Cultivating Complex Analysis:
Definition (of analytic functions) (2.4.1)
Analytic functions are holomorphic (2.4.2)

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Analytic functions are functions equal to a convergent power series near every point.

Definition

Let $U \subset \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *analytic* if for every $p \in U$, there exists an $r > 0$ and a power series $\sum c_n (z - p)^n$ converging to f on $\Delta_r(p) \subset U$.

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Remark: A subtle point is that it is not immediate that a convergent power series is analytic.

Proposition

Let $f : \Delta_R(p) \longrightarrow \mathbb{C}$ *be defined by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad \textit{converging in } \Delta_R(p).$$

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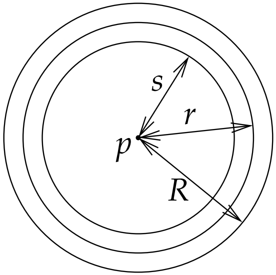

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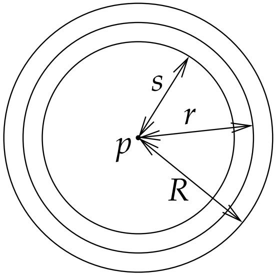

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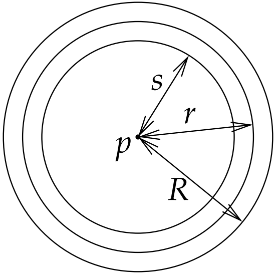

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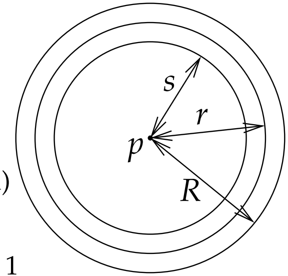

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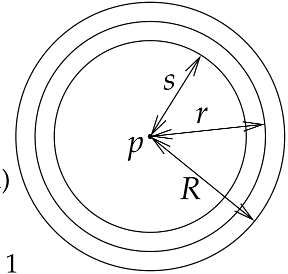

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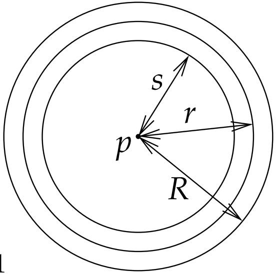

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Applied to the analytic functions we get:

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An analytic function is infinitely complex differentiable, and each derivative is analytic.