

Cultivating Complex Analysis:  
Harmonic functions  
Identity and the maximum principle (7.1.2)

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$Z$  is also closed and  $U$  is connected  $\Rightarrow Z = U$ .



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**Exercise:** Suppose  $U \subset \mathbb{C}$  is a domain and  $f: U \rightarrow \mathbb{R}$  is harmonic. Prove that  $f(U)$  is an open interval or a single point.