

# 13. Second order linear ODEs (Notes on Diffy Qs, 2.1)

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The textbook: <https://www.jirka.org/diffyqs/>

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**Examples:**

$$y'' + k^2y = 0$$

Two solutions are:  $y_1 = \cos(kx)$ ,  $y_2 = \sin(kx)$ .

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**Exercise:** Verify that

$$\cosh 0 = 1,$$

$$\frac{d}{dt} [\cosh t] = \sinh t,$$

$$\cosh^2 t - \sinh^2 t = 1.$$

$$\sinh 0 = 0,$$

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Similarly,  $y'' - k^2y = 0$  with  $y(0) = b_0$  and  $y'(0) = b_1$  has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

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$\sin$  and  $\cos$  are linearly independent: If  $\sin x = A \cos x$  for some constant  $A$ , then let  $x = 0$  to get  $A = 0$ .

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$\sin$  and  $\cos$  are linearly independent: If  $\sin x = A \cos x$  for some constant  $A$ , then let  $x = 0$  to get  $A = 0$ . But then  $\sin x = 0$  for all  $x$ , that's nonsense.

Two functions  $y_1$  and  $y_2$  are *linearly independent* if one is not a constant multiple of the other.

### Theorem

Let  $p, q$  be continuous functions. Let  $y_1$  and  $y_2$  be two linearly independent solutions to the homogeneous equation  $y'' + p(x)y' + q(x)y = 0$ . Then every other solution is of the form

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$\Rightarrow y = C_1x^2 + C_2\frac{1}{x}$  is the general solution.

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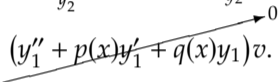
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We can even just write down a formula

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{(y_1(x))^2} dx$$

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**Exercise:** For  $x^2y'' - xy' = 0$ , find two solutions, show that they are linearly independent and find the general solution.

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Equations of the form  $ax^2y'' + bxy' + cy = 0$  are called *Euler's equations* or *Cauchy–Euler equations*.