

# Cultivating Complex Analysis: Schwarz's lemma (3.5.1)

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b) We'll see later that every domain "without holes" (except  $\mathbb{C}$ ) is biholomorphic to  $\mathbb{D}$ , so it tells us about global behavior as well.

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As  $g(0) = f'(0)$ , the same conclusion holds if  $|f'(0)| = 1$ .



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Schwarz's lemma says all holomorphic functions behave this way, not just  $z^n$ .

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For (ii) use Cauchy estimates on the first nonzero nonlinear term of  $f^\ell$ .