Cultivating Complex Analysis: Line integrals (3.1 part 2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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Proposition (Reparametrization)

Suppose $\gamma\colon [a,b] \to \mathbb{C}$ *and* $\alpha\colon [c,d] \to \mathbb{C}$ *are piecewise-*C^1 *paths such that* $\gamma\big([a,b]\big) = \alpha\big([c,d]\big)$. *Suppose either*

(i) γ *and* α *are injective, or*

(ii) $\gamma|_{(a,b]}$ *and* $\alpha|_{(c,d]}$ *are injective and* $\gamma(a) = \alpha(c) = \gamma(b) = \alpha(d)$ *(simple closed paths).*

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Then there exists a strictly monotone continuous $h\colon [c,d] \to [a,b]$ *such that* $\gamma\big(h(t)\big) = \alpha(t)$ *for all* $t \in [c,d]$. *Furthermore:*

(i) *If* h *is increasing, then for every* f *continuous on the path,* $\displaystyle\int_\gamma f(z)\,dz = \int_\alpha f(z)\,dz$.

(ii) *If* h *is decreasing, then for every* f *continuous on the path,* $\displaystyle\int_\gamma f(z)\,dz = -\int_\alpha f(z)\,dz$.

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and then

$$\int_{\partial\Delta_r(p)} f(z)\,dz = \int_\gamma f(z)\,dz.$$

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Proposition

Let $U, V \subset \mathbb{C}$ be open, $\gamma\colon [a,b] \to V$ piecewise-C^1 path, $g\colon V \to U$ holomorphic (assume g' is continuous), and $f\colon U \to \mathbb{C}$ continuous.

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For example,

$$\int_\gamma |dz| = \int_a^b |\gamma'(t)|\,dt \qquad \left(= \int_\gamma ds\right)$$

is the length of γ.

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Suppose $\gamma : [a, b] \to \mathbb{C}$ *is a piecewise-*C^1 *path and* f *is a continuous function on* γ*. Then*

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Remark: Uniform convergence of the functions, $f_n \to f$, passes under the integral:

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz \qquad \lim_{n\to\infty} \int_\gamma f_n(z)\,|dz| = \int_\gamma f(z)\,|dz|$$

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For this you would also need γ_n' to also converge (uniformly).