Cultivating Complex Analysis:
Types of singularities and Riemann extension (5.2.1)

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Examples: Pole: ${}^1\!/_z$, essential: $e^{1/z}$.

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Then $f(z) = (z-p)^{k-2} h(z)$, that is, p is a removable singularity. $\square$

Exercise: Prove that if $f : \mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$ is an automorphism, then $f(z) = e^{i\theta} z$ for some θ.

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Exercise: Prove that $\frac{xy}{x^2+y^2}$ is a bounded infinitely (real) differentiable function on $\mathbb{R}^2 \setminus \{(0,0)\}$ with an isolated singularity, and this function does not extend through the singularity even continuously.

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$$f(z) = \frac{1}{\left(g(z)\right)^k}.$$