Cultivating Complex Analysis: Laurent series (4.4 part 1)

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For Laurent series we generally have absolute convergence and the limit can be taken in any way, but it is still useful to split the series like this.

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E.g., the first part converges in $\Delta_{r_2}(p)$, and the second in $\mathbb{C} \setminus \overline{\Delta_{r_1}(p)}$, so the full series converges (uniformly absolutely on compact subsets) in $\operatorname{ann}(p; r_1, r_2)$ if $r_1 < r_2$.

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One of the main applications of complex analysis in engineering is to compute integrals by computing certain coefficients of the Laurent series by other means than integration.