Cultivating Complex Analysis: Holomorphic functions are analytic (3.3.1)

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Moreover,

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-p)^{n+1}}\, dz,$$

where γ is any circle of radius r, $0 < r < R$, centered at p oriented counterclockwise.

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The last equality held because the sum converges uniformly in $\zeta \in \partial\Delta_r(p)$ (we'll justify that on the next slide).

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We get the same series for every r and it converges in $\Delta_R(p)$. $\square$

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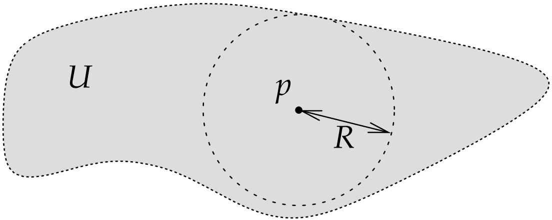
Besides analyticity, we also proved that the radius of convergence is at least R, where R is the maximum R such that $\Delta_R(p) \subset U$.



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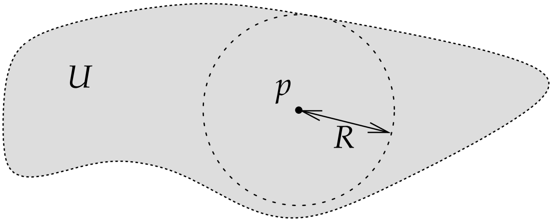

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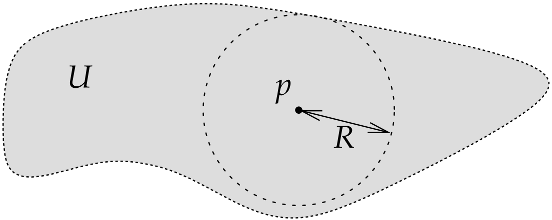

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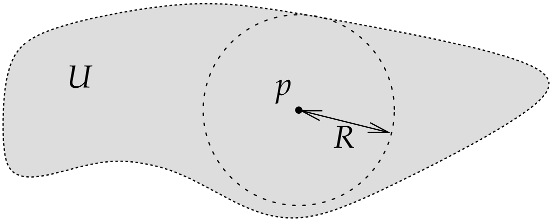

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Remark: Nothing like this is true for real-analytic functions such as $\varphi(x) = \frac{1}{1+x^2}$ whose radius of convergence at $x = 0$ is 1, but $\varphi \colon \mathbb{R} \to \mathbb{R}$ is (real) analytic everywhere.

Let us restate the main conclusion:

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We have also finally proved the following:
A convergent power series defines an analytic function.