Cultivating Complex Analysis: The argument principle (5.4.1)

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Terminology: *zeros/poles counted with multiplicity*: $f(z) = z^2(z-1)^3$ has the zeros $z_1, z_2, z_3, z_4, z_5 = 0, 0, 1, 1, 1$.

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Theorem (Argument principle)

Suppose $U \subset \mathbb{C}$ is open and Γ is a cycle in U homologous to zero in U. Suppose $f : U \to \mathbb{C}_\infty$ is a meromorphic function with no zeros or poles on Γ. Let $z_1, \ldots, z_n$ denote the zeros of f counted with multiplicity, and let $p_1, \ldots, p_\ell$ denote the poles of f counted with multiplicity. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{n} n(\Gamma; z_k) \quad - \quad \sum_{k=1}^{\ell} n(\Gamma; p_k).$$

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Furthermore, if $h\colon U \to \mathbb{C}$ is holomorphic, then

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# of poles/zero normally countable, but can assume finite above.

Suppose $n(\Gamma; z) = 1$ or 0 for all $z \in U$.

The "inside of Γ" are the points where $n(\Gamma; z) = 1$.

If there are n zeros and ℓ poles (counting multiplicity) inside Γ, then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz = n - \ell.$$

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Another interpretation:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{\zeta}\,d\zeta = n(f\circ\gamma; 0).$$

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$$\frac{1}{2\pi i}\int_{\Gamma} h(z)\frac{f'(z)}{f(z)}\,dz = \sum_{p\in S} n(\Gamma;p)\operatorname{Res}\left(h\frac{f'}{f};p\right).$$

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Write $f(z) = z^m F(z)$ where $F(0) \neq 0$ and $h(z) = h(0) + zH(z)$.

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Everything except $m\,h(0)\frac{1}{z}$ is holomorphic. So

$$\operatorname{Res}\left(h\frac{f'}{f};0\right) = m\,h(0) \qquad \square$$

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If there is one simple zero z_0 of f within Γ, then

$$\frac{1}{2\pi i}\int_\Gamma z\frac{f'(z)}{f(z)}\,dz = z_0.$$