

Cultivating Complex Analysis: Power series (2.3 part 3)

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Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z - p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z - p|^n$ converges uniformly for $z \in K$.

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So a uniformly absolutely convergent series converges uniformly.

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The result follows. □

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If $\sum M_n < \infty$, then $\sum f_n(x)$ converges uniformly absolutely on X .

Exercise: Suppose $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have a radius of convergence at least $r > 0$. Show that $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ has a radius of convergence at least r and converges to the sum of the two series.