

# Cultivating Complex Analysis: Line integrals (3.1 part 2)

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Suppose  $\gamma: [a, b] \rightarrow \mathbb{C}$  and  $\alpha: [c, d] \rightarrow \mathbb{C}$  are piecewise- $C^1$  paths such that  $\gamma([a, b]) = \alpha([c, d])$ . Suppose either

- (i)  $\gamma$  and  $\alpha$  are injective, or
- (ii)  $\gamma|_{(a,b]}$  and  $\alpha|_{(c,d]}$  are injective and  $\gamma(a) = \alpha(c) = \gamma(b) = \alpha(d)$  (simple closed paths).

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Then there exists a strictly monotone continuous  $h: [c, d] \rightarrow [a, b]$  such that  $\gamma(h(t)) = \alpha(t)$  for all  $t \in [c, d]$ .

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- (i) If  $h$  is increasing, then for every  $f$  continuous on the path,  $\int_{\gamma} f(z) dz = \int_{\alpha} f(z) dz$ .
- (ii) If  $h$  is decreasing, then for every  $f$  continuous on the path,  $\int_{\gamma} f(z) dz = - \int_{\alpha} f(z) dz$ .



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$$\int_{\partial\Delta_r(p)} f(z) dz = \int_{\gamma} f(z) dz.$$

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For example,

$$\int_{\gamma} |dz| = \int_a^b |\gamma'(t)| dt \quad \left( = \int_{\gamma} ds \right)$$

is the length of  $\gamma$ .

### Proposition (Triangle inequality for line integrals)

Suppose  $\gamma: [a, b] \rightarrow \mathbb{C}$  is a piecewise- $C^1$  path and  $f$  is a continuous function on  $\gamma$ . Then

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**Remark:** Uniform convergence of the functions,  $f_n \rightarrow f$ , passes under the integral:

$$\lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} f(z) dz \qquad \lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) |dz| = \int_{\gamma} f(z) |dz|$$



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For this you would also need  $\gamma'_n$  to also converge (uniformly).