

Cultivating Complex Analysis: Line integrals (3.1 part 1)

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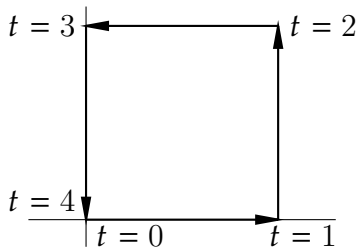
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E.g., we say γ is in U or $\gamma \subset U$ if $\gamma([a, b]) \subset U$.

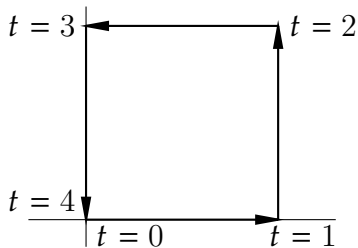
Example: Consider $\gamma: [0, 4] \rightarrow \mathbb{C}$,

$$\gamma(t) = \begin{cases} t & \text{if } t \in [0, 1], \\ 1 + i(t - 1) & \text{if } t \in (1, 2], \\ 3 - t + i & \text{if } t \in (2, 3], \\ i(4 - t) & \text{if } t \in (3, 4]. \end{cases}$$



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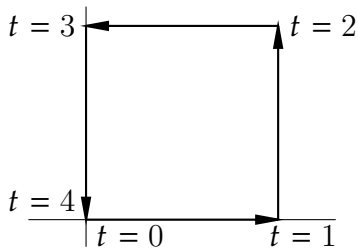
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Similarly $\lim_{t \downarrow 1} \gamma'(t) = i$, etc.

Given a piecewise- C^1 path $\gamma: [a, b] \rightarrow \mathbb{C}$ and a continuous function f on γ , we define the *line integral* (or *path integral*, *curve integral*, *contour integral*)

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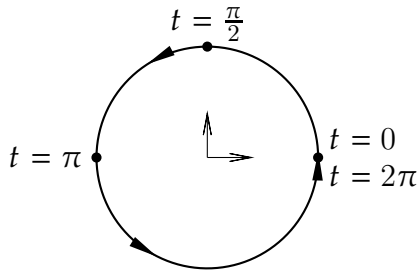
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The definition makes sense even if $\gamma'(t)$ is zero somewhere.

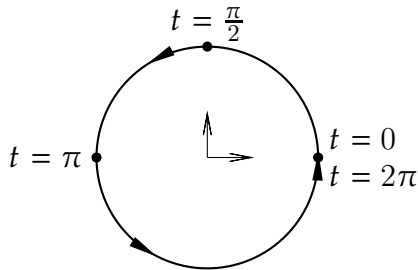
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$$\int_{\gamma} z^n dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

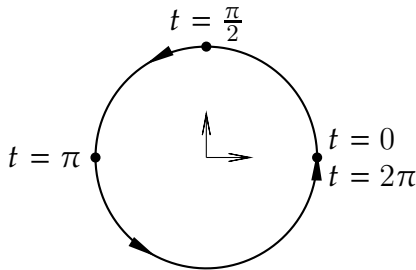


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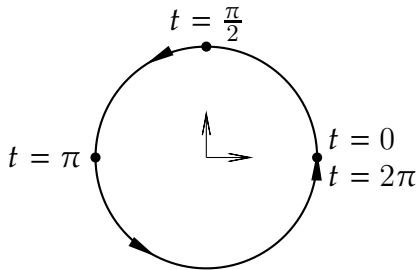
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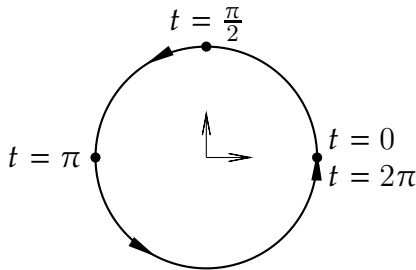
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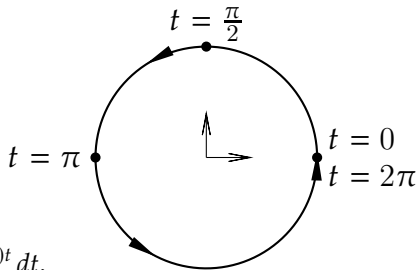
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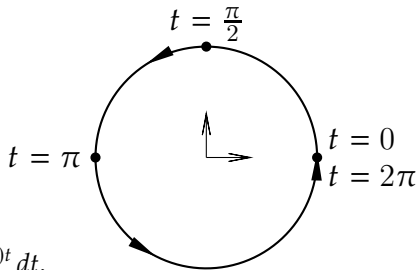
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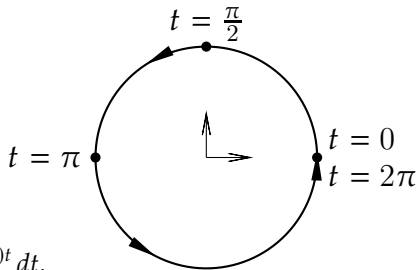
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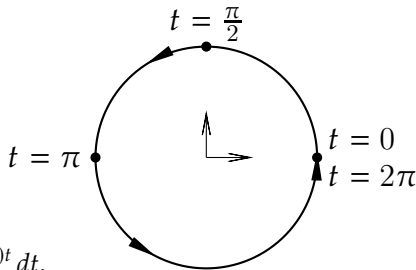
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Note that the value of the integral does not depend on r .



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In fact, if you also write $d\bar{z} = dx - i dy$, you can write any integral

$$\int_{\gamma} P dx + Q dy \quad \text{as} \quad \int_{\gamma} F dz + G d\bar{z}$$

and vice versa (exercise).