

# Cultivating Complex Analysis: Rouché's theorem (5.4.2)

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**Example:** 1 and  $\frac{z-\epsilon}{z+\epsilon}$  are close on  $\partial\mathbb{D}$ .

## Theorem (Rouché)

*Suppose  $U \subset \mathbb{C}$  is open,  $\Gamma$  is a cycle in  $U$  homologous to zero in  $U$ , and  $n(\Gamma; z)$  is either 0 or 1 for all  $z \notin \Gamma$ .*

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Let  $V = \{z \in U \setminus \Gamma : n(\Gamma; z) = 1\}$ . Let  $N_f, N_g$  be the number of zeros in  $V$  and  $P_f, P_g$  the number of poles in  $V$  (both counting multiplicity) of  $f$  and  $g$  respectively.



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### Corollary (Rouché)

Let  $U, \Gamma$  and  $V$  be as in the theorem. Suppose  $f: U \rightarrow \mathbb{C}$  and  $g: U \rightarrow \mathbb{C}$  are holomorphic such that  $|f(z) - g(z)| < |f(z)| + |g(z)|$  for all  $z \in \Gamma$ . Then  $f$  and  $g$  have the same number of zeros (counting multiplicity) in  $V$ .

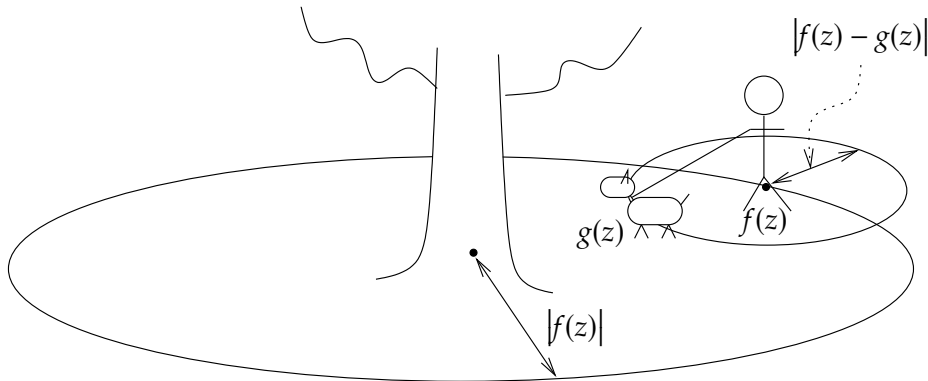
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It has a nice geometric interpretation:



**Proof:**

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By Cauchy's theorem for derivatives, together with the argument principle:

$$0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi'(z)}{\varphi(z)} dz$$

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On  $\partial\Delta_{1+\epsilon}(0)$ ,

$$|P(z) - z^n| = 1 < |z^n|.$$

By Rouché,  $P(z)$  and  $z^n$  have the same number of zeros in  $\Delta_{1+\epsilon}(0)$ .

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(Actually the largest zero of  $P$  has modulus less than 10).