

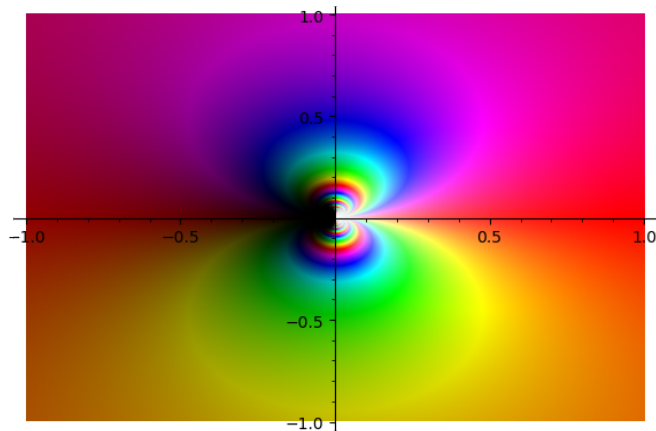
Cultivating Complex Analysis:
Wild world of essential singularities, Casorati–Weierstrass
(5.2.3)

Jiří Lebl

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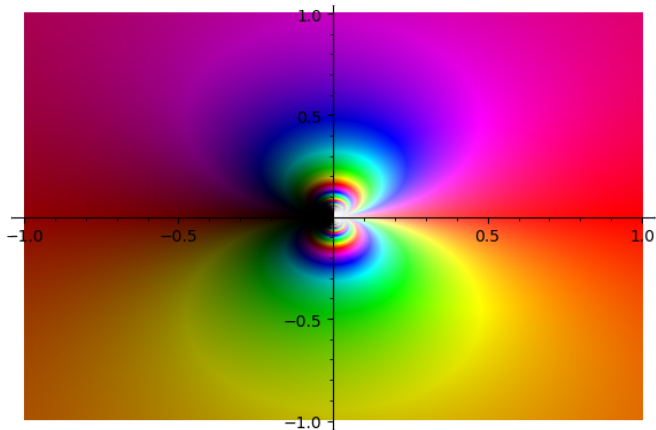
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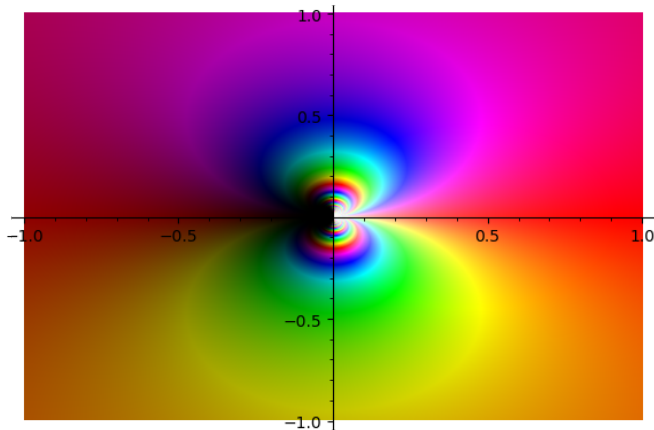


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We actually get $\mathbb{C} \setminus \{0\}$ as
the image of any neighborhood of 0.



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Picard's theorem says

$$f(\Delta_r(p) \setminus \{p\}) = \mathbb{C} \text{ or}$$

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In either case, f does not have an essential singularity at p .



Exercise: Prove the converse of Casorati–Weierstrass. Let $U \subset \mathbb{C}$ be open, $p \in U$, and $f: U \setminus \{p\} \rightarrow \mathbb{C}$ holomorphic. Prove that if $f(\Delta_r(p) \setminus \{p\})$ is dense in \mathbb{C} for all $r > 0$ such that $\Delta_r(p) \subset U$, then f has an essential singularity at p .

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Remark: The so-called “little Picard theorem” says that $f(\mathbb{C})$ is actually everything minus possibly one point, but that is much harder to prove.