Cultivating Complex Analysis:
Harmonic functions
Identity and the maximum principle (7.1.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f\colon U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h\colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h\colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Rightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Longrightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.
$\Longrightarrow$ h is constant on an open subset of $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h : \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Rightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Rightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on $\Delta_r(p)$.
$\Rightarrow$ f is constant on $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h\colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Rightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on $\Delta_r(p)$.
$\Rightarrow$ f is constant on $\Delta_r(p)$.
$\Rightarrow$ f is zero on $\Delta_r(p)$.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f\colon U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h\colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Rightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on $\Delta_r(p)$.
$\Rightarrow$ f is constant on $\Delta_r(p)$.
$\Rightarrow$ f is zero on $\Delta_r(p)$.
$\Rightarrow$ Z is open.

A zero set of a harmonic f can have limit points: E.g., $\operatorname{Re} z$. But no open sets.

Theorem (Identity)

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{R}$ a harmonic function. Suppose $V \subset U$ is a nonempty open subset and $f = 0$ on V. Then $f \equiv 0$.

Proof: Let $Z_f = f^{-1}(0)$. Let Z be the closure of the interior of Z_f (subspace topology on U).

Suppose Z is nonempty and $p \in Z$.
If $\Delta_r(p) \subset U$, f is zero on some open subset of $\Delta_r(p)$.

$\exists$ a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$ on $\Delta_r(p)$.
$\Rightarrow$ h is holomorphic and purely imaginary on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on an open subset of $\Delta_r(p)$.
$\Rightarrow$ h is constant on $\Delta_r(p)$.
$\Rightarrow$ f is constant on $\Delta_r(p)$.
$\Rightarrow$ f is zero on $\Delta_r(p)$.
$\Rightarrow$ Z is open.

Z is also closed and U is connected $\Rightarrow$ $Z = U$. $\square$

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (*or a local minimum*) *in U, then f is constant.*

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f: U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h\colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

h takes $\Delta_r(p)$ to $X = \left\{ w \in \mathbb{C} : \operatorname{Re} w \leq f(p) \right\}$.

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

h takes $\Delta_r(p)$ to $X = \left\{w \in \mathbb{C} : \operatorname{Re} w \leq f(p)\right\}$.

$h(p)$ is on the boundary of X (as $\operatorname{Re} h(p) = f(p)$).

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h\colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

h takes $\Delta_r(p)$ to $X = \{w \in \mathbb{C} : \operatorname{Re} w \leq f(p)\}$.

$h(p)$ is on the boundary of X (as $\operatorname{Re} h(p) = f(p)$).

$\Rightarrow \quad h(\Delta_r(p))$ is not open

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f: U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h: \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

h takes $\Delta_r(p)$ to $X = \left\{w \in \mathbb{C} : \operatorname{Re} w \leq f(p)\right\}$.

$h(p)$ is on the boundary of X (as $\operatorname{Re} h(p) = f(p)$).

$\Rightarrow \quad h\left(\Delta_r(p)\right)$ is not open $\quad \Rightarrow \quad h$ is constant (open mapping theorem)

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

h takes $\Delta_r(p)$ to $X = \{w \in \mathbb{C} : \operatorname{Re} w \leq f(p)\}$.

$h(p)$ is on the boundary of X (as $\operatorname{Re} h(p) = f(p)$).

$\Rightarrow \quad h(\Delta_r(p))$ is not open $\quad \Rightarrow \quad h$ is constant (open mapping theorem)

$\Rightarrow \quad f$ is constant on $\Delta_r(p)$

Theorem (Maximum principle)

Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R}$ is harmonic. If f attains a local maximum (or a local minimum) in U, then f is constant.

Proof: WLOG assume f has a local maximum at $p \in U$ (otherwise consider $-f$).

It is the global maximum on some $\Delta_r(p) \subset U$.

There exists a holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $f = \operatorname{Re} h$.

h takes $\Delta_r(p)$ to $X = \{w \in \mathbb{C} : \operatorname{Re} w \leq f(p)\}$.

$h(p)$ is on the boundary of X (as $\operatorname{Re} h(p) = f(p)$).

$\Rightarrow \quad h(\Delta_r(p))$ is not open $\quad \Rightarrow \quad h$ is constant (open mapping theorem)

$\Rightarrow \quad f$ is constant on $\Delta_r(p) \quad \Rightarrow \quad f$ is constant on U by identity. $\square$

Exercise: Prove that the maximum principle for harmonic functions implies the maximum modulus principle for holomorphic functions. Hint: Consider $\log|f(z)|$.

Exercise: Prove that the maximum principle for harmonic functions implies the maximum modulus principle for holomorphic functions. Hint: Consider $\log|f(z)|$.

Exercise: Prove the second version of the maximum principle: If $U \subset \mathbb{C}$ is a bounded domain and $f \colon \overline{U} \to \mathbb{R}$ is continuous and harmonic on U, then f achieves both its maximum and its minimum on the boundary ∂U.

Exercise: Prove that the maximum principle for harmonic functions implies the maximum modulus principle for holomorphic functions. Hint: Consider $\log|f(z)|$.

Exercise: Prove the second version of the maximum principle: If $U \subset \mathbb{C}$ is a bounded domain and $f \colon \overline{U} \to \mathbb{R}$ is continuous and harmonic on U, then f achieves both its maximum and its minimum on the boundary ∂U.

Exercise: Suppose $U \subset \mathbb{C}$ is a domain and $f \colon U \to \mathbb{R}$ is harmonic. Prove that $f(U)$ is an open interval or a single point.