Cultivating Complex Analysis: Line integrals, chains (3.1 part 3)

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A *chain* in $U \subset \mathbb{C}$ is an expression $\Gamma = a_1\gamma_1 + \cdots + a_n\gamma_n$, where $a_1, \ldots, a_n \in \mathbb{Z}$ and $\gamma_1, \ldots, \gamma_n$ are piecewise-C^1 paths in U.

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Define the *zero chain* 0 by defining $\int_0 f(z)\,dz = 0$ for all continuous $f\colon U \to \mathbb{C}$.

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Remark: The domain of the continuous f is not a big deal. Whether on U, $\Gamma_1 \cup \Gamma_2$, or $\mathbb{C}$. By Tietze's extension theorem every continuous function on a closed subset of $\mathbb{C}$ (e.g., $\Gamma_1 \cup \Gamma_2$) extends to a continuous function on $\mathbb{C}$.

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Remark: Equivalence is for all *continuous* functions. We will show later that for many U and many Γ, $\int_\Gamma f(z)\, dz = 0$ for all holomorphic f.

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Most often used paths are composed of segments and arcs of circles.