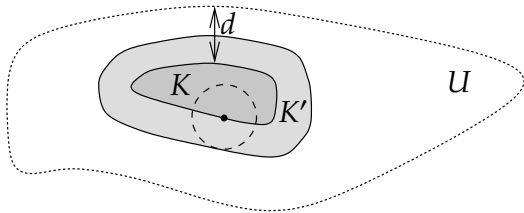
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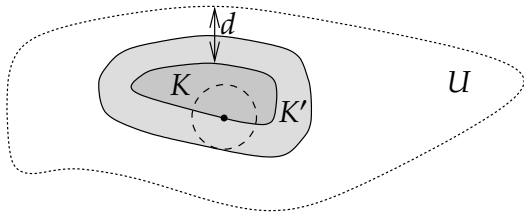
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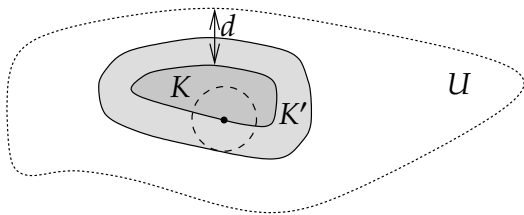
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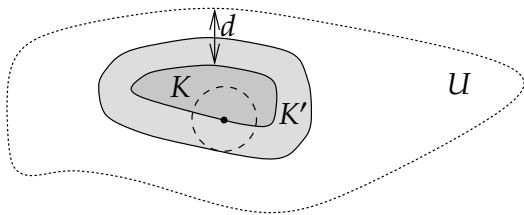
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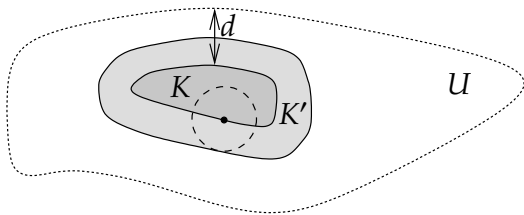
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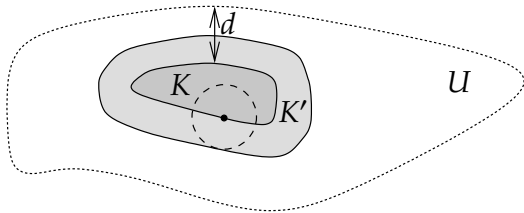
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Integration is a far nicer operation than differentiation, and for holomorphic functions, we can differentiate by integrating.