# Cultivating Complex Analysis: The geometry and topology of the plane (1.1.2) Complex-valued functions (1.1.3)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \stackrel{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

### Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

### Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$0 \leq |z\bar{w} - \bar{z}w|^2$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \le |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \le |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$0 \le |z\bar{w} - \bar{z}w|^2 = (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w})$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$0 \leq |z\bar{w} - \bar{z}w|^2 = (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$\begin{aligned} 0 \leq |z\bar{w} - \bar{z}w|^2 &= (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2 \\ &= 4z\bar{z}w\bar{w} - (z\bar{w} + \bar{z}w)^2 \end{aligned}$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$\begin{aligned} 0 \leq |z\bar{w} - \bar{z}w|^2 &= (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2 \\ &= 4z\bar{z}w\bar{w} - (z\bar{w} + \bar{z}w)^2 = \big(2|z||w|\big)^2 - \big(2\operatorname{Re} z\bar{w}\big)^2. \end{aligned}$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$\begin{aligned} 0 \leq |z\bar{w} - \bar{z}w|^2 &= (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2 \\ &= 4z\bar{z}w\bar{w} - (z\bar{w} + \bar{z}w)^2 = \big(2|z||w|\big)^2 - \big(2\operatorname{Re} z\bar{w}\big)^2. \end{aligned}$$

Triangle inequality:

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w})$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$\begin{aligned} 0 \leq |z\bar{w} - \bar{z}w|^2 &= (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2 \\ &= 4z\bar{z}w\bar{w} - (z\bar{w} + \bar{z}w)^2 = \big(2|z||w|\big)^2 - \big(2\operatorname{Re} z\bar{w}\big)^2. \end{aligned}$$

Triangle inequality:

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$     (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$     (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$\begin{aligned} 0 \leq |z\bar{w} - \bar{z}w|^2 &= (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2 \\ &= 4z\bar{z}w\bar{w} - (z\bar{w} + \bar{z}w)^2 = \big(2|z||w|\big)^2 - \big(2 \operatorname{Re} z\bar{w}\big)^2. \end{aligned}$$

Triangle inequality:

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \leq z\bar{z} + w\bar{w} + 2|z||w|$$

We measure distance with the *modulus* (euclidean distance from zero):

$$|z| \overset{\text{def}}{=} \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

The (euclidean) distance $|z - w|$ makes $\mathbb{C}$ into a complete metric space.

## Proposition (Cauchy–Schwarz and the triangle inequality)

*If $z, w \in \mathbb{C}$, then*

(i) $|\operatorname{Re} z\bar{w}| \leq |z||w|$ (*Cauchy–Schwarz inequality, note*: $\operatorname{Re} z\bar{w}$ *is the real dot product*),

(ii) $|z + w| \leq |z| + |w|$ (*Triangle inequality*).

**Proof:** Cachy–Schwarz:

$$\begin{aligned} 0 \leq |z\bar{w} - \bar{z}w|^2 &= (z\bar{w} - \bar{z}w)(\bar{z}w - z\bar{w}) = 2z\bar{z}w\bar{w} - z^2\bar{w}^2 - \bar{z}^2w^2 \\ &= 4z\bar{z}w\bar{w} - (z\bar{w} + \bar{z}w)^2 = \big(2|z||w|\big)^2 - \big(2\operatorname{Re} z\bar{w}\big)^2. \end{aligned}$$

Triangle inequality:

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \leq z\bar{z} + w\bar{w} + 2|z||w| = \big(|z| + |w|\big)^2. \qquad \square$$

## Proposition

*Complex addition, multiplication, division, and conjugation are continuous: Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences of complex numbers. Then,*

(i) $\lim_{n\to\infty}(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$,

(ii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$,

(iii) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$, *as long as* $\lim_{n\to\infty} a_n \neq 0$,

(iv) $\lim_{n\to\infty} \bar{a}_n = \overline{\lim_{n\to\infty} a_n}$.

### Proposition

*Complex addition, multiplication, division, and conjugation are continuous: Suppose $\{a_n\}$ and $\{b_n\}$ are two convergent sequences of complex numbers. Then,*

(i) $\lim_{n\to\infty}(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$,

(ii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$,

(iii) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$, *as long as* $\lim_{n\to\infty} a_n \neq 0$,

(iv) $\lim_{n\to\infty} \bar{a}_n = \overline{\lim_{n\to\infty} a_n}$.

All these operations are defined in terms of operations on the real an imaginary parts which are continuous. Details left as exercise.

If $p \in \mathbb{C}$ and $r > 0$, define the *disc* of radius $r$ around $p$ as

$$\Delta_r(p) \overset{\text{def}}{=} \{z \in \mathbb{C} : |z - p| < r\}.$$

If $p \in \mathbb{C}$ and $r > 0$, define the *disc* of radius $r$ around $p$ as

$$\Delta_r(p) \stackrel{\text{def}}{=} \left\{ z \in \mathbb{C} : |z - p| < r \right\}.$$

$$\mathbb{D} \stackrel{\text{def}}{=} \Delta_1(0) = \left\{ z \in \mathbb{C} : |z| < 1 \right\} \qquad \textit{unit disc}.$$

If $p \in \mathbb{C}$ and $r > 0$, define the *disc* of radius $r$ around $p$ as

$$\Delta_r(p) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - p| < r\}.$$

$$\mathbb{D} \stackrel{\text{def}}{=} \Delta_1(0) = \{z \in \mathbb{C} : |z| < 1\} \qquad \textit{unit disc}.$$

A useful "version" of $\mathbb{D}$ is the *upper half-plane*:

$$\mathbb{H} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

If $p \in \mathbb{C}$ and $r > 0$, define the *disc* of radius $r$ around $p$ as

$$\Delta_r(p) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z - p| < r\}.$$

$$\mathbb{D} \stackrel{\text{def}}{=} \Delta_1(0) = \{z \in \mathbb{C} : |z| < 1\} \qquad \textit{unit disc}.$$

A useful "version" of $\mathbb{D}$ is the *upper half-plane*:

$$\mathbb{H} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

**Definition**

An open and connected set $U \subset \mathbb{C}$ is called a *domain*.

If $f\colon X \longrightarrow \mathbb{C}$, write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$: $\qquad f = u + iv.$

If $f\colon X \longrightarrow \mathbb{C}$, write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$: $\qquad f = u + iv.$

If $X \subset \mathbb{C}$ ($z = x + iy$): $\qquad \dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ $\qquad$ and $\qquad \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}.$

If $f\colon X \longrightarrow \mathbb{C}$, write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$: $\qquad f = u + iv.$

If $X \subset \mathbb{C}$ ($z = x + iy$): $\qquad \dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} \qquad$ and $\qquad \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}.$

If $X \subset \mathbb{R}$: $\qquad f' = u' + iv'.$

If $f\colon X \longrightarrow \mathbb{C}$, write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$: $\qquad f = u + iv.$

If $X \subset \mathbb{C}$ ($z = x + iy$): $\qquad \dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} \qquad$ and $\qquad \dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}.$

If $X \subset \mathbb{R}$: $\qquad f' = u' + iv'.$

If $f\colon [a, b] \longrightarrow \mathbb{C}$, $f$ is (Riemann) integrable if $u$ and $v$ are, and

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

If $f\colon X \to \mathbb{C}$, write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$: $\quad f = u + iv.$

If $X \subset \mathbb{C}$ ($z = x + iy$): $\quad \dfrac{\partial f}{\partial x} = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial f}{\partial y} = \dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}.$

If $X \subset \mathbb{R}$: $\quad f' = u' + iv'.$

If $f\colon [a, b] \to \mathbb{C}$, $f$ is (Riemann) integrable if $u$ and $v$ are, and

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

## Proposition

*Suppose $f\colon [a, b] \to \mathbb{C}$ is (Riemann) integrable. Then $|f|$ is (Riemann) integrable and*

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b \left|f(t)\right|\,dt.$$