

Cultivating Complex Analysis: Residue theorem, applications (5.3 part 2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Recall that the residue theorem says that given a finite $S \subset U$, Γ a cycle in $U \setminus S$ homologous to zero in U , and $f: U \setminus S \rightarrow \mathbb{C}$ holomorphic,

$$\frac{1}{2\pi i} \int_{\Gamma} f(z) dz = \sum_{p \in S} n(\Gamma; p) \operatorname{Res}(f; p),$$

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c_{-1} has a formula in terms of an integral. So how is the residue theorem useful if it takes an integral and replaces it with integrals?

Because we have lots of tricks to compute c_{-1} . We'll go over a few.

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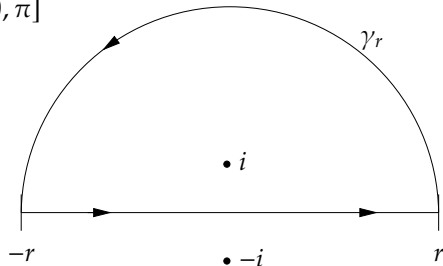
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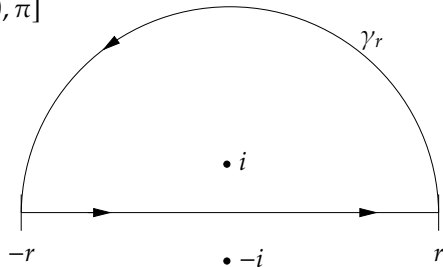
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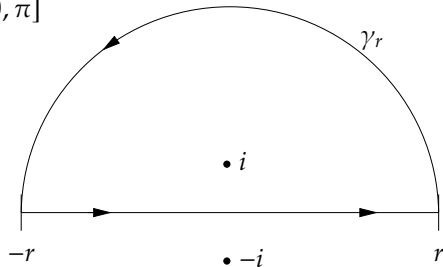
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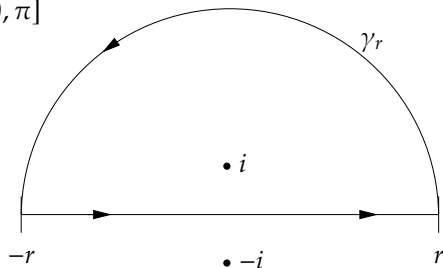
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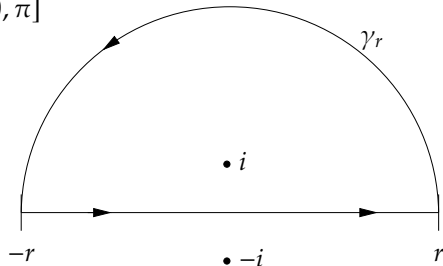
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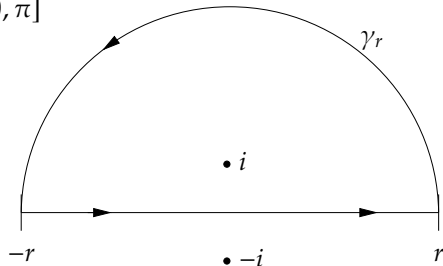
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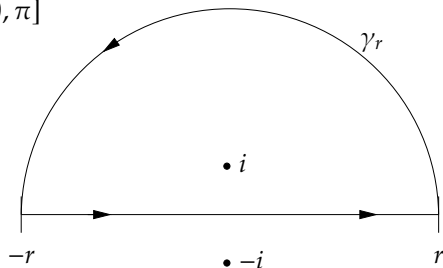
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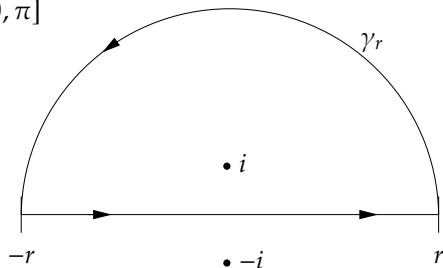
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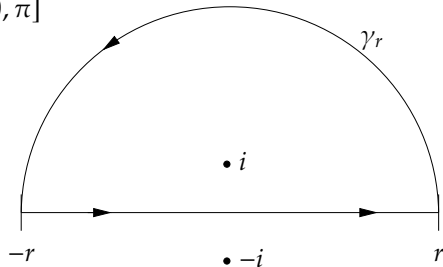
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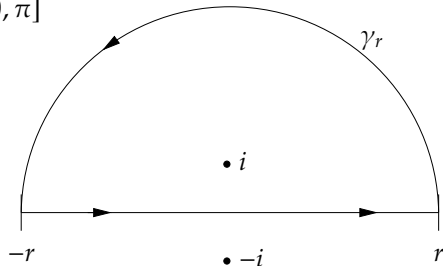
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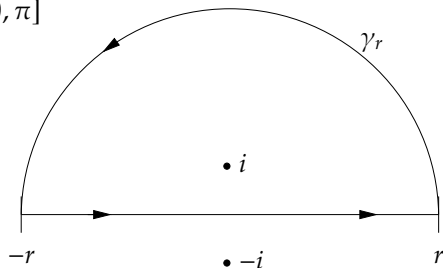
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Minor technicality: Why the symmetric limit is sufficient?

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$\frac{1}{z^2 + 2cz + 1}$ has two poles: $-c \pm \sqrt{c^2 - 1}$, one inside and one outside the unit circle.

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Example: Suppose $c > 1$.

$$\int_0^{2\pi} \frac{1}{c + \cos \theta} d\theta = \int_{\partial \mathbb{D}} \frac{1}{c + \frac{z+1/z}{2}} \frac{1}{iz} dz = -2i \int_{\partial \mathbb{D}} \frac{1}{z^2 + 2cz + 1} dz.$$

$\frac{1}{z^2 + 2cz + 1}$ has two poles: $-c \pm \sqrt{c^2 - 1}$, one inside and one outside the unit circle.

$$\int_0^{2\pi} \frac{1}{c + \cos \theta} d\theta = (-2i)(2\pi i) \operatorname{Res} \left(\frac{1}{z^2 + 2cz + 1}; -c + \sqrt{c^2 - 1} \right)$$

Sometimes we just recognize a path integral.

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A common computation via the residue theorem are inverse Laplace transforms. *Mellin's inversion formula* says that given a transform $F(s)$, the original $f(t)$ is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{c-ir}^{c+ir} e^{st} F(s) ds$$

for some $c \in \mathbb{R}$ (usually $c \geq 0$) is the inverse.

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As an exercise, try your hand at computing a few. Say

$$\mathcal{L}^{-1} \left[\frac{1}{s(s+1)} \right], \quad \text{or} \quad \mathcal{L}^{-1} \left[\frac{s^2}{(s+2)^2(s^2+1)} \right].$$

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Hint: Pick the correct vertical line (pick a c) and an arc that goes around all the poles.