15. Constant coefficient second order linear ODEs (part 2)
(Notes on Diffy Qs, 2.2)

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The textbook: `https://www.jirka.org/diffyqs/`

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If $r = r_1 = r_2$, then e^{rx} and xe^{rx} are linearly independent solutions.

But, what if the characteristic equation has no real roots? E.g., $r^2 + 1 = 0$.

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Remark: Most trig identities follow from $e^{z+w} = e^z e^w$ for complex numbers.

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Theorem

If the characteristic equation has the roots $\alpha \pm i\beta$ (i.e., $b^2 - 4ac < 0$), then the general solution is

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Complex numbers are useful also for the Cauchy–Euler equations.

Exercise: Suppose $(b-a)^2 - 4ac < 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$. Hint: Try $y = x^r$ and note $x^r = e^{r \ln x}$.

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Try it with something simple like $x^2y'' + y = 0$.