Cultivating Complex Analysis:
Derivative is holomorphic and Morera’s theorem (3.3.2)

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E.g., $f(x) = \int_c^x g(t)\, dt$ for $c \in (a, b)$.

Even worse, the real derivative could even be discontinuous.

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We passed x and y derivatives under the integral sign (Leibniz rule), which is valid as the x and y derivatives of $\frac{f(\zeta)}{(\zeta - z)^{k+1}}$ are continuous functions of $(z, \zeta) \in \Delta_r(p) \times \partial\Delta_r(p)$. □

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Exercise: Suppose $f(z,t)$ is a continuous function of $(z,t) \in U \times (a,b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a,b)$, the function $z \mapsto f(z,t)$ is holomorphic.

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But $\frac{\partial f}{\partial x}$ is not continuous as a function of both x and t.

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$f = F'$ in U, and complex derivatives are holomorphic. □

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Exercise: Show that if $f : \mathbb{C} \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{C} \setminus \mathbb{R}$, then f is holomorphic everywhere.