

# Cultivating Complex Analysis: Linear fractional transformations (1.4 part 1)

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So takes the Riemann sphere to Riemann sphere.

It is an easy exercise that  $f: \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$  is bijective and continuous.

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If  $c = 0$ , assume  $d = 1$  and  $f(z) = az + b$ :

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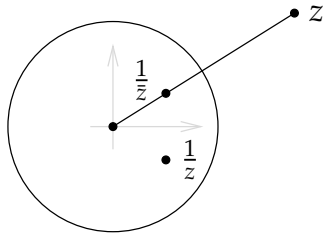
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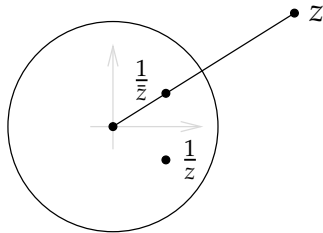
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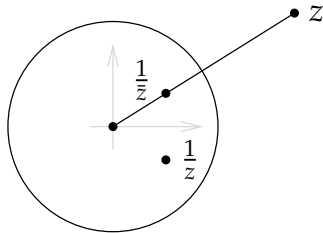
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**Remark:** A straight line is really a very large circle through  $\infty$ .

