

Cultivating Complex Analysis:  
The exponential (1.2.1)  
Polar coordinates (1.2.2)

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Define the exponential  $e^z$  for  $z \in \mathbb{C}$  (using the real exponential and sin/cos):

$$\exp(z) = e^z = e^{x+iy} \stackrel{\text{def}}{=} e^x e^{iy} = e^x \cos y + ie^x \sin y.$$

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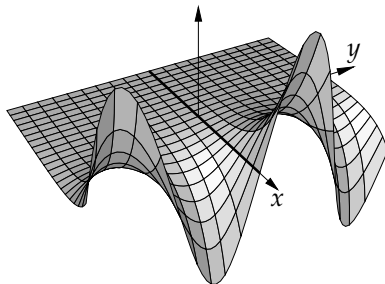
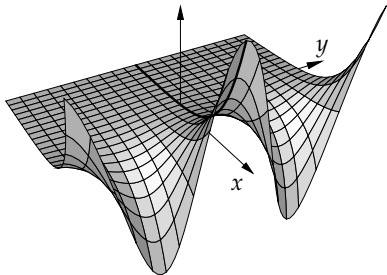
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Graphs of the real and imaginary part:



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We define  $\sin$  and  $\cos$  for  $z \in \mathbb{C}$  accordingly:

$$\cos z \stackrel{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z \stackrel{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i}.$$

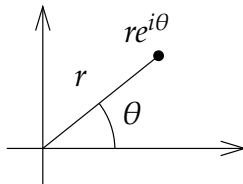


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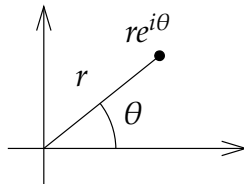


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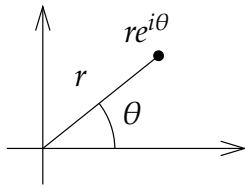
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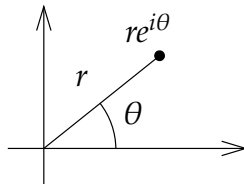
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The downside is that the polar form is terrible for addition.