

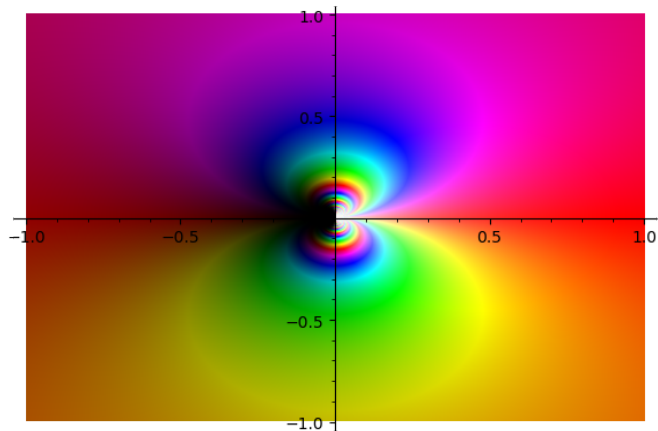
Cultivating Complex Analysis:
Wild world of essential singularities, Casorati–Weierstrass
(5.2.3)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Consider a “plot” of $e^{1/z}$.

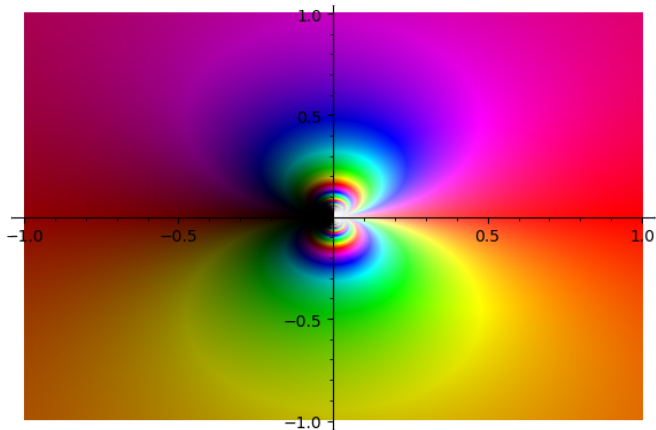
Values of $e^{1/z}$ are shown using
modulus as brightness,
argument as hue (color).



Consider a “plot” of $e^{1/z}$.

Values of $e^{1/z}$ are shown using
modulus as brightness,
argument as hue (color).

It seems like near zero,
we achieve every value.

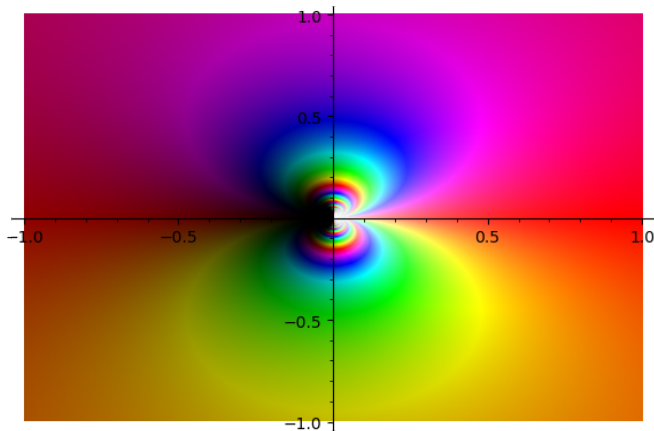


Consider a “plot” of $e^{1/z}$.

Values of $e^{1/z}$ are shown using
modulus as brightness,
argument as hue (color).

It seems like near zero,
we achieve every value.

We actually get $\mathbb{C} \setminus \{0\}$ as
the image of any neighborhood of 0.



Theorem (Casorati–Weierstrass)

Suppose $U \subset \mathbb{C}$ is open and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at $p \in U$.

Theorem (Casorati–Weierstrass)

Suppose $U \subset \mathbb{C}$ is open and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at $p \in U$. Then for every punctured disc $\Delta_r(p) \setminus \{p\} \subset U$, the image

$$f(\Delta_r(p) \setminus \{p\}) = \{w \in \mathbb{C} : w = f(z), z \in \Delta_r(p) \setminus \{p\}\} \quad \text{is dense in } \mathbb{C}.$$

Theorem (Casorati–Weierstrass)

Suppose $U \subset \mathbb{C}$ is open and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at $p \in U$. Then for every punctured disc $\Delta_r(p) \setminus \{p\} \subset U$, the image

$$f(\Delta_r(p) \setminus \{p\}) = \{w \in \mathbb{C} : w = f(z), z \in \Delta_r(p) \setminus \{p\}\} \quad \text{is dense in } \mathbb{C}.$$

Intuitive idea of the proof:

If a whole disc $\Delta_s(q)$ is missing from the image, take q to ∞ by an LFT.

Theorem (Casorati–Weierstrass)

Suppose $U \subset \mathbb{C}$ is open and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at $p \in U$. Then for every punctured disc $\Delta_r(p) \setminus \{p\} \subset U$, the image

$$f(\Delta_r(p) \setminus \{p\}) = \{w \in \mathbb{C} : w = f(z), z \in \Delta_r(p) \setminus \{p\}\} \quad \text{is dense in } \mathbb{C}.$$

Intuitive idea of the proof:

If a whole disc $\Delta_s(q)$ is missing from the image, take q to ∞ by an LFT.

$\Delta_s(q)$ goes to the complement of a bounded closed disc, and Riemann extension applies.

Theorem (Casorati–Weierstrass)

Suppose $U \subset \mathbb{C}$ is open and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at $p \in U$. Then for every punctured disc $\Delta_r(p) \setminus \{p\} \subset U$, the image

$$f(\Delta_r(p) \setminus \{p\}) = \{w \in \mathbb{C} : w = f(z), z \in \Delta_r(p) \setminus \{p\}\} \quad \text{is dense in } \mathbb{C}.$$

Intuitive idea of the proof:

If a whole disc $\Delta_s(q)$ is missing from the image, take q to ∞ by an LFT.

$\Delta_s(q)$ goes to the complement of a bounded closed disc, and Riemann extension applies.

Remark: There is a stronger (and much harder to prove) theorem:

Theorem (Casorati–Weierstrass)

Suppose $U \subset \mathbb{C}$ is open and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic with an essential singularity at $p \in U$. Then for every punctured disc $\Delta_r(p) \setminus \{p\} \subset U$, the image

$$f(\Delta_r(p) \setminus \{p\}) = \{w \in \mathbb{C} : w = f(z), z \in \Delta_r(p) \setminus \{p\}\} \text{ is dense in } \mathbb{C}.$$

Intuitive idea of the proof:

If a whole disc $\Delta_s(q)$ is missing from the image, take q to ∞ by an LFT.

$\Delta_s(q)$ goes to the complement of a bounded closed disc, and Riemann extension applies.

Remark: There is a stronger (and much harder to prove) theorem:

Picard's theorem says

$$f(\Delta_r(p) \setminus \{p\}) = \mathbb{C} \text{ or}$$

$$f(\Delta_r(p) \setminus \{p\}) = \mathbb{C} \setminus \{z_0\} \text{ for some } z_0.$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

Define $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ by $g(z) = \frac{1}{f(z) - q}$.

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

Define $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ by $g(z) = \frac{1}{f(z) - q}$.

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

Define $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ by $g(z) = \frac{1}{f(z) - q}$.

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow |g(z)| \leq 1/s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that $\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\})$.

Define $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ by $g(z) = \frac{1}{f(z) - q}$.

$|f(z) - q| \geq s$ for $z \in \Delta_r(p) \setminus \{p\}$

$\Rightarrow |g(z)| \leq 1/s$ for $z \in \Delta_r(p) \setminus \{p\}$

$\Rightarrow g$ has a removable singularity at p by Riemann extension.

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that $\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\})$.

Define $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ by $g(z) = \frac{1}{f(z) - q}$.

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow |g(z)| \leq 1/s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$\Rightarrow g$ has a removable singularity at p by Riemann extension.

So assume g is defined in $\Delta_r(p)$.

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

Define $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ by $g(z) = \frac{1}{f(z) - q}$.

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow |g(z)| \leq 1/s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$\Rightarrow g$ has a removable singularity at p by Riemann extension.

So assume g is defined in $\Delta_r(p)$.

$$f(z) = \frac{1}{g(z)} + q \quad \text{for } z \in \Delta_r(p) \setminus \{p\}.$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

$$\text{Define } g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C} \quad \text{by} \quad g(z) = \frac{1}{f(z) - q}.$$

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow |g(z)| \leq 1/s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow g \text{ has a removable singularity at } p \text{ by Riemann extension.}$$

So assume g is defined in $\Delta_r(p)$.

$$f(z) = \frac{1}{g(z)} + q \quad \text{for } z \in \Delta_r(p) \setminus \{p\}.$$

$$\text{If } g(p) = 0, \quad \Rightarrow f \text{ has a pole at } p.$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

$$\text{Define } g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C} \quad \text{by} \quad g(z) = \frac{1}{f(z) - q}.$$

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow |g(z)| \leq 1/s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow g \text{ has a removable singularity at } p \text{ by Riemann extension.}$$

So assume g is defined in $\Delta_r(p)$.

$$f(z) = \frac{1}{g(z)} + q \quad \text{for } z \in \Delta_r(p) \setminus \{p\}.$$

$$\text{If } g(p) = 0, \quad \Rightarrow f \text{ has a pole at } p.$$

$$\text{If } g(p) \neq 0, \quad \Rightarrow f \text{ has a removable singularity at } p.$$

Proof (of Casorati–Weierstrass):

Suppose $f: U \setminus \{p\} \rightarrow \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

$$\Delta_s(q) \subset \mathbb{C} \setminus f(\Delta_r(p) \setminus \{p\}).$$

$$\text{Define } g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C} \quad \text{by} \quad g(z) = \frac{1}{f(z) - q}.$$

$$|f(z) - q| \geq s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow |g(z)| \leq 1/s \text{ for } z \in \Delta_r(p) \setminus \{p\}$$

$$\Rightarrow g \text{ has a removable singularity at } p \text{ by Riemann extension.}$$

So assume g is defined in $\Delta_r(p)$.

$$f(z) = \frac{1}{g(z)} + q \quad \text{for } z \in \Delta_r(p) \setminus \{p\}.$$

$$\text{If } g(p) = 0, \quad \Rightarrow f \text{ has a pole at } p.$$

$$\text{If } g(p) \neq 0, \quad \Rightarrow f \text{ has a removable singularity at } p.$$

In either case, f does not have an essential singularity at p .



Exercise: Prove the converse of Casorati–Weierstrass. Let $U \subset \mathbb{C}$ be open, $p \in U$, and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ holomorphic. Prove that if $f(\Delta_r(p) \setminus \{p\})$ is dense in \mathbb{C} for all $r > 0$ such that $\Delta_r(p) \subset U$, then f has an essential singularity at p .

Exercise: Prove the converse of Casorati–Weierstrass. Let $U \subset \mathbb{C}$ be open, $p \in U$, and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ holomorphic. Prove that if $f(\Delta_r(p) \setminus \{p\})$ is dense in \mathbb{C} for all $r > 0$ such that $\Delta_r(p) \subset U$, then f has an essential singularity at p .

Exercise: Suppose that $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ has an isolated singularity. Prove $f(z) = e^{g(z)}$ has either a removable or an essential singularity.

Exercise: Prove the converse of Casorati–Weierstrass. Let $U \subset \mathbb{C}$ be open, $p \in U$, and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ holomorphic. Prove that if $f(\Delta_r(p) \setminus \{p\})$ is dense in \mathbb{C} for all $r > 0$ such that $\Delta_r(p) \subset U$, then f has an essential singularity at p .

Exercise: Suppose that $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ has an isolated singularity. Prove $f(z) = e^{g(z)}$ has either a removable or an essential singularity.

Exercise: Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and nonconstant. Prove $f(\mathbb{C})$ is dense in \mathbb{C} .

Exercise: Prove the converse of Casorati–Weierstrass. Let $U \subset \mathbb{C}$ be open, $p \in U$, and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ holomorphic. Prove that if $f(\Delta_r(p) \setminus \{p\})$ is dense in \mathbb{C} for all $r > 0$ such that $\Delta_r(p) \subset U$, then f has an essential singularity at p .

Exercise: Suppose that $g: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$ has an isolated singularity. Prove $f(z) = e^{g(z)}$ has either a removable or an essential singularity.

Exercise: Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and nonconstant. Prove $f(\mathbb{C})$ is dense in \mathbb{C} .

Remark: The so-called “little Picard theorem” says that $f(\mathbb{C})$ is actually everything minus possibly one point, but that is much harder to prove.