

# Cultivating Complex Analysis: Holomorphic functions (2.1.1)

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**Key point:** The limits are “as a *complex*  $h$  goes to 0.”

We want functions with such a derivative. That is, functions approximated by  $c_0 + c_1h$ :

$$f(z_0 + h) = \underbrace{f(z_0)}_{c_0} + \underbrace{\xi h}_{c_1 h} + o(|h|) \quad \text{for some } \xi \in \mathbb{C}.$$

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### Definition

Let  $U \subset \mathbb{C}$  be open. A function  $f: U \rightarrow \mathbb{C}$  is *complex differentiable* at  $z_0 \in U$  if the limit

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$f: U \rightarrow \mathbb{C}$  is *holomorphic* if it is complex differentiable at every point.

We proved above:

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An exercise:

### Proposition

*If  $U \subset \mathbb{C}$  is open and  $f: U \rightarrow \mathbb{C}$  is holomorphic, then  $f$  is continuous.*