Cultivating Complex Analysis: Cauchy's formula in a disc (3.2.4)

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Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, $\overline{\Delta_r(p)} \subset U$. Then for all $z \in \Delta_r(p)$,

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A quick (but hardly only) application is to compute integrals of expressions such as $\frac{\cos(z^2)}{z(z-1)}$ that blow up somewhere inside the cycle.

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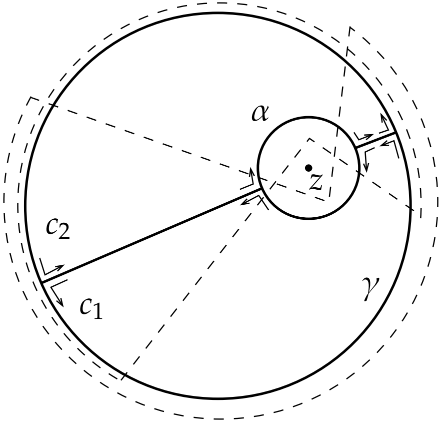
Connect α to γ via two straight lines
to give two closed paths c_1 and c_2



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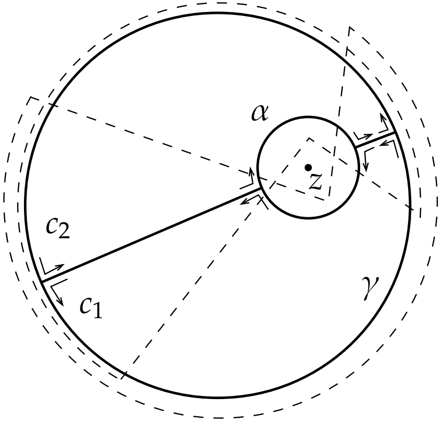

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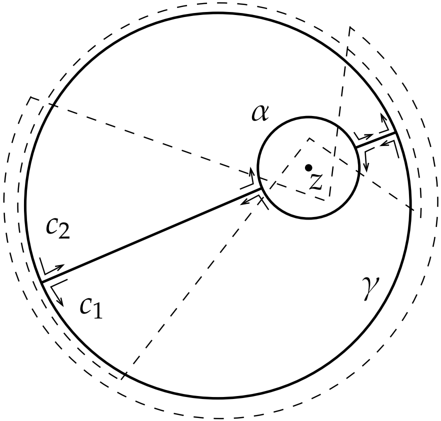

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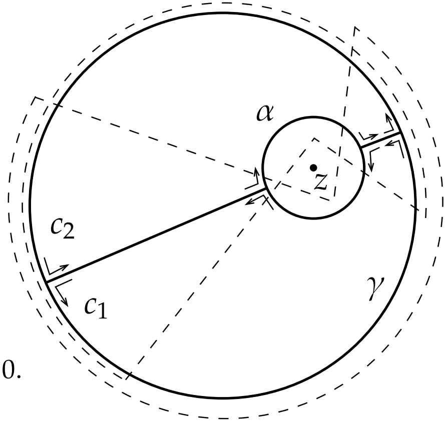

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$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \int_\alpha \frac{f(\zeta)}{\zeta - z}\,d\zeta =$$

$$\int_{c_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{c_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta = 0 + 0.$$

(by Cauchy's theorem)

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(by continuity of f at z) □

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Exercise: Suppose f is holomorphic in an open neighborhood of $\overline{\Delta_r(p)}$. Show that f at p is the average of the values on $\partial\Delta_r(p)$. That is, show

$$f(p) = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{it})\, dt.$$