Cultivating Complex Analysis: Cauchy estimates, Liouville, and the fundamental theorem of algebra (3.3.4)

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1. The "triangle inequality" on the integral formula for the coefficients of the power series gives estimates on their size: **Cauchy's estimates**.
2. Cauchy's estimates imply **Liouville's theorem**: Bounded entire (defined on all of $\mathbb{C}$) holomorphic functions are constant.
3. Liouville's theorem gives **the fundamental theorem of algebra**: Every nonconstant polynomial has a root.

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Let $U \subset \mathbb{C}$ be open, $f \colon U \to \mathbb{C}$ be holomorphic, and $\overline{\Delta_r(p)} \subset U$ be a closed disc. Expand $f(z) = \sum c_n (z-p)^n$.

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Liouville says that $R(z)$ and hence $P(z)$ must be constant, a contradiction. $\square$