Cultivating Complex Analysis: Harmonic functions Mean-value property (7.2.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

$f : \mathbb{R} \to \mathbb{R}$ is harmonic if

$f : \mathbb{R} \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} f = f'' = 0.$$

$f\colon \mathbb{R} \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} f = f'' = 0.$$

So $f(x) = Ax + B$ (affine linear).

$f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} f = f'' = 0.$$

So $f(x) = Ax + B$ (affine linear).

For any interval $[a, b]$, $a < b$,

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2}. \tag{*}$$

$f\colon \mathbb{R} \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} f = f'' = 0.$$

So $f(x) = Ax + B$ (affine linear).

For any interval $[a, b]$, $a < b$,

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2}. \tag{*}$$

"Mean-value property":

Exercise: A continuous $f\colon \mathbb{R} \to \mathbb{R}$ is harmonic (affine linear) $\quad\Leftrightarrow\quad$ (*) holds for all $[a, b]$.

Theorem (Mean-value property)

Suppose $U \subset \mathbb{C}$ is open. A continuous $f : U \longrightarrow \mathbb{R}$ is harmonic if and only if for every $p \in U$ there is an $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$ and

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta \qquad \text{for all } r < R_p.$$

Moreover, if f is harmonic we may choose any $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$.

Theorem (Mean-value property)

Suppose $U \subset \mathbb{C}$ is open. A continuous $f : U \to \mathbb{R}$ is harmonic if and only if for every $p \in U$ there is an $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$ and

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta \qquad \text{for all } r < R_p.$$

Moreover, if f is harmonic we may choose any $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$.

Proof:

$\Rightarrow$)

Theorem (Mean-value property)

Suppose $U \subset \mathbb{C}$ is open. A continuous $f : U \to \mathbb{R}$ is harmonic if and only if for every $p \in U$ there is an $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$ and

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta \qquad \text{for all } r < R_p.$$

Moreover, if f is harmonic we may choose any $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$.

Proof:

$\Rightarrow$) Suppose f is harmonic and $\overline{\Delta_r(p)} \subset U$.

Theorem (Mean-value property)

Suppose $U \subset \mathbb{C}$ is open. A continuous $f: U \to \mathbb{R}$ is harmonic if and only if for every $p \in U$ there is an $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$ and

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta \qquad \text{for all } r < R_p.$$

Moreover, if f is harmonic we may choose any $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$.

Proof:

$\Rightarrow$) Suppose f is harmonic and $\overline{\Delta_r(p)} \subset U$. ($R_p$ as large as wanted)

Theorem (Mean-value property)

Suppose $U \subset \mathbb{C}$ is open. A continuous $f : U \to \mathbb{R}$ is harmonic if and only if for every $p \in U$ there is an $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$ and

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta \qquad \textit{for all } r < R_p.$$

Moreover, if f is harmonic we may choose any $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$.

Proof:

$\Rightarrow$) Suppose f is harmonic and $\overline{\Delta_r(p)} \subset U$. ($R_p$ as large as wanted)

Solve the Dirichlet problem in $\Delta_r(p)$ (at p)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{it})\, dt = P\left[f|_{\partial \Delta_r(p)}\right](p)$$

Theorem (Mean-value property)

Suppose $U \subset \mathbb{C}$ is open. A continuous $f : U \to \mathbb{R}$ is harmonic if and only if for every $p \in U$ there is an $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$ and

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta \qquad \textit{for all } r < R_p.$$

Moreover, if f is harmonic we may choose any $R_p > 0$ such that $\Delta_{R_p}(p) \subset U$.

Proof:

$\Rightarrow$) Suppose f is harmonic and $\overline{\Delta_r(p)} \subset U$. ($R_p$ as large as wanted)

Solve the Dirichlet problem in $\Delta_r(p)$ (at p)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{it})\, dt = P\big[f|_{\partial \Delta_r(p)}\big](p) = f(p).$$

$\Leftarrow$)

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$.

$\Leftarrow)$ Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

$\Longleftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$.

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous,

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$,

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial \Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial \Delta_s(q)$, and satisfies the mean-value property on same circles as f.

$\Leftarrow$)   Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$.   Let $h = P\big[f|_{\partial\Delta_s(q)}\big]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$\Longleftarrow$)   Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$.   Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$X = \left\{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\right\}$ is compact.

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$X = \left\{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\right\}$ is compact.

Assume p is the point of X closest to $\partial\Delta_s(q)$.

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\big[f|_{\partial\Delta_s(q)}\big]$.

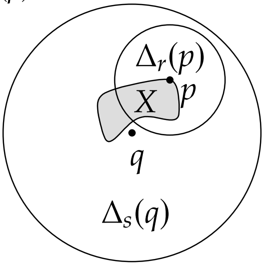
Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$X = \big\{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\big\}$ is compact.

Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),



$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\big[f|_{\partial\Delta_s(q)}\big]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

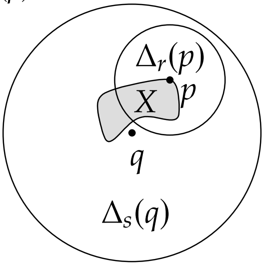
Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$X = \{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\}$ is compact.

Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),

$\varphi \leq C < \varphi(p)$ (for some $C \in \mathbb{R}$) on an open subset of $\partial\Delta_r(p)$



$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

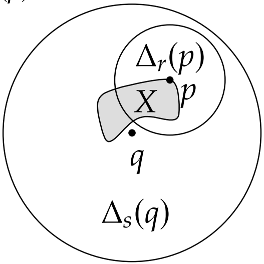
$X = \{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\}$ is compact.

Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),

$\varphi \le C < \varphi(p)$ (for some $C \in \mathbb{R}$) on an open subset of $\partial\Delta_r(p)$

So $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(p + re^{i\theta})\, d\theta < \varphi(p).$$



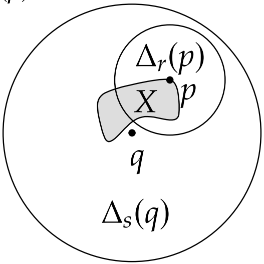
$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$X = \left\{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\right\}$ is compact.



Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),

$\varphi \leq C < \varphi(p)$ (for some $C \in \mathbb{R}$) on an open subset of $\partial\Delta_r(p)$

So $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(p + re^{i\theta})\, d\theta < \varphi(p)$. A contradiction!

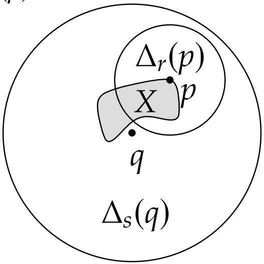
$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

$X = \left\{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\right\}$ is compact.



Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),

$\varphi \leq C < \varphi(p)$ (for some $C \in \mathbb{R}$) on an open subset of $\partial\Delta_r(p)$

So $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(p + re^{i\theta})\,d\theta < \varphi(p)$. A contradiction! So $\varphi \leq 0$.

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

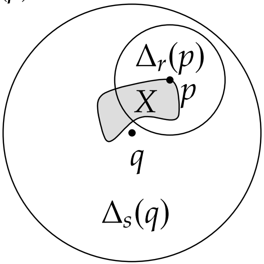
$X = \{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\}$ is compact.

Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),

$\varphi \leq C < \varphi(p)$ (for some $C \in \mathbb{R}$) on an open subset of $\partial\Delta_r(p)$

So $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(p + re^{i\theta})\, d\theta < \varphi(p)$. A contradiction! So $\varphi \leq 0$.



Similarly $\varphi \geq 0$.

$\Leftarrow$) Suppose f is continuous and satisfies the mean-value property.

Suppose $\overline{\Delta_s(q)} \subset U$. Let $h = P\left[f|_{\partial\Delta_s(q)}\right]$.

Let $\varphi = f - h$. φ is continuous, identically zero on $\partial\Delta_s(q)$, and satisfies the mean-value property on same circles as f.

Suppose (for contradiction) φ attains a maximum at $p \in \Delta_s(q)$ and $\varphi(p) > 0$.

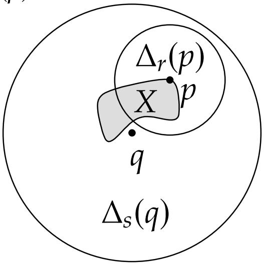
$X = \{z \in \Delta_s(q) : \varphi(z) = \varphi(p)\}$ is compact.

Assume p is the point of X closest to $\partial\Delta_s(q)$.

Suppose $\partial\Delta_r(p) \subset \Delta_s(q)$ (and $r < R_p$),

$\varphi \leq C < \varphi(p)$ (for some $C \in \mathbb{R}$) on an open subset of $\partial\Delta_r(p)$

So $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \varphi(p + re^{i\theta})\, d\theta < \varphi(p)$. A contradiction! So $\varphi \leq 0$.



Similarly $\varphi \geq 0$.

So $f = h$ and f is harmonic. $\square$

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

Proof: Firstly, f is continuous.

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

Proof: Firstly, f is continuous.

If $\overline{\Delta_r(p)} \subset U$, then $\{f_n\}$ converges uniformly on $\partial\Delta_r(p)$. So

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

Proof: Firstly, f is continuous.

If $\overline{\Delta_r(p)} \subset U$, then $\{f_n\}$ converges uniformly on $\partial\Delta_r(p)$. So

$$f(p) = \lim_{n\to\infty} f_n(p)$$

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

Proof: Firstly, f is continuous.

If $\overline{\Delta_r(p)} \subset U$, then $\{f_n\}$ converges uniformly on $\partial\Delta_r(p)$. So

$$f(p) = \lim_{n\to\infty} f_n(p) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(p + re^{i\theta})\, d\theta$$

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

Proof: Firstly, f is continuous.

If $\overline{\Delta_r(p)} \subset U$, then $\{f_n\}$ converges uniformly on $\partial\Delta_r(p)$. So

$$f(p) = \lim_{n\to\infty} f_n(p) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(p + re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta.$$

For C^2 functions, just because $f_n \to f$ uniformly, doesn't mean that $\nabla^2 f_n$ goes to $\nabla^2 f$.

But it does for harmonic functions.

Theorem (Harnack's first)

Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{R}$ be a sequence of harmonic functions converging uniformly on compact subsets to $f : U \to \mathbb{R}$. Then f is harmonic.

Proof: Firstly, f is continuous.

If $\overline{\Delta_r(p)} \subset U$, then $\{f_n\}$ converges uniformly on $\partial\Delta_r(p)$. So

$$f(p) = \lim_{n\to\infty} f_n(p) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(p + re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + re^{i\theta})\, d\theta.$$

Done by mean-value property. □

Exercise: Prove the maximum principle for harmonic functions directly from the mean-value property.

Exercise: Prove the maximum principle for harmonic functions directly from the mean-value property.

Exercise: Let $U \subset \mathbb{C}$ be open. Prove that a continuous $f\colon U \to \mathbb{R}$ is harmonic if and only if it satisfies the *disc mean-value property* for every $\overline{\Delta_r(p)} \subset U$:

$$f(p) = \frac{1}{\pi r^2} \int_{\overline{\Delta_r(p)}} f(z)\, dA.$$