

Cultivating Complex Analysis: Cauchy for star-like sets (3.2.3)

Jiří Lebl

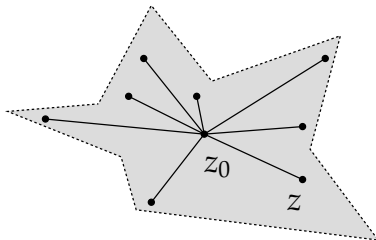
Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Definition

A set $U \subset \mathbb{C}$ is called *star-like* (or *star-like with respect to z_0*) if there exists a point $z_0 \in U$ such that the segment $[z_0, z] \subset U$ for every $z \in U$.

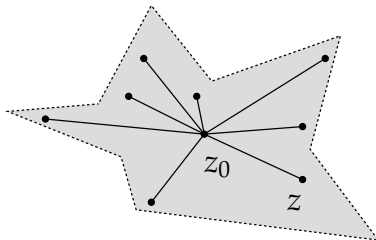
Definition

A set $U \subset \mathbb{C}$ is called *star-like* (or *star-like with respect to z_0*) if there exists a point $z_0 \in U$ such that the segment $[z_0, z] \subset U$ for every $z \in U$.



Definition

A set $U \subset \mathbb{C}$ is called *star-like* (or *star-like with respect to z_0*) if there exists a point $z_0 \in U$ such that the segment $[z_0, z] \subset U$ for every $z \in U$.



A convex set is star-like, but not vice versa.

Proposition

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is continuous, and

$$\int_{\partial T} f(z) dz = 0 \quad \text{for every triangle } T \subset U.$$

Then f has a primitive: There exists a holomorphic $F: U \rightarrow \mathbb{C}$ such that $F' = f$.

Proposition

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is continuous, and

$$\int_{\partial T} f(z) dz = 0 \quad \text{for every triangle } T \subset U.$$

Then f has a primitive: There exists a holomorphic $F: U \rightarrow \mathbb{C}$ such that $F' = f$.

Proof: Suppose U is star-like with respect to $z_0 \in U$.

Proposition

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is continuous, and

$$\int_{\partial T} f(z) dz = 0 \quad \text{for every triangle } T \subset U.$$

Then f has a primitive: There exists a holomorphic $F: U \rightarrow \mathbb{C}$ such that $F' = f$.

Proof: Suppose U is star-like with respect to $z_0 \in U$. Define

$$F(z) = \int_{[z_0, z]} f(\zeta) d\zeta.$$

Proposition

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is continuous, and

$$\int_{\partial T} f(z) dz = 0 \quad \text{for every triangle } T \subset U.$$

Then f has a primitive: There exists a holomorphic $F: U \rightarrow \mathbb{C}$ such that $F' = f$.

Proof: Suppose U is star-like with respect to $z_0 \in U$. Define

$$F(z) = \int_{[z_0, z]} f(\zeta) d\zeta.$$

Consider a disc $\Delta_r(z) \subset U$,
and $|h| < r$ so that $z + h \in \Delta_r(z)$.

Proposition

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is continuous, and

$$\int_{\partial T} f(z) dz = 0 \quad \text{for every triangle } T \subset U.$$

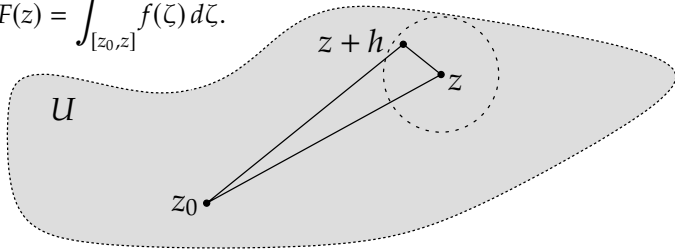
Then f has a primitive: There exists a holomorphic $F: U \rightarrow \mathbb{C}$ such that $F' = f$.

Proof: Suppose U is star-like with respect to $z_0 \in U$. Define

$$F(z) = \int_{[z_0, z]} f(\zeta) d\zeta.$$

Consider a disc $\Delta_r(z) \subset U$,
and $|h| < r$ so that $z + h \in \Delta_r(z)$.

U is star-like w.r.t. $z_0 \Rightarrow$
the entire triangle with vertices
 z_0 , z , and $z + h$ is in U .



By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$.

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\frac{F(z+h) - F(z)}{h}$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta = \frac{1}{h} \int_{[z, z+h]} f(\zeta) d\zeta$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta = \frac{1}{h} \int_{[z, z+h]} f(\zeta) d\zeta \\ &= \frac{1}{h} \int_0^1 f(z+th)h dt\end{aligned}$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta = \frac{1}{h} \int_{[z, z+h]} f(\zeta) d\zeta \\ &= \frac{1}{h} \int_0^1 f(z+th)h dt = \int_0^1 f(z+th) dt.\end{aligned}$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta = \frac{1}{h} \int_{[z, z+h]} f(\zeta) d\zeta \\ &= \frac{1}{h} \int_0^1 f(z+th)h dt = \int_0^1 f(z+th) dt.\end{aligned}$$

In other words,

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \int_0^1 f(z+th) dt - \int_0^1 f(z) dt \right|$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta = \frac{1}{h} \int_{[z, z+h]} f(\zeta) d\zeta \\ &= \frac{1}{h} \int_0^1 f(z+th)h dt = \int_0^1 f(z+th) dt.\end{aligned}$$

In other words,

$$\begin{aligned}\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \int_0^1 f(z+th) dt - \int_0^1 f(z) dt \right| \\ &\leq \int_0^1 |f(z+th) - f(z)| dt.\end{aligned}$$

By hypothesis $\int_{[z_0, z] + [z, z+h] - [z_0, z+h]} f(\zeta) d\zeta = 0$. So

$$\begin{aligned}\frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_{[z_0, z+h] - [z_0, z]} f(\zeta) d\zeta = \frac{1}{h} \int_{[z, z+h]} f(\zeta) d\zeta \\ &= \frac{1}{h} \int_0^1 f(z+th)h dt = \int_0^1 f(z+th) dt.\end{aligned}$$

In other words,

$$\begin{aligned}\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \int_0^1 f(z+th) dt - \int_0^1 f(z) dt \right| \\ &\leq \int_0^1 |f(z+th) - f(z)| dt.\end{aligned}$$

By continuity of f at z ,

$$\lim_{h \rightarrow 0} \frac{F(z+h) - F(z)}{h} = f(z). \quad \square$$

Cauchy–Goursat (the integral around triangles is zero if f is holomorphic) implies

Cauchy–Goursat (the integral around triangles is zero if f is holomorphic) implies

Corollary

Suppose $U \subset \mathbb{C}$ is open and star-like and $f: U \rightarrow \mathbb{C}$ is holomorphic. Then f has a primitive.

Cauchy–Goursat (the integral around triangles is zero if f is holomorphic) implies

Corollary

Suppose $U \subset \mathbb{C}$ is open and star-like and $f: U \rightarrow \mathbb{C}$ is holomorphic. Then f has a primitive.

Theorem (Cauchy's theorem for star-like domains)

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is holomorphic, and Γ is a cycle in U . Then

$$\int_{\Gamma} f(z) dz = 0.$$

Cauchy–Goursat (the integral around triangles is zero if f is holomorphic) implies

Corollary

Suppose $U \subset \mathbb{C}$ is open and star-like and $f: U \rightarrow \mathbb{C}$ is holomorphic. Then f has a primitive.

Theorem (Cauchy's theorem for star-like domains)

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is holomorphic, and Γ is a cycle in U . Then

$$\int_{\Gamma} f(z) dz = 0.$$

Proof: The Corollary above says there is a primitive $F: U \rightarrow \mathbb{C}$.

Cauchy–Goursat (the integral around triangles is zero if f is holomorphic) implies

Corollary

Suppose $U \subset \mathbb{C}$ is open and star-like and $f: U \rightarrow \mathbb{C}$ is holomorphic. Then f has a primitive.

Theorem (Cauchy's theorem for star-like domains)

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is holomorphic, and Γ is a cycle in U . Then

$$\int_{\Gamma} f(z) dz = 0.$$

Proof: The Corollary above says there is a primitive $F: U \rightarrow \mathbb{C}$.

By Cauchy's theorem for derivatives, the integral is zero.



Cauchy–Goursat (the integral around triangles is zero if f is holomorphic) implies

Corollary

Suppose $U \subset \mathbb{C}$ is open and star-like and $f: U \rightarrow \mathbb{C}$ is holomorphic. Then f has a primitive.

Theorem (Cauchy's theorem for star-like domains)

Suppose $U \subset \mathbb{C}$ is open and star-like, $f: U \rightarrow \mathbb{C}$ is holomorphic, and Γ is a cycle in U . Then

$$\int_{\Gamma} f(z) dz = 0.$$

Proof: The Corollary above says there is a primitive $F: U \rightarrow \mathbb{C}$.

By Cauchy's theorem for derivatives, the integral is zero. □

Remark: \mathbb{C} -valued function gives a vector-field on \mathbb{R}^2 .

The corollary is a special case of a theorem from vector calculus:

In a star-like domain $U \subset \mathbb{R}^2$, if a C^1 vector field $(u, v): U \rightarrow \mathbb{R}^2$ satisfies $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ (irrotational), then there exists a real-valued $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\nabla f = (u, v)$ (conservative vector field).