

# Cultivating Complex Analysis: Power series (2.3 part 1)

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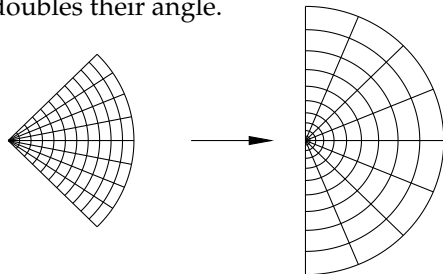
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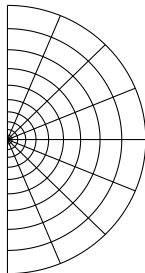
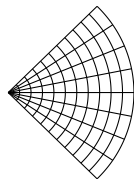
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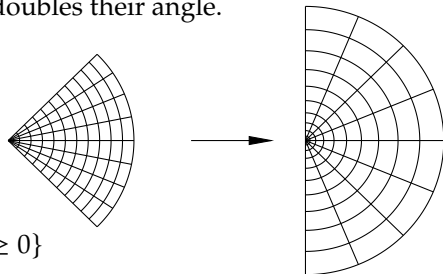
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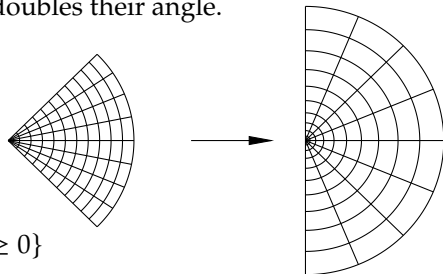
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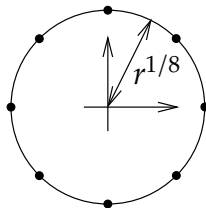
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E.g., in the picture,  $n = 8$ ,  $\theta = 0$ :

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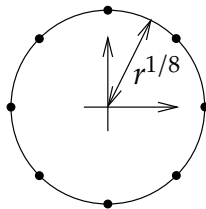
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The roots of  $w = 1$  are called the *roots of unity*.



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If  $z \neq 1$  is such that  $|z| = 1$ , then  $z^n$  diverges as  $n \rightarrow \infty$ .



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We say a power series is *convergent* if it converges for any  $z \neq p$ .

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which follows by expanding  $(1-z)(1+z+z^2+\cdots+z^m)$ .



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Thus, an absolutely convergent series converges.