Cultivating Complex Analysis: Primitives, cycles, and Cauchy for derivatives (3.2.1)

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Definition

Let $U \subset \mathbb{C}$ be open and $f\colon U \to \mathbb{C}$ a function. A holomorphic $F\colon U \to \mathbb{C}$ with $f = F'$ is called a (holomorphic) *primitive* (or an *antiderivative*) of f.

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Primitives do not always exist, but if they do, then they are unique up to a constant.

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Suppose $U \subset \mathbb{C}$ is a domain, and $F : U \to \mathbb{C}$ and $G : U \to \mathbb{C}$ are holomorphic such that $F' = G'$. Then $F(z) = G(z) + C$ for some $C \in \mathbb{C}$.

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Proof is an exercise.

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Remark: The hypothesis that $f = F'$ is continuous is extraneous (we will prove later that f is better than continuous, it is holomorphic).

Definition

A chain Γ that is equivalent to $a_1\gamma_1 + \cdots + a_n\gamma_n$, where $\gamma_1, \ldots, \gamma_n$ are closed piecewise-C^1 paths and $a_1, \ldots, a_n \in \mathbb{Z}$, is called a *cycle*.

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Corollary (Cauchy's theorem for derivatives)

Suppose $U \subset \mathbb{C}$ is open and $f\colon U \to \mathbb{C}$ is continuous with a primitive $F\colon U \to \mathbb{C}$. Let Γ be a cycle in U. Then

$$\int_\Gamma f(z)\,dz = 0.$$

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We can now prove a very neat version of the theorem (exercise):

Corollary (Cauchy's theorem for polynomials)

Suppose $P(z)$ is a polynomial and Γ is a cycle (in $\mathbb{C}$). Then

$$\int_\Gamma P(z)\, dz = 0.$$