

# Cultivating Complex Analysis: Singularities and the Laurent series (5.2.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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The Laurent series at  $z = 0$  is just  $1/z$ ,  
and all coefficients of order less than  $-1$  are zero.

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Again, pole of order 1 at  $z = 0$ ,  
and all coefficients in the series of order less than  $-1$  are zero.

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A pole of order 3,  
and all coefficients in the series of order less than  $-3$  are zero.



$$e^{1/z} = \sum_{n=-\infty}^0 \frac{1}{(-n)!} z^n$$

has an essential singularity at  $z = 0$ ,  
and has nonzero coefficients of all negative orders.

It is not difficult to prove the general statement:

### Proposition

Suppose  $f: \Delta_r(p) \setminus \{p\} \rightarrow \mathbb{C}$  is holomorphic, and

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - p)^n$$

is the corresponding Laurent series.

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Hint: Laurent series is unique, and for a removable singularity equals the power series.

## Definition

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*Observation:* If  $P(z)$  is the principal part of  $f(z)$  at  $p$ , then  $f(z) - P(z)$  has a removable singularity at  $p$ .

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$e^z$  has an essential singularity at infinity,  
because  $e^{1/z}$  has an essential singularity at  $0$ .

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**Exercise:** Show that if  $f: \mathbb{C} \rightarrow \mathbb{C}$  is an automorphism, then  $f(z) = az + b$  for some constants  $a \neq 0$  and  $b$ . Hint: Show that  $f$  has a simple pole at infinity.