Cultivating Complex Analysis: Singularities and the Laurent series (5.2.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

$\frac{1}{z}$ has an isolated singularity at $z = 0$.

$\frac{1}{z}$ has an isolated singularity at $z = 0$.

It is a pole of order 1.

$\frac{1}{z}$ has an isolated singularity at $z = 0$.

It is a pole of order 1.

The Laurent series at $z = 0$ is just ${}^{1}/{}_{z}$,
and all coefficients of order less than -1 are zero.

OK, we could make this more complicated:

$$\frac{1}{z} + \frac{1}{1-z} = \sum_{n=-1}^{\infty} z^n$$

OK, we could make this more complicated:

$$\frac{1}{z} + \frac{1}{1-z} = \sum_{n=-1}^{\infty} z^n$$

Again, pole of order 1 at $z = 0$,
and all coefficients in the series of order less than -1 are zero.

Or even more complicated:

$$\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z} = \sum_{n=-3}^{\infty} z^n$$

Or even more complicated:

$$\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z} = \sum_{n=-3}^{\infty} z^n$$

A pole of order 3,
and all coefficients in the series of order less than -3 are zero.

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

has an essential singularity at $z = 0$,
and has nonzero coefficients of all negative orders.

It is not difficult to prove the general statement:

Proposition

Suppose $f : \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ *is holomorphic, and*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

is the corresponding Laurent series.

It is not difficult to prove the general statement:

Proposition

Suppose $f : \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ *is holomorphic, and*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

is the corresponding Laurent series. The singularity at p *is*

(i) *removable if and only if* $c_n = 0$ *for all* $n < 0$*,*

It is not difficult to prove the general statement:

Proposition

Suppose $f : \Delta_r(p) \setminus \{p\} \longrightarrow \mathbb{C}$ *is holomorphic, and*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

is the corresponding Laurent series. The singularity at p *is*

(i) *removable if and only if* $c_n = 0$ *for all* $n < 0$*,*

(ii) *a pole of order* $k \in \mathbb{N}$ *if and only if* $c_n = 0$ *for all* $n < -k$ *and* $c_{-k} \neq 0$*,*

It is not difficult to prove the general statement:

Proposition

Suppose $f : \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ *is holomorphic, and*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

is the corresponding Laurent series. The singularity at p *is*

(i) *removable if and only if* $c_n = 0$ *for all* $n < 0$,

(ii) *a pole of order* $k \in \mathbb{N}$ *if and only if* $c_n = 0$ *for all* $n < -k$ *and* $c_{-k} \neq 0$,

(iii) *essential if and only if* $c_n \neq 0$ *for infinitely many negative* n.

It is not difficult to prove the general statement:

Proposition

Suppose $f\colon \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ is holomorphic, and

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

is the corresponding Laurent series. The singularity at p is

(i) *removable if and only if $c_n = 0$ for all $n < 0$,*

(ii) *a pole of order $k \in \mathbb{N}$ if and only if $c_n = 0$ for all $n < -k$ and $c_{-k} \neq 0$,*

(iii) *essential if and only if $c_n \neq 0$ for infinitely many negative n.*

The proof is left as an exercise.

It is not difficult to prove the general statement:

Proposition

Suppose $f : \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ *is holomorphic, and*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-p)^n$$

is the corresponding Laurent series. The singularity at p *is*

(i) *removable if and only if* $c_n = 0$ *for all* $n < 0$,

(ii) *a pole of order* $k \in \mathbb{N}$ *if and only if* $c_n = 0$ *for all* $n < -k$ *and* $c_{-k} \neq 0$,

(iii) *essential if and only if* $c_n \neq 0$ *for infinitely many negative* n.

The proof is left as an exercise.
Hint: Laurent series is unique, and for a removable singularity equals the power series.

Definition

At an isolated singularity, the negative part of the Laurent series

$$\sum_{n=-\infty}^{-1} c_n (z-p)^n$$

is called the *principal part*.

Definition

At an isolated singularity, the negative part of the Laurent series

$$\sum_{n=-\infty}^{-1} c_n (z-p)^n$$

is called the *principal part*.

Observation: If $P(z)$ is the principal part of $f(z)$ at p, then $f(z) - P(z)$ has a removable singularity at p.

The example $e^{1/z}$ motivates the following concept.

The example $e^{1/z}$ motivates the following concept.

Given an entire $f : \mathbb{C} \to \mathbb{C}$, we talk about its *singularity at infinity.*

The example $e^{1/z}$ motivates the following concept.

Given an entire $f : \mathbb{C} \to \mathbb{C}$, we talk about its *singularity at infinity.*

$\mathbb{C} \subset \mathbb{C}_\infty$ and ${}^1\!/\!{}_z$ is a self mapping of $\mathbb{C}_\infty$ that swaps ∞ and 0.

The example $e^{1/z}$ motivates the following concept.

Given an entire $f : \mathbb{C} \to \mathbb{C}$, we talk about its *singularity at infinity.*

$\mathbb{C} \subset \mathbb{C}_\infty$ and $1/z$ is a self mapping of $\mathbb{C}_\infty$ that swaps ∞ and 0.

$z \mapsto f(1/z)$ has an isolated singularity at 0, and that's the "singularity of f at ∞."

The example $e^{1/z}$ motivates the following concept.

Given an entire $f : \mathbb{C} \to \mathbb{C}$, we talk about its *singularity at infinity.*

$\mathbb{C} \subset \mathbb{C}_\infty$ and ${}^1\!/\!_z$ is a self mapping of $\mathbb{C}_\infty$ that swaps ∞ and 0.

$z \mapsto f({}^1\!/\!_z)$ has an isolated singularity at 0, and that's the "singularity of f at ∞."

e^z has an essential singularity at infinity,
because $e^{1/z}$ has an essential singularity at 0.

Exercise: Prove that if f has a pole at the origin and g has an essential singularity at the origin, then $f + g$ has an essential singularity at the origin.

Exercise: Prove that if f has a pole at the origin and g has an essential singularity at the origin, then $f + g$ has an essential singularity at the origin.

Exercise: If f has a pole at p, then $e^{f(z)}$ has an essential singularity at p.
Hint: First do it for a simple pole.

Exercise: Prove that if f has a pole at the origin and g has an essential singularity at the origin, then $f+g$ has an essential singularity at the origin.

Exercise: If f has a pole at p, then $e^{f(z)}$ has an essential singularity at p.
Hint: First do it for a simple pole.

Exercise: Show that an entire holomorphic $f\colon \mathbb{C} \to \mathbb{C}$ has a pole at infinity if and only if it is a nonconstant polynomial. The order of the pole is the degree of the polynomial.

Exercise: Prove that if f has a pole at the origin and g has an essential singularity at the origin, then $f + g$ has an essential singularity at the origin.

Exercise: If f has a pole at p, then $e^{f(z)}$ has an essential singularity at p.
Hint: First do it for a simple pole.

Exercise: Show that an entire holomorphic $f : \mathbb{C} \to \mathbb{C}$ has a pole at infinity if and only if it is a nonconstant polynomial. The order of the pole is the degree of the polynomial.

Exercise: Show that if $f : \mathbb{C} \to \mathbb{C}$ is an automorphism, then $f(z) = az + b$ for some constants $a \neq 0$ and b. Hint: Show that f has a simple pole at infinity.