Cultivating Complex Analysis: Wirtinger operators (2.2.2)

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The operators are really determined by wanting

$$\frac{\partial}{\partial z} z = 1, \quad \frac{\partial}{\partial z} \bar{z} = 0, \quad \frac{\partial}{\partial \bar{z}} z = 0, \quad \frac{\partial}{\partial \bar{z}} \bar{z} = 1.$$

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Proposition

Let $U \subset \mathbb{C}$ be open. Then $f : U \to \mathbb{C}$ is holomorphic if and only if f is (real) differentiable and

$$\frac{\partial f}{\partial \bar{z}} \equiv 0.$$

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So for a holomorphic function

$$f' = \frac{\partial f}{\partial z}.$$

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For polynomials the operators work as if z and $\bar{z}$ were separate variables. E.g. (exercise)

$$\frac{\partial}{\partial z}\left[z^2\bar{z}^3 + z^{10}\right] = 2z\bar{z}^3 + 10z^9 \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}}\left[z^2\bar{z}^3 + z^{10}\right] = z^2(3\bar{z}^2) + 0.$$

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Caution: Note that $\frac{d}{dz}\left[z^2\bar{z}^3 + z^{10}\right]$ does not exist, while $\frac{\partial}{\partial z}\left[z^2\bar{z}^3 + z^{10}\right]$ does.

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Remark 3: The chain rule for real differentiable functions can be written as

$$\left.\frac{\partial (g \circ f)}{\partial z}\right|_p = \left.\frac{\partial g}{\partial z}\right|_{f(p)} \left.\frac{\partial f}{\partial z}\right|_p + \left.\frac{\partial g}{\partial \bar{z}}\right|_{f(p)} \left.\frac{\partial \bar{f}}{\partial z}\right|_p \qquad \text{and} \qquad \left.\frac{\partial (g \circ f)}{\partial \bar{z}}\right|_p = \left.\frac{\partial g}{\partial z}\right|_{f(p)} \left.\frac{\partial f}{\partial \bar{z}}\right|_p + \left.\frac{\partial g}{\partial \bar{z}}\right|_{f(p)} \left.\frac{\partial \bar{f}}{\partial \bar{z}}\right|_p.$$