

# Cultivating Complex Analysis: Simply connected domains (4.3 part 2)

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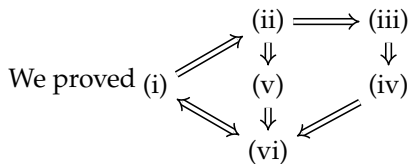
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**Remark:** It is important to use  $\mathbb{C}_\infty$  and not  $\mathbb{C}$ : If  $U = \mathbb{C} \setminus \{0\}$  (not simply connected), then  $\mathbb{C} \setminus U = \{0\}$  is connected, but  $\mathbb{C}_\infty \setminus U = \{0, \infty\}$  is not connected.

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2) If  $U_1 \cap U_2$  is nonempty and connected, then  $U_1 \cup U_2$  is simply connected.