Cultivating Complex Analysis: Power series (2.3 part 2)

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Proposition (Cauchy–Hadamard theorem)

$\sum c_n (z-p)^n$ converges absolutely if $|z-p| < R$ and diverges if $|z-p| > R$.

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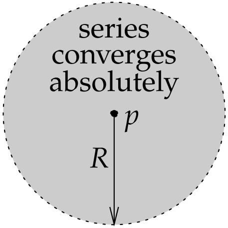

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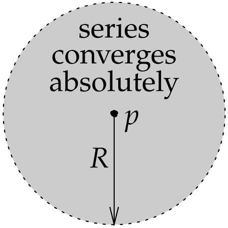

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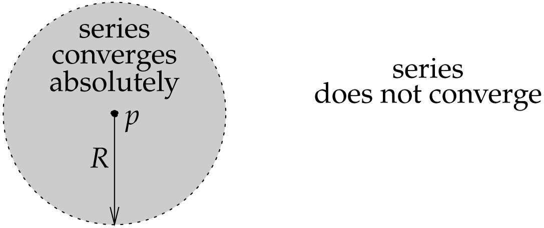

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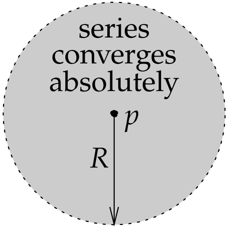

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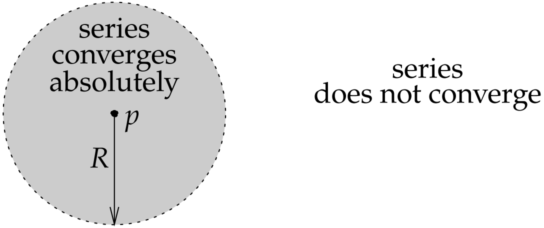

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R is called the *radius of convergence*.

Proposition

The series $\sum c_n(z-p)^n$ *converges in* $\Delta_R(p)$ *for some* $R > 0$ *if and only if for every* r *with* $0 < r < R$*, there exists an* $M > 0$ *such that*

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However, $\{nr^n\}$ is bounded for every $r < 1$.

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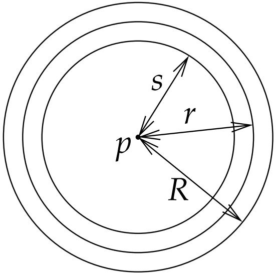

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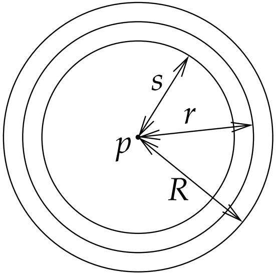

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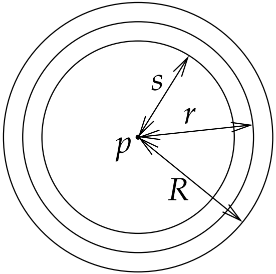

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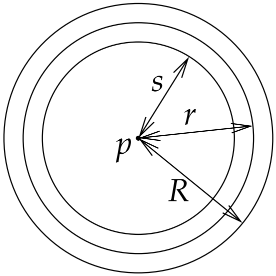

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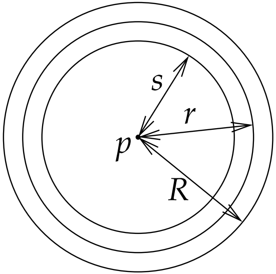

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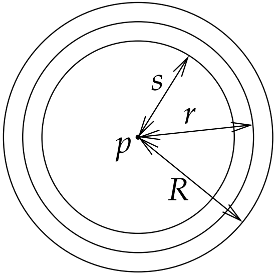

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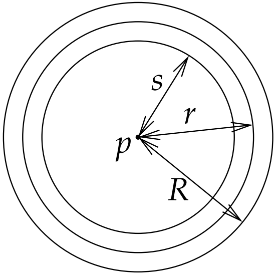

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As s and r with $0 < s < r < R$ were arbitrary, the series converges (absolutely) in $\Delta_R(p)$. $\square$