

# Cultivating Complex Analysis: Automorphisms of the disc (3.5.2)

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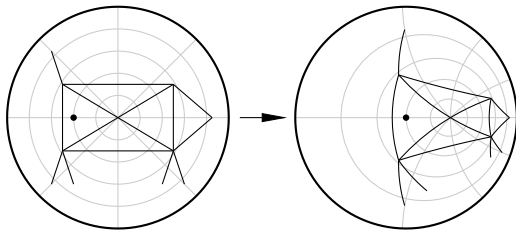
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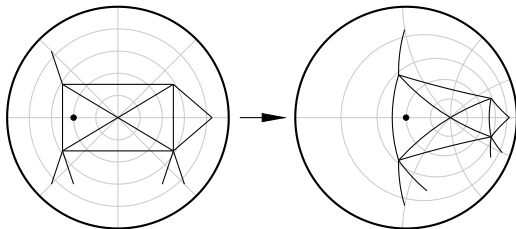
Proof is an exercise.

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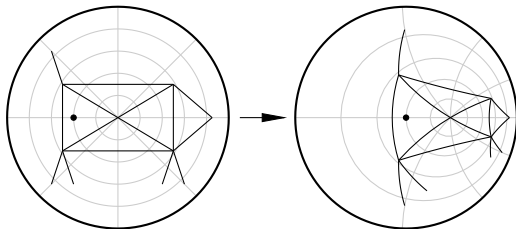
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If  $f \in \text{Aut}(\mathbb{D})$ , then there exists an  $a \in \mathbb{D}$  and  $\theta \in \mathbb{R}$  such that

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} = e^{i\theta} \varphi_a(z).$$

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Apply  $\varphi_{-a}$  to both sides of  $e^{i\theta}z = \varphi_a \circ f$  to find  $f(z) = \varphi_{-a}(ze^{i\theta}) = e^{i\theta} \varphi_{-ae^{-i\theta}}(z)$ .



Some good exercises to try:

**Exercise:** Given distinct  $a, b \in \mathbb{D}$ , show that there exists a unique  $f \in \text{Aut}(\mathbb{D})$  such that  $f(a) = b$  and  $f(b) = a$ .

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Hint: The idea is to show that you can divide by finitely many  $\varphi_a(z)$  for various  $a$  until you get something that has no zeros in  $\mathbb{D}$  and will have to be a constant.

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Conversely, if  $f \in \text{Aut}(\mathbb{D})$ , then equality holds in both inequalities for all  $z, \zeta \in \mathbb{D}$ .

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If equality holds in one of the inequalities for some  $z \neq \zeta$ , then  $f \in \text{Aut}(\mathbb{D})$ .

Conversely, if  $f \in \text{Aut}(\mathbb{D})$ , then equality holds in both inequalities for all  $z, \zeta \in \mathbb{D}$ .

Proof is an exercise.

Using automorphisms one proves an “invariant version” of the Schwarz lemma.

### Lemma (Schwarz–Pick)

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Proof is an exercise.

The Schwarz–Pick lemma gives a bound on the derivative at all points:

If  $f: \mathbb{D} \rightarrow \mathbb{D}$  is holomorphic, nonconstant, and  $f(a) = b$ , then

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

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Equality  $\Rightarrow f(z) = \varphi_{-b}(e^{i\theta} \varphi_a(z))$  for some  $\theta \in \mathbb{R}$ .