26. Transforms of derivatives and ODEs, part 2
(Notes on Diffy Qs, 6.2)

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The textbook: https://www.jirka.org/diffyqs/

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For any other input $f(t)$, the output (in s-space) is again simply

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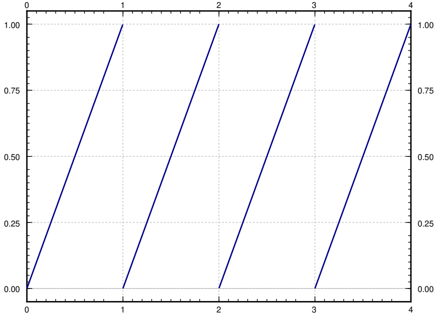

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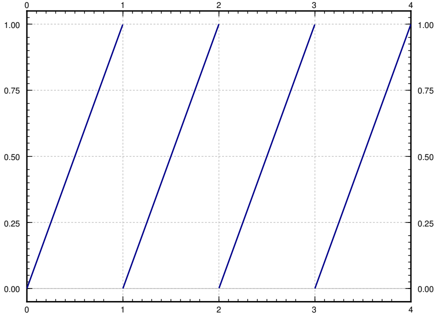

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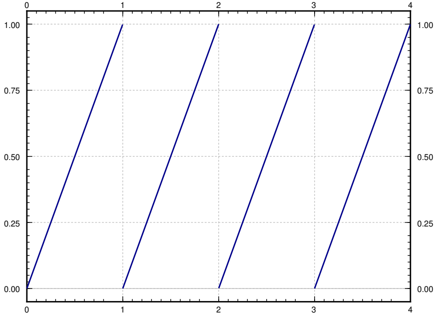

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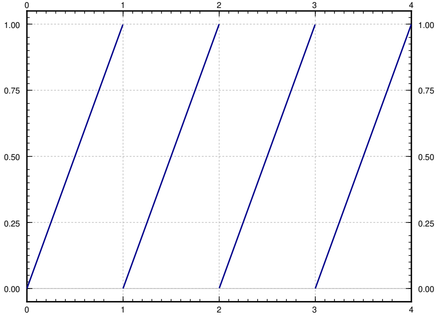

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