

7. Linear equations and the integrating factor (Notes on Diffy Qs, 1.4)

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The textbook: <https://www.jirka.org/diffyqs/>

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We have an explicit formula for the solution:

$$y(x) = e^{-\int_{x_0}^x p(s) ds} \left(\int_{x_0}^x e^{\int_{x_0}^t p(s) ds} f(t) dt + y_0 \right) \quad (*)$$

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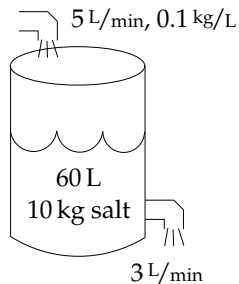
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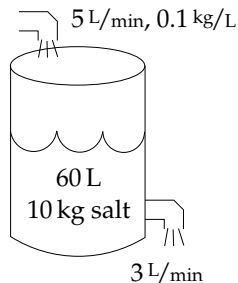
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Remark 4: There is a stronger version of Picard's theorem for linear equations: Formula (*) says that if $f(x)$ and $p(x)$ are continuous on an interval (a, b) , the solution also exists and is continuous on (a, b) . Nothing like that weird nonlinear $y' = y^2$ we saw before.

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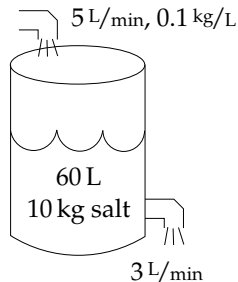


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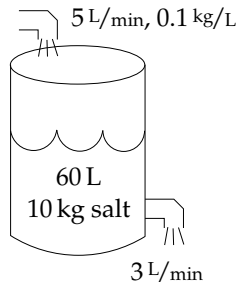
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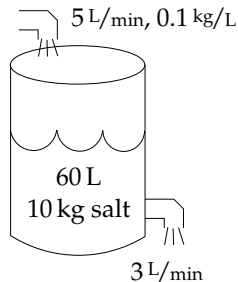


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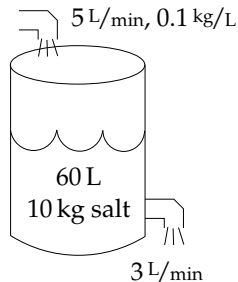
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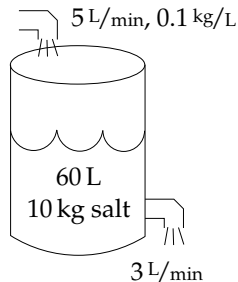
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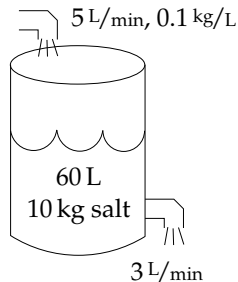
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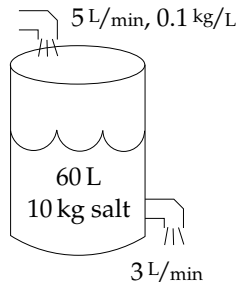
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Compute

$$(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x = 0.5(60+2t)^{3/2}$$

For $\frac{dx}{dt} + \frac{3}{60+2t}x = 0.5$, the integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t}dt\right) = \exp\left(\frac{3}{2}\ln(60+2t)\right) = (60+2t)^{3/2}.$$

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$$\begin{aligned}(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2} \\ \frac{d}{dt}\left[(60+2t)^{3/2}x\right] &= 0.5(60+2t)^{3/2}\end{aligned}$$

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$$x = \frac{60+2t}{10} + C(60+2t)^{-3/2}$$

At $t = 0, x = 10$: $10 = x(0)$

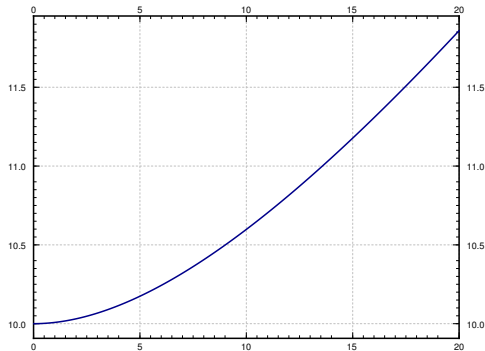
At $t = 0, x = 10$: $10 = x(0) = \frac{60}{10} + C(60)^{-3/2}$

$$\text{At } t = 0, x = 10: \quad 10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2} \quad \Rightarrow$$

At $t = 0, x = 10$: $10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2} \Rightarrow C = 4(60^{3/2}) \approx 1859.03.$

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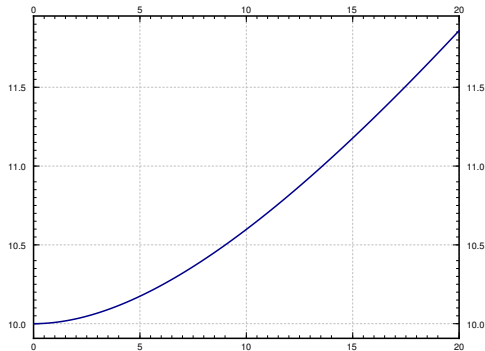
Here's a graph:



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So when is the tank full?

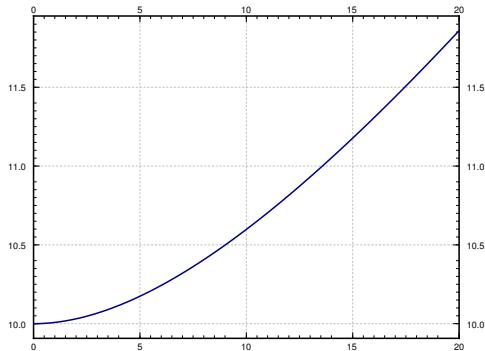


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The tank is full when $60 + 2t = 100$, or $t = 20$.



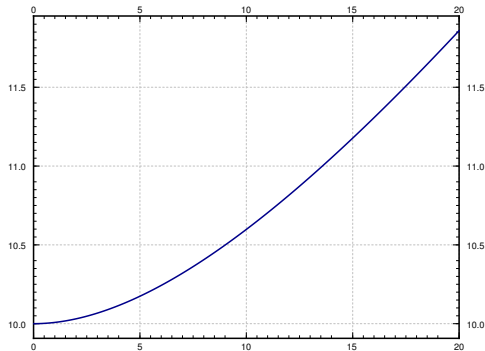
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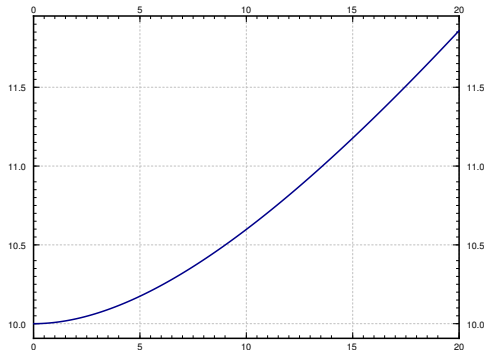
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What is x when tank is full?

$$x(20) = \frac{60 + 40}{10} + C(60 + 40)^{-3/2}$$



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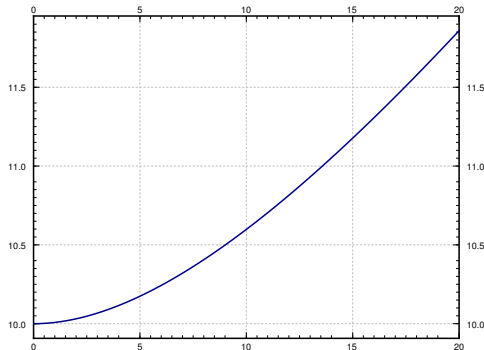
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$$\begin{aligned} x(20) &= \frac{60 + 40}{10} + C(60 + 40)^{-3/2} \\ &\approx 10 + 1859.03(100)^{-3/2} \approx 11.86 \end{aligned}$$



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The concentration when the tank is full is approx. $11.86/100 = 0.1186$ kg/liter.

(we started with $1/6$ or 0.1667 kg/liter.)

