

Cultivating Complex Analysis:  
Harmonic functions  
Harnack's inequality and principle (7.2.3–7.2.4)

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## Theorem (Harnack's inequality)

Suppose  $f: \Delta_R(p) \rightarrow \mathbb{R}$  is harmonic and nonnegative, and suppose  $0 < r < R$ . Then

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As  $S < R$  was arbitrary, the theorem follows by taking a limit.



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Also assume  $\Delta_r(z_j) \cap \Delta_r(z_{j+1}) \neq \emptyset$

for all  $j = 1, \dots, n-1$  ( $K$  connected).



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**Proof:** WLOG assume that  $K$  is connected: (e.g. cover by finitely many closed discs so that  $K$  has finitely many components, then connect with paths as  $U$  is path connected).

Let  $r > 0$  be less than half the distance from  $K$  to  $\partial U$ .

$\exists \Delta_r(z_1), \dots, \Delta_r(z_N)$  that cover  $K$  and  $\Delta_{2r}(z_j) \subset U$  for every  $j$ .

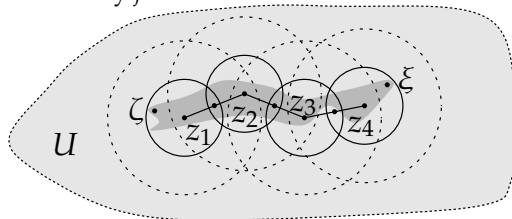
Fix  $\zeta, \xi \in K$ . Assume  $\zeta \in \Delta_r(z_1)$

and  $\xi \in \Delta_r(z_n)$

(for some  $n \leq N$ .)

Also assume  $\Delta_r(z_j) \cap \Delta_r(z_{j+1}) \neq \emptyset$

for all  $j = 1, \dots, n-1$  ( $K$  connected).



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**Remark:** We got an explicit (if not optimal)  $C$ .

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**Exercise:** Use Harnack's inequality to prove Liouville's theorem for harmonic functions: If  $f: \mathbb{C} \rightarrow \mathbb{R}$  is harmonic and nonnegative, then  $f$  is constant.

## Theorem (Harnack's principle)

*Let  $U \subset \mathbb{C}$  be a domain and  $\{f_n\}$  a sequence of harmonic functions on  $U$  such that  $f_1 \leq f_2 \leq f_3 \leq \dots$ . Then either  $f_n \rightarrow +\infty$  uniformly on compact subsets, or  $f_n \rightarrow f$  for a harmonic  $f: U \rightarrow \mathbb{R}$  uniformly on compact subsets.*

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## Theorem (Harnack's principle)

Let  $U \subset \mathbb{C}$  be a domain and  $\{f_n\}$  a sequence of harmonic functions on  $U$  such that  $f_1 \leq f_2 \leq f_3 \leq \dots$ . Then either  $f_n \rightarrow +\infty$  uniformly on compact subsets, or  $f_n \rightarrow f$  for a harmonic  $f: U \rightarrow \mathbb{R}$  uniformly on compact subsets.

**Proof:** WLOG assume  $f_n \geq 0$  for all  $n$ . (Otherwise apply to  $f_n - f_1$ ).

By the monotonicity,  $\{f_n\}$  converges pointwise (possibly to  $+\infty$ ).

If  $\lim f_n(p) = +\infty$  for some  $p$ , let  $K \subset U$  be any compact and let  $K' = K \cup \{p\}$ .

Harnack's inequality  $\Rightarrow f_n(p) \leq \sup_{z \in K'} f_n(z) \leq C \inf_{z \in K'} f_n(z) \leq C \inf_{z \in K} f_n(z)$ .

$\Rightarrow f_n(z) \rightarrow +\infty$  uniformly for  $z \in K$ .

Suppose  $f(z) = \lim f_n(z) < +\infty$  for every  $z \in U$ .

Let  $K \subset U$  be compact, take the  $C$  from Harnack's, and take any  $p \in K$ .

Given  $\epsilon > 0$ , suppose  $m > n$  are such that  $f_m(p) - f_n(p) < \epsilon/C$ , then

$$\sup_{z \in K} (f_m(z) - f_n(z)) \leq C \inf_{z \in K} (f_m(z) - f_n(z)) \leq C(f_m(p) - f_n(p)) < \epsilon.$$

$\Rightarrow \{f_n\}$  is uniformly Cauchy on  $K \Rightarrow$  converges uniformly.

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$f$  is harmonic by Harnack's first theorem.





**Exercise:** Prove yet another version of Harnack's principle. Suppose  $U \subset \mathbb{C}$  is a domain,  $\{f_n\}$  is a sequence of nonnegative harmonic functions on  $U$ , and  $p \in U$  is fixed.

- a) If  $f_n(p) \rightarrow +\infty$ , then  $\{f_n\}$  converges to  $+\infty$  uniformly on compact subsets.
- b) If  $f: U \rightarrow \mathbb{R}$  is harmonic,  $f_n(z) \leq f(z)$  for all  $z \in U$ , and  $f_n(p) \rightarrow f(p)$ , then  $\{f_n\}$  converges to  $f$  uniformly on compact subsets.

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**Exercise:** Prove a Montel-like theorem for harmonic functions. Suppose  $U \subset \mathbb{C}$  is a domain and  $\{f_n\}$  is a sequence of nonnegative harmonic functions. Show that at least one (or both) of the following are true:

- (i)  $\exists$  a subsequence converging to  $+\infty$  uniformly on compact subsets.
- (ii)  $\exists$  a subsequence converging to a harmonic function uniformly on compact subsets.