Cultivating Complex Analysis: Winding numbers (4.1.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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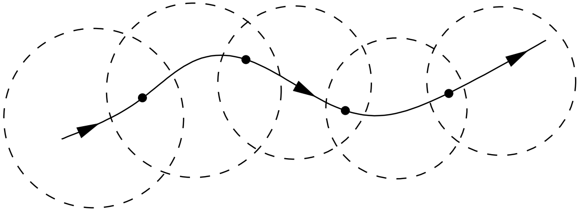

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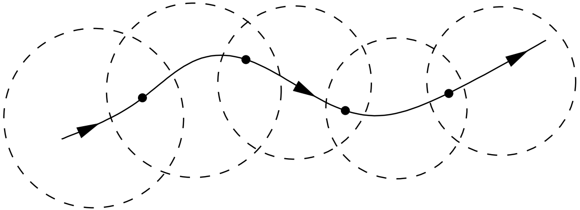

(cover the whole closed curve, of course)

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their difference is $2\pi k i$ for some $k \in \mathbb{Z}$. $\square$

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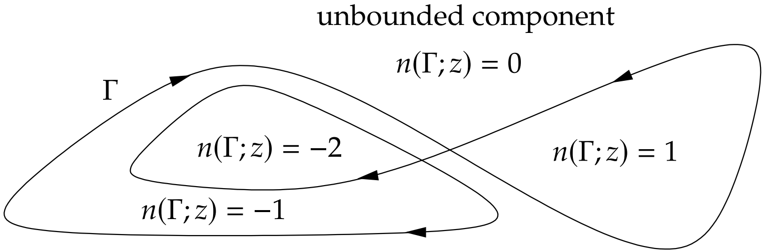

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Fix $p_0 \in \mathbb{C} \setminus \Gamma$, and let $d = d(p_0, \Gamma)$ be the distance from p_0 to Γ ($d > 0$ as Γ is compact).

If $p \in \Delta_{d/2}(p_0)$, then $|z - p| \geq {}^d\!/2$ for $z \in \Gamma$.

Let $\ell = \int_\Gamma |dz|$ (length of Γ). Then

$$|n(\Gamma; p_0) - n(\Gamma; p)| = \left| \frac{1}{2\pi i} \int_\Gamma \frac{p_0 - p}{(z-p_0)(z-p)}\, dz \right| \leq \frac{1}{2\pi} \int_\Gamma \frac{|p_0 - p|}{|z-p_0||z-p|}\, |dz| \leq \frac{\ell}{\pi d^2} |p_0 - p|.$$

So, $p \mapsto n(\Gamma; p)$ is a continuous.

Being continuous and integer-valued, it is constant on every component of $\mathbb{C} \setminus \Gamma$.

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So $n(\Gamma; p) = 0$ on this component. $\square$

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