Cultivating Complex Analysis: Linear fractional transformations (1.4 part 2)

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Define the bijection $\Psi \colon \mathbb{C}_\infty \to \mathbb{CP}^1$ as

$$\Psi(z) = \begin{cases} [z : 1] & \text{if } z \in \mathbb{C}, \\ [1 : 0] & \text{if } z = \infty. \end{cases}$$

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As $M \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} az+bw \\ cz+dw \end{smallmatrix}\right]$, the function f corresponds to the linear map $v \mapsto Mv$ on $\mathbb{C}^2$.

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We have the commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^2 \setminus \{0\} \arrow[r, "M"] \arrow[d, "\pi"] & \mathbb{C}^2 \setminus \{0\} \arrow[d, "\pi"] \\
\mathbb{CP}^1 \arrow[r, "\Psi \circ f \circ \Psi^{-1}"] \arrow[d, shift left, "\Psi^{-1}"] & \mathbb{CP}^1 \arrow[d, shift left, "\Psi^{-1}"] \\
\mathbb{C}_\infty \arrow[r, "f"] \arrow[u, shift left, "\Psi"] & \mathbb{C}_\infty \arrow[u, shift left, "\Psi"]
\end{tikzcd}$$

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The group is generated by T_a, D_a, and I for $a \in \mathbb{C}$.

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