Cultivating Complex Analysis: Homology versions of Cauchy (4.2)

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Theorem (Cauchy integral formula (homology version))

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and Γ is a cycle in U homologous to zero in U. Then for $z \in U \setminus \Gamma$,

$$n(\Gamma; z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

Proof: Define $g\colon U \times U \to \mathbb{C}$ by

$$g(\zeta, z) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{if } \zeta \neq z, \\ f'(\zeta) & \text{if } \zeta = z. \end{cases}$$

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So $h\colon \mathbb{C} \to \mathbb{C}$ is well-defined.

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Let M be such that $|f(\zeta)| \le M$ for $\zeta \in \Gamma$ and ℓ be the length of Γ.

$$|h(z)| = \left| \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| \le \int_\Gamma \left| \frac{f(\zeta)}{\zeta - z} \right| |d\zeta| \le \frac{M\ell}{d(z, \Gamma)} \qquad (d(z, \Gamma) \text{ is the distance of } z \text{ and } \Gamma).$$

$$z \to \infty \quad \Rightarrow \quad d(z, \Gamma) \to \infty \quad \Rightarrow \quad h(z) \to 0$$

So h is bounded, by Liouville h is constant, and the constant is zero.

Suppose $z \in U \setminus \Gamma$. Then

$$0 = h(z) = \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\,(2\pi i)\, n(\Gamma; z). \qquad \square$$

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Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and Γ is a cycle in U homologous to zero in U. Then

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Remark: Cauchy integral formula and Cauchy's theorem are equivalent logically (if you prove one the other follows).

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Remark: Being “homologous” is an equivalence relation and the set of equivalence classes of cycles is an abelian group (under the cycle addition, exercise). This group is called the first *homology group* of U.