Cultivating Complex Analysis: Inverses of holomorphic functions (5.6)

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Let us restate the inverse function theorem.

Theorem (Inverse function theorem for holomorphic functions)

Suppose $U \subset \mathbb{C}$ *is open,* $f\colon U \to \mathbb{C}$ *is holomorphic,* $p \in U$*, and* $f'(p) \neq 0$*. Then there exist open sets* $V, W \subset \mathbb{C}$ *such that* $p \in V \subset U$*,* $f(V) = W$*, the restriction* $f|_V$ *is injective* (*one-to-one*)*, and hence a* $g\colon W \to V$ *exists such that* $g(w) = (f|_V)^{-1}(w)$ *for all* $w \in W$*. Furthermore,* g *is holomorphic and*

$$g'(w) = \frac{1}{f'\big(g(w)\big)} \qquad \textit{for all } w \in W.$$

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Only local: $f(z) = z^2$ maps $\mathbb{C} \setminus \{0\}$ to itself, f' does not vanish, but f is 2-to-1 globally.

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$\Rightarrow \quad f(z) - w$ has at least two distinct zeros $\quad \Rightarrow \quad f$ is not injective. $\square$

We can actually compute the inverse:

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If $f : U \to \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f\left(\Delta_r(p)\right)$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z) z}{f(z) - w} \, dz.$$

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Consequently, f^{-1} is holomorphic without even using the inverse function theorem.

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If $U \subset \mathbb{C}$ is open and $f : U \longrightarrow \mathbb{C}$ is holomorphic and injective, then $f(U)$ is open, f' is never zero on U, and $f^{-1} : f(U) \longrightarrow U$ is holomorphic.

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By the other (or IFT), f^{-1} is holomorphic. $\square$