# Cultivating Complex Analysis: The identity theorem (2.4.4)

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*If the function is zero on a nonempty open subset, then* $f \equiv 0$*.*

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$z^k$ is only zero at $0 \Rightarrow 0$ is an isolated zero of $f$.

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$Z'_f$ is open and closed and $U$ is connected $\Rightarrow$ either $U = Z'_f$ or $Z'_f = \emptyset$. $\square$

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$$f(z) = (z - a)^k g(z),$$

where $g(z)$ is a power series at $a$ such that $g(a) \neq 0$.