

# Cultivating Complex Analysis: Winding numbers (4.1.2)

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**Example:**  $\gamma(t) = e^{-it}$  for  $t \in [0, 2\pi]$  goes once around in the clockwise direction,

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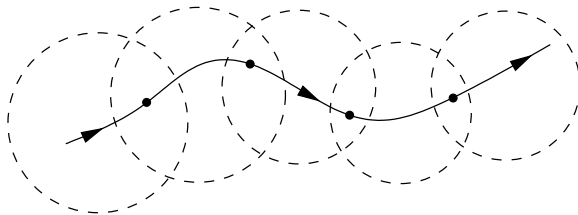
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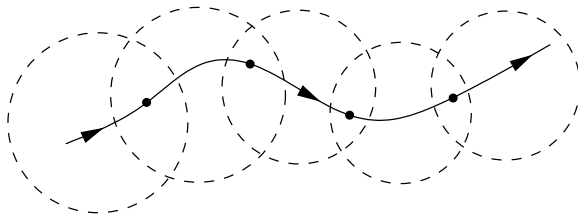
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(cover the whole closed curve, of course)

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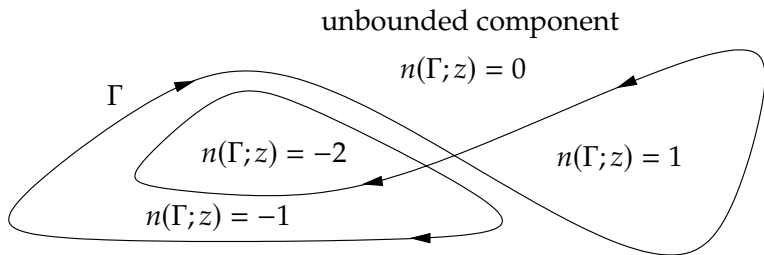
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**Example:**



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So,  $p \mapsto n(\Gamma; p)$  is a continuous.

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