

Cultivating Complex Analysis:
Harmonic functions
Real and imaginary parts of holomorphic functions (7.1.1)

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Definition

Let $U \subset \mathbb{C}$ be open. A twice continuously differentiable $f: U \rightarrow \mathbb{R}$ is *harmonic* if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{on } U.$$

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Convenient to write using Wirtinger operators:

$$4 \frac{\partial^2}{\partial \bar{z} \partial z} f = 4 \left[\frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right] \left[\frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \right] f$$

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$$f \text{ is harmonic} \quad \Leftrightarrow \quad \frac{\partial f}{\partial \bar{z}} \text{ is holomorphic.}$$

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Similarly f is the imaginary part of some holomorphic function.

Conversely suppose $f(z) = \operatorname{Re} \varphi(z) = \frac{1}{2}(\varphi(z) + \overline{\varphi(z)})$ (φ holomorphic).

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We proved:

Proposition

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{R}$ a function.

- (i) *The function f is harmonic if and only if for every $p \in U$ there exists an open neighborhood V of p and a holomorphic $\varphi: V \rightarrow \mathbb{C}$ such that $f = \operatorname{Re} \varphi$ on V .*

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- (ii) The function f is harmonic if and only if for every $p \in U$, there exists a power series expansion

$$f(z) = c_0 + \sum_{n=1}^{\infty} c_n(z-p)^n + \bar{c}_n(\overline{z-p})^n$$

converging uniformly absolutely on every closed disc $\overline{\Delta_r(p)} \subset U$.

A quick corollary:

Proposition

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{R}$ is harmonic, then f is infinitely (real) differentiable.

Proof: Holomorphic functions are infinitely differentiable.

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Proof: $\frac{(f + ig_1) - (f + ig_2)}{i} = g_1 - g_2$ is holomorphic, real-valued \Rightarrow constant.

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Proposition

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic and never zero. Then

$$z \mapsto \log|f(z)|$$

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Proof: Exercise.

Exercise: Suppose $U \subset \mathbb{C}$ is a simply connected domain and $f: U \rightarrow \mathbb{R}$ harmonic. Prove there exists a holomorphic $\varphi: U \rightarrow \mathbb{C}$ such that $f(z) = \log|\varphi(z)|$.

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Exercise: Let $U, V \subset \mathbb{C}$ be open sets and $f: U \rightarrow V$ be holomorphic. Prove:

- a) If $g: V \rightarrow \mathbb{R}$ is harmonic, then $g \circ f$ is harmonic.
- b) If f is a biholomorphism, then $g: V \rightarrow \mathbb{R}$ is harmonic if and only if $g \circ f$ is harmonic.

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Conversely, if $\log|f(z)| \geq 0$, then $\frac{1}{f(z)}$ is bounded,

and if $\operatorname{Re} f(z) \geq 0$, then $\frac{f(z) - 1}{f(z) + 1}$ is bounded.

Remark: Writing $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial \bar{z} \partial z}$ so that we can integrate twice may sound familiar.

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The wave operator is (using (x, t) for tradition's sake):

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Remark: Writing $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial \bar{z} \partial z}$ so that we can integrate twice may sound familiar.

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Write $\mu = x + t$ and $\eta = x - t$ (characteristic coordinates), then

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Two waves travelling in opposite directions.