

Cultivating Complex Analysis:
The argument (1.2.3)
Mapping properties of the exponential (1.2.4)

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Note that $\arg 0$ is undefined.

To try to solve the multivaluedness, we define the *principal branch of arg*:

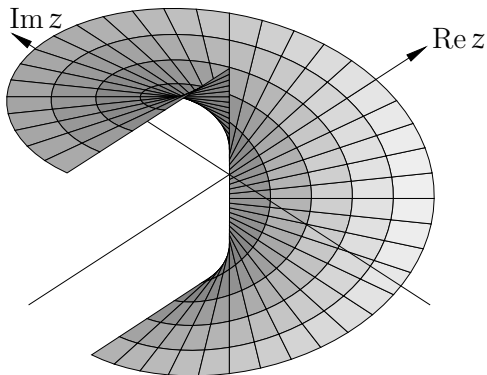
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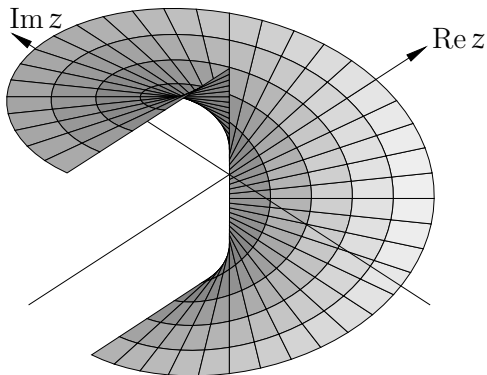
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Remark: Some authors make principal
branch take values in $[0, 2\pi)$.



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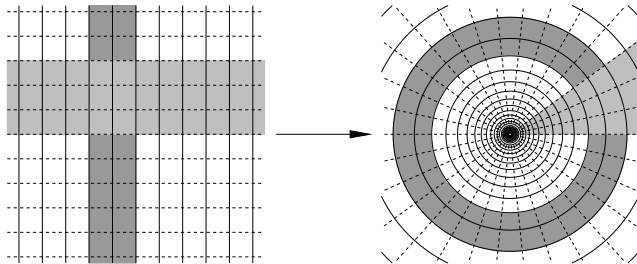
The complex exponential is not one-to-one, it is infinitely-many-to-one. For any integer k ,

$$e^{z+ik2\pi} = e^z e^{ik2\pi} = e^z.$$

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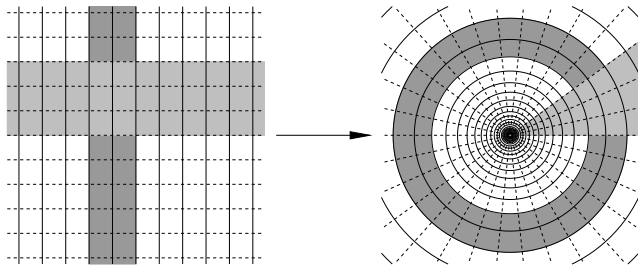
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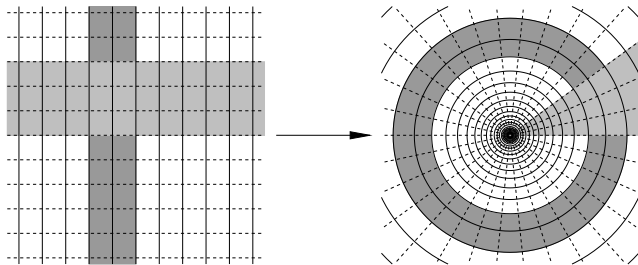


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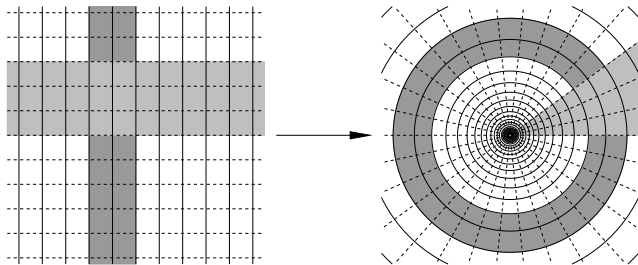
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Remark: e^z takes the set given by $2k\pi < \operatorname{Im} z \leq 2(k+1)\pi$ in a one-to-one way onto $\mathbb{C} \setminus \{0\}$.

