

Cultivating Complex Analysis: Derivative is holomorphic and Morera (3.3.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Last time we proved holomorphic functions are analytic.

Last time we proved holomorphic functions are analytic.

Let us restate a theorem that we proved for analytic functions for holomorphic functions.

Last time we proved holomorphic functions are analytic.

Let us restate a theorem that we proved for analytic functions for holomorphic functions.

Theorem

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ holomorphic. Then f is infinitely complex differentiable.

Last time we proved holomorphic functions are analytic.

Let us restate a theorem that we proved for analytic functions for holomorphic functions.

Theorem

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ holomorphic. Then f is infinitely complex differentiable. In particular, f' is holomorphic.

Last time we proved holomorphic functions are analytic.

Let us restate a theorem that we proved for analytic functions for holomorphic functions.

Theorem

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ holomorphic. Then f is infinitely complex differentiable. In particular, f' is holomorphic.

Nothing like this is true for real differentiable functions.

Last time we proved holomorphic functions are analytic.

Let us restate a theorem that we proved for analytic functions for holomorphic functions.

Theorem

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ holomorphic. Then f is infinitely complex differentiable. In particular, f' is holomorphic.

Nothing like this is true for real differentiable functions.

Any continuous $g: (a, b) \rightarrow \mathbb{R}$ is the derivative of a real differentiable function

E.g., $f(x) = \int_c^x g(t) dt$ for $c \in (a, b)$.

Last time we proved holomorphic functions are analytic.

Let us restate a theorem that we proved for analytic functions for holomorphic functions.

Theorem

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ holomorphic. Then f is infinitely complex differentiable. In particular, f' is holomorphic.

Nothing like this is true for real differentiable functions.

Any continuous $g: (a, b) \rightarrow \mathbb{R}$ is the derivative of a real differentiable function

E.g., $f(x) = \int_c^x g(t) dt$ for $c \in (a, b)$.

Even worse, the real derivative could even be discontinuous.

In complex analysis, we can differentiate by integrating.

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$.

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$f^{(k+1)}(z)$$

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$f^{(k+1)}(z) = \frac{\partial}{\partial z} [f^{(k)}(z)]$$

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$f^{(k+1)}(z) = \frac{\partial}{\partial z} [f^{(k)}(z)] = \frac{\partial}{\partial z} \left[\frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right]$$

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$\begin{aligned} f^{(k+1)}(z) &= \frac{\partial}{\partial z} [f^{(k)}(z)] = \frac{\partial}{\partial z} \left[\frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right] \\ &= \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} f(\zeta) \frac{\partial}{\partial z} \left[\frac{1}{(\zeta - z)^{k+1}} \right] d\zeta \end{aligned}$$

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$\begin{aligned} f^{(k+1)}(z) &= \frac{\partial}{\partial z} [f^{(k)}(z)] = \frac{\partial}{\partial z} \left[\frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right] \\ &= \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} f(\zeta) \frac{\partial}{\partial z} \left[\frac{1}{(\zeta - z)^{k+1}} \right] d\zeta = \frac{(k+1)!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+2}} d\zeta. \end{aligned}$$

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$\begin{aligned} f^{(k+1)}(z) &= \frac{\partial}{\partial z} [f^{(k)}(z)] = \frac{\partial}{\partial z} \left[\frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right] \\ &= \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} f(\zeta) \frac{\partial}{\partial z} \left[\frac{1}{(\zeta - z)^{k+1}} \right] d\zeta = \frac{(k+1)!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+2}} d\zeta. \end{aligned}$$

We passed x and y derivatives under the integral sign (Leibniz rule),

In complex analysis, we can differentiate by integrating.

Theorem (Cauchy integral formula for derivatives)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $k \in \mathbb{N}$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \quad \text{for all } z \in \Delta_r(p).$$

Proof: All complex derivatives exist.

We can compute them by the Wirtinger operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, (where $z = x + iy$).

Suppose conclusion holds for some k (Cauchy formula is $k = 0$) and fix some $z \in \Delta_r(p)$.

$$\begin{aligned} f^{(k+1)}(z) &= \frac{\partial}{\partial z} [f^{(k)}(z)] = \frac{\partial}{\partial z} \left[\frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta \right] \\ &= \frac{k!}{2\pi i} \int_{\partial\Delta_r(p)} f(\zeta) \frac{\partial}{\partial z} \left[\frac{1}{(\zeta - z)^{k+1}} \right] d\zeta = \frac{(k+1)!}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{(\zeta - z)^{k+2}} d\zeta. \end{aligned}$$

We passed x and y derivatives under the integral sign (Leibniz rule), which is valid as the x and y derivatives of $\frac{f(\zeta)}{(\zeta - z)^{k+1}}$ are continuous functions of $(z, \zeta) \in \Delta_r(p) \times \partial\Delta_r(p)$. □

As an aside we mention a result that will be needed later.

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

Prove that $\frac{\partial f}{\partial z}$ is a continuous function of $U \times (a, b)$.

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

Prove that $\frac{\partial f}{\partial z}$ is a continuous function of $U \times (a, b)$.

Then show $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

Prove that $\frac{\partial f}{\partial z}$ is a continuous function of $U \times (a, b)$.

Then show $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

The above is not true for real differentiable functions:

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

Prove that $\frac{\partial f}{\partial z}$ is a continuous function of $U \times (a, b)$.

Then show $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

The above is not true for real differentiable functions:

Let $f(x, t) = t \sin(x/t)$ for $t \neq 0$ and $f(x, 0) = 0$.

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

Prove that $\frac{\partial f}{\partial z}$ is a continuous function of $U \times (a, b)$.

Then show $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

The above is not true for real differentiable functions:

Let $f(x, t) = t \sin(x/t)$ for $t \neq 0$ and $f(x, 0) = 0$.

Then (exercise) f is continuous on \mathbb{R}^2 and $x \mapsto f(x, t)$ is differentiable for each fixed t .

As an aside we mention a result that will be needed later.

Exercise: Suppose $f(z, t)$ is a continuous function of $(z, t) \in U \times (a, b)$, where $U \subset \mathbb{C}$ is open, and for every fixed $t \in (a, b)$, the function $z \mapsto f(z, t)$ is holomorphic.

Prove that $\frac{\partial f}{\partial z}$ is a continuous function of $U \times (a, b)$.

Then show $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

The above is not true for real differentiable functions:

Let $f(x, t) = t \sin(x/t)$ for $t \neq 0$ and $f(x, 0) = 0$.

Then (exercise) f is continuous on \mathbb{R}^2 and $x \mapsto f(x, t)$ is differentiable for each fixed t .

But $\frac{\partial f}{\partial x}$ is not continuous as a function of both x and t .

That f' is holomorphic gives us a very useful converse to Cauchy.

That f' is holomorphic gives us a very useful converse to Cauchy.

Theorem (Morera)

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ continuous. Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for every triangle such that $T \subset U$.

That f' is holomorphic gives us a very useful converse to Cauchy.

Theorem (Morera)

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ continuous. Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for every triangle such that $T \subset U$. Then f is holomorphic.

That f' is holomorphic gives us a very useful converse to Cauchy.

Theorem (Morera)

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ continuous. Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for every triangle such that $T \subset U$. Then f is holomorphic.

It is far easier to integrate a continuous f than to show that f' exists.

That f' is holomorphic gives us a very useful converse to Cauchy.

Theorem (Morera)

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ continuous. Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for every triangle such that $T \subset U$. Then f is holomorphic.

It is far easier to integrate a continuous f than to show that f' exists.

Proof: Holomorphicity is local, so assume U is a disc.

That f' is holomorphic gives us a very useful converse to Cauchy.

Theorem (Morera)

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ continuous. Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for every triangle such that $T \subset U$. Then f is holomorphic.

It is far easier to integrate a continuous f than to show that f' exists.

Proof: Holomorphicity is local, so assume U is a disc.

A disc is star-like, and the hypothesis is precisely what we used to show that f has a primitive F in a star-like U .

That f' is holomorphic gives us a very useful converse to Cauchy.

Theorem (Morera)

Let $U \subset \mathbb{C}$ be open and $f: U \rightarrow \mathbb{C}$ continuous. Suppose that

$$\int_{\partial T} f(z) dz = 0$$

for every triangle such that $T \subset U$. Then f is holomorphic.

It is far easier to integrate a continuous f than to show that f' exists.

Proof: Holomorphicity is local, so assume U is a disc.

A disc is star-like, and the hypothesis is precisely what we used to show that f has a primitive F in a star-like U .

$f = F'$ in U , and complex derivatives are holomorphic.



The reduction to a disc is necessary:

E.g., $1/z$ does not have a primitive in $U = \mathbb{C} \setminus \{0\}$,
but does satisfy hypotheses of Morera.

The reduction to a disc is necessary:

E.g., $1/z$ does not have a primitive in $U = \mathbb{C} \setminus \{0\}$,
but does satisfy hypotheses of Morera.

Typical application of Morera is something like the following exercise:

The reduction to a disc is necessary:

E.g., $1/z$ does not have a primitive in $U = \mathbb{C} \setminus \{0\}$,
but does satisfy hypotheses of Morera.

Typical application of Morera is something like the following exercise:

Exercise: Show that if $f: \mathbb{C} \rightarrow \mathbb{C}$ is continuous and holomorphic on $\mathbb{C} \setminus \mathbb{R}$, then f is holomorphic everywhere.