Cultivating Complex Analysis: Basic calculus (2.2.1)

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The proof is just the standard real result, since $f'(z) = 0$ implies that the real derivative is also zero (a zero 2×2 matrix).

Proposition (Chain rule)

Let $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$ be open, $f\colon U \to V$ complex differentiable at $z \in U$, and $g\colon V \to \mathbb{C}$ complex differentiable at $f(z)$. Then the composition $g \circ f$ is complex differentiable at z and
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Multiplication is continuous, so take the limit $h \to 0$ to finish. □

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So $D(g \circ f)|_z$ corresponds to the pertinent complex number. $\square$

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(i) $f + g$ *is holomorphic and* $\frac{d}{dz}\left[f(z) + g(z)\right] = f'(z) + g'(z)$.

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(iii) *${}^1/_g$ is holomorphic on* $\left\{z \in U : g(z) \neq 0\right\}$ *and* $\frac{d}{dz}\left[\frac{1}{g(z)}\right] = \frac{-g'(z)}{\left(g(z)\right)^2}$.

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Proof: Exercise. Just adapt the one real variable proof.

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Remark: A holomorphic function is continuous so $\left\{z \in U : g(z) \neq 0\right\}$ is open.

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(i) *For every integer n, the function $z \mapsto z^n$ is holomorphic where defined (outside the origin if n negative) and* $\frac{d}{dz}\left[z^n\right] = nz^{n-1}$ *if* $n \neq 0$ *and* $\frac{d}{dz}\left[z^0\right] = 0$.

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Proof: Again exercise.

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The key is of course that f is not differentiable (neither real nor complex) at the origin.