

# Cultivating Complex Analysis: Primitives, cycles, and Cauchy for derivatives (3.2.1)

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## Definition

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Suppose  $U \subset \mathbb{C}$  is a domain, and  $F: U \rightarrow \mathbb{C}$  and  $G: U \rightarrow \mathbb{C}$  are holomorphic such that  $F' = G'$ . Then  $F(z) = G(z) + C$  for some  $C \in \mathbb{C}$ .

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Proof is an exercise.

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**Remark:** The hypothesis that  $f = F'$  is continuous is extraneous (we will prove later that  $f$  is better than continuous, it is holomorphic).

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A chain  $\Gamma$  that is equivalent to  $a_1\gamma_1 + \cdots + a_n\gamma_n$ , where  $\gamma_1, \dots, \gamma_n$  are closed piecewise- $C^1$  paths and  $a_1, \dots, a_n \in \mathbb{Z}$ , is called a *cycle*.

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## Corollary (Cauchy's theorem for derivatives)

Suppose  $U \subset \mathbb{C}$  is open and  $f: U \rightarrow \mathbb{C}$  is continuous with a primitive  $F: U \rightarrow \mathbb{C}$ . Let  $\Gamma$  be a cycle in  $U$ . Then

$$\int_{\Gamma} f(z) dz = 0.$$

We will prove several versions of Cauchy, though usually we will put restrictions on  $U$  or the  $\Gamma$  rather than the function.

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We can now prove a very neat version of the theorem (exercise):

### Corollary (Cauchy’s theorem for polynomials)

*Suppose  $P(z)$  is a polynomial and  $\Gamma$  is a cycle (in  $\mathbb{C}$ ). Then*

$$\int_{\Gamma} P(z) dz = 0.$$