Cultivating Complex Analysis: Inverse function theorem and automorphisms (2.2.3)

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Example: For any $a, b \in \mathbb{C}$, $a \neq 0$, the function $az + b$ is an automorphism of $\mathbb{C}$.

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Note that we also found that a holomorphic f preserves orientation (positive $\det Df$).

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Theorem (Inverse function theorem for holomorphic functions)

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, $p \in U$, and $f'(p) \neq 0$. Suppose further that f is continuously differentiable.

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(Although we don't yet have enough machinery to prove these statements.)