Cultivating Complex Analysis: Harmonic functions Real and imaginary parts of holomorphic functions (7.1.1)

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Definition

Let $U \subset \mathbb{C}$ be open. A twice continuously differentiable $f : U \to \mathbb{R}$ is *harmonic* if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{on } U.$$

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Convenient to write using Wirtinger operators:

$$4\frac{\partial^2}{\partial \bar{z}\partial z} f = 4\left[\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right]\left[\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right] f$$

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$$f \text{ is harmonic} \quad \Leftrightarrow \quad \frac{\partial f}{\partial z} \text{ is holomorphic.}$$

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Similarly f is the imaginary part of some holomorphic function.

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We proved:

Proposition

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{R}$ a function.

(i) *The function f is harmonic if and only if for every $p \in U$ there exists an open neighborhood V of p and a holomorphic $\varphi : V \to \mathbb{C}$ such that $f = \operatorname{Re}\varphi$ on V.*

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(ii) *The function f is harmonic if and only if for every $p \in U$, there exists a power series expansion*

$$f(z) = c_0 + \sum_{n=1}^{\infty} c_n(z-p)^n + \bar{c}_n(\overline{z-p})^n$$

converging uniformly absolutely on every closed disc $\overline{\Delta_r(p)} \subset U$.

A quick corollary:

Proposition

If $U \subset \mathbb{C}$ is open and $f : U \to \mathbb{R}$ is harmonic, then f is infinitely (real) differentiable.

Proof: Holomorphic functions are infinitely differentiable.

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Proof: $\dfrac{(f + ig_1) - (f + ig_2)}{i} = g_1 - g_2$ is holomorphic, real-valued $\implies$ constant.

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Proposition

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic and never zero. Then

$$z \mapsto \log|f(z)|$$

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Proof: Exercise.

Exercise: Suppose $U \subset \mathbb{C}$ is a simply connected domain and $f\colon U \to \mathbb{R}$ harmonic. Prove there exists a holomorphic $\varphi\colon U \to \mathbb{C}$ such that $f(z) = \log|\varphi(z)|$.

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Exercise: Let $U, V \subset \mathbb{C}$ be open sets and $f : U \to V$ be holomorphic. Prove:
a) If $g : V \to \mathbb{R}$ is harmonic, then $g \circ f$ is harmonic.
b) If f is a biholomorphism, then $g : V \to \mathbb{R}$ is harmonic if and only if $g \circ f$ is harmonic.

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then $\log|f(z)+M|$ or $\operatorname{Re} f(z)+M$ is nonnegative for large enough M.

Conversely, if $\log|f(z)| \geq 0$, then $\dfrac{1}{f(z)}$ is bounded,

and if $\operatorname{Re} f(z) \geq 0$, then $\dfrac{f(z)-1}{f(z)+1}$ is bounded.

Remark: Writing $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \bar{z} \partial z}$ so that we can integrate twice may sound familiar.

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The wave operator is (using (x, t) for tradition's sake):

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The wave operator is (using (x, t) for tradition's sake):

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left[\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right].$$

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A solution f to the wave equation is

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Two waves travelling in opposite directions.