

Cultivating Complex Analysis: The Riemann sphere (1.3)

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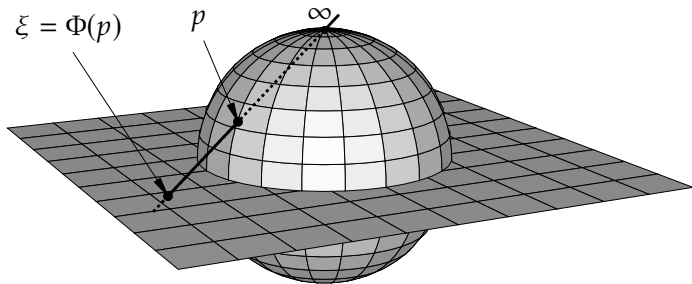
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We'll use Riemann sphere sense unless otherwise noted or obvious. We may use $+\infty$ to distinguish from Riemann sphere ∞ if confusion could arise.

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Note that this is different from the extended real arithmetic.