13. Second order linear ODEs
(Notes on Diffy Qs, 2.1)

Jiří Lebl

Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

Second order linear differential equation:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Second order linear differential equation:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Or

$$y'' + p(x)y' + q(x)y = f(x).$$

Second order linear differential equation:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Or

$$y'' + p(x)y' + q(x)y = f(x).$$

Linear means that the equation contains no powers nor functions of y, y', and y''.

Second order linear differential equation:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Or

$$y'' + p(x)y' + q(x)y = f(x).$$

Linear means that the equation contains no powers nor functions of y, y', and y''.

If $f(x) = 0$, then the equation is *homogeneous*:

$$y'' + p(x)y' + q(x)y = 0.$$

Second order linear differential equation:

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Or

$$y'' + p(x)y' + q(x)y = f(x).$$

Linear means that the equation contains no powers nor functions of y, y', and y''.

If $f(x) = 0$, then the equation is *homogeneous*:

$$y'' + p(x)y' + q(x)y = 0.$$

Examples:

$y'' + k^2 y = 0$ Two solutions are: $y_1 = \cos(kx), \quad y_2 = \sin(kx).$

$y'' - k^2 y = 0$ Two solutions are: $y_1 = e^{kx}, \quad y_2 = e^{-kx}.$

Theorem (Superposition)

Suppose y_1 *and* y_2 *are two solutions of the homogeneous equation* $y'' + p(x)y' + q(x)y = 0$.

Theorem (Superposition)

Suppose y_1 *and* y_2 *are two solutions of the homogeneous equation* $y'' + p(x)y' + q(x)y = 0$. *Then*

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

also solves the equation for arbitrary constants C_1 *and* C_2.

Theorem (Superposition)

Suppose y_1 and y_2 are two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

also solves the equation for arbitrary constants C_1 and C_2.

We call $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2.

Theorem (Superposition)

Suppose y_1 and y_2 are two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

also solves the equation for arbitrary constants C_1 and C_2.

We call $C_1y_1 + C_2y_2$ a *linear combination* of y_1 and y_2.

Proof: Let $y = C_1y_1 + C_2y_2$.

Theorem (Superposition)

Suppose y_1 and y_2 are two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

also solves the equation for arbitrary constants C_1 and C_2.

We call $C_1y_1 + C_2y_2$ a *linear combination* of y_1 and y_2.

Proof: Let $y = C_1y_1 + C_2y_2$. Then

$$y'' + py' + qy = (C_1y_1 + C_2y_2)'' + p(C_1y_1 + C_2y_2)' + q(C_1y_1 + C_2y_2)$$

Theorem (Superposition)

Suppose y_1 and y_2 are two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

also solves the equation for arbitrary constants C_1 and C_2.

We call $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2.

Proof: Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \end{aligned}$$

Theorem (Superposition)

Suppose y_1 and y_2 are two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

also solves the equation for arbitrary constants C_1 and C_2.

We call $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2.

Proof: Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \\ &= C_1 (y_1'' + p y_1' + q y_1) + C_2 (y_2'' + p y_2' + q y_2) \end{aligned}$$

Theorem (Superposition)

Suppose y_1 and y_2 are two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

also solves the equation for arbitrary constants C_1 and C_2.

We call $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2.

Proof: Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \\ &= C_1(y_1'' + p y_1' + q y_1) + C_2(y_2'' + p y_2' + q y_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \end{aligned}$$

$\square$

An *operator* is a thing that eats functions and spits out functions.

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

The operator L is *linear* if

$$L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 \quad \text{(almost like multiplying by } L\text{)}$$

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

The operator L is *linear* if

$$L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 \quad \text{(almost like multiplying by } L\text{)}$$

Nicer proof of the theorem:

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

The operator L is *linear* if

$$L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 \quad \text{(almost like multiplying by } L\text{)}$$

Nicer proof of the theorem:

Suppose that $Ly_1 = 0$ and $Ly_2 = 0$.

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

The operator L is *linear* if

$$L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 \quad \text{(almost like multiplying by } L\text{)}$$

Nicer proof of the theorem:

Suppose that $Ly_1 = 0$ and $Ly_2 = 0$. Then

$$Ly = L(C_1 y_1 + C_2 y_2)$$

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

The operator L is *linear* if

$$L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 \quad \text{(almost like multiplying by } L\text{)}$$

Nicer proof of the theorem:

Suppose that $Ly_1 = 0$ and $Ly_2 = 0$. Then

$$Ly = L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2$$

An *operator* is a thing that eats functions and spits out functions.

Define the operator L by

$$Ly = y'' + py' + qy.$$

The homogeneous differential equation becomes $Ly = 0$.

The operator L is *linear* if

$$L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 \quad \text{(almost like multiplying by } L\text{)}$$

Nicer proof of the theorem:

Suppose that $Ly_1 = 0$ and $Ly_2 = 0$. Then

$$Ly = L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 = C_1 \cdot 0 + C_2 \cdot 0 = 0.$$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\quad \cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\quad \cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\quad \cosh(kx)$

Example: $y'' - k^2y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2}$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx}$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\quad \cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\quad \cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

And $\quad \sinh(kx)$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\quad \cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\quad \cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

And $\quad \sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2}$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}$, $\sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

And $\sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2} = (1/2)e^{kx} + (-1/2)e^{-kx}$

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}$, $\sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

And $\sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2} = (1/2)e^{kx} + (-1/2)e^{-kx} = (1/2)y_1 + (-1/2)y_2$ is a solution

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

And $\sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2} = (1/2)e^{kx} + (-1/2)e^{-kx} = (1/2)y_1 + (-1/2)y_2$ is a solution

sinh and cosh are sometimes more convenient than the exponential.

Example: $y'' - k^2 y = 0$ has solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$.

Recall that $\cosh t = \dfrac{e^t + e^{-t}}{2}, \quad \sinh t = \dfrac{e^t - e^{-t}}{2}$

So $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2} = (1/2)e^{kx} + (1/2)e^{-kx} = (1/2)y_1 + (1/2)y_2$ is a solution

And $\sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2} = (1/2)e^{kx} + (-1/2)e^{-kx} = (1/2)y_1 + (-1/2)y_2$ is a solution

sinh and cosh are sometimes more convenient than the exponential.

Exercise: Verify that

$$\cosh 0 = 1, \qquad \sinh 0 = 0,$$
$$\frac{d}{dt}\Big[\cosh t\Big] = \sinh t, \qquad \frac{d}{dt}\Big[\sinh t\Big] = \cosh t,$$
$$\cosh^2 t - \sinh^2 t = 1.$$

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants.

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Example: $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Example: $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

Verify IC: $\quad y(0) = b_0 \cos(0) + \dfrac{b_1}{k} \sin(0) = b_0.$

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Example: $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

Verify IC: $\quad y(0) = b_0 \cos(0) + \dfrac{b_1}{k} \sin(0) = b_0.$

$y'(x) = -kb_0 \sin(kx) + b_1 \cos(kx)$

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Example: $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

Verify IC: $\quad y(0) = b_0 \cos(0) + \frac{b_1}{k} \sin(0) = b_0.$

$$y'(x) = -kb_0 \sin(kx) + b_1 \cos(kx) \quad \Rightarrow \quad y'(0) = -kb_0 \sin(0) + b_1 \cos(0) = b_1.$$

Theorem (Existence and uniqueness)

Suppose p, q, f are continuous functions on some interval I, a is a number in I, and b_0, b_1 are constants. Then the equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

Example: $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

Verify IC: $\quad y(0) = b_0 \cos(0) + \frac{b_1}{k} \sin(0) = b_0.$

$y'(x) = -kb_0 \sin(kx) + b_1 \cos(kx) \quad \Longrightarrow \quad y'(0) = -kb_0 \sin(0) + b_1 \cos(0) = b_1.$

Similarly, $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$.

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2. \qquad \text{(\textit{i.e., that's the general solution})}$$

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2. \qquad (\textit{i.e., that's the general solution})$$

Example: $y_1 = \sin x$ and $y_2 = \cos x$ solve $y'' + y = 0$.

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2. \qquad \text{(\textit{i.e., that's the general solution})}$$

Example: $y_1 = \sin x$ and $y_2 = \cos x$ solve $y'' + y = 0$.

sin and cos are linearly independent:

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2. \qquad (\textit{i.e., that's the general solution})$$

Example: $y_1 = \sin x$ and $y_2 = \cos x$ solve $y'' + y = 0$.

sin and cos are linearly independent: If $\sin x = A \cos x$ for some constant A, then let $x = 0$ to get $A = 0$.

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2. \qquad (\textit{i.e., that's the general solution})$$

Example: $y_1 = \sin x$ and $y_2 = \cos x$ solve $y'' + y = 0$.

sin and cos are linearly independent: If $\sin x = A \cos x$ for some constant A, then let $x = 0$ to get $A = 0$. But then $\sin x = 0$ for all x, that's nonsense.

Two functions y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem

Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2. \qquad \text{(\textit{i.e., that's the general solution})}$$

Example: $y_1 = \sin x$ and $y_2 = \cos x$ solve $y'' + y = 0$.

sin and cos are linearly independent: If $\sin x = A \cos x$ for some constant A, then let $x = 0$ to get $A = 0$. But then $\sin x = 0$ for all x, that's nonsense.

y_1 and y_2 are linearly independent and

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

To see y_1 and y_2 are linearly independent, suppose $y_1 = Ay_2$.

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

To see y_1 and y_2 are linearly independent, suppose $y_1 = Ay_2$.

Solve for A

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

To see y_1 and y_2 are linearly independent, suppose $y_1 = Ay_2$.

Solve for $A = \dfrac{y_1}{y_2} = x^3$.

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

To see y_1 and y_2 are linearly independent, suppose $y_1 = Ay_2$.

Solve for $A = \dfrac{y_1}{y_2} = x^3$. That's not a constant!

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

To see y_1 and y_2 are linearly independent, suppose $y_1 = Ay_2$.

Solve for $A = \dfrac{y_1}{y_2} = x^3$. That's not a constant! $\quad\Rightarrow\quad$ y_1 and y_2 are linearly independent.

Example: $y'' - 2x^{-2}y = 0$ has solutions $y_1 = x^2$ and $y_2 = \dfrac{1}{x}$.

To see y_1 and y_2 are linearly independent, suppose $y_1 = Ay_2$.

Solve for $A = \dfrac{y_1}{y_2} = x^3$. That's not a constant! $\quad\Rightarrow\quad$ y_1 and y_2 are linearly independent.

$\Rightarrow \quad y = C_1 x^2 + C_2 \dfrac{1}{x}$ is the general solution.

If you have one solution we can find the second via the *reduction of order method*.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2$$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}v.$$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0.$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \underbrace{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}_{0}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0$.

Solve for w, and find v by antidifferentiating.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\big(y_1'' + p(x)y_1' + q(x)y_1\big)}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0$.

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0$.

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

$\Rightarrow \quad xw' + 3w = 0$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \underbrace{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}_{0}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0$.

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

$\Rightarrow \quad xw' + 3w = 0 \quad \Rightarrow \quad w = Cx^{-3}$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \underbrace{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}_{0}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0.$

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

$$\Rightarrow \quad xw' + 3w = 0 \quad \Rightarrow \quad w = Cx^{-3} \quad \Rightarrow \quad v = \frac{-C}{2x^2}$$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\big(y_1'' + p(x)y_1' + q(x)y_1\big)}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0$.

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

$$\Rightarrow \quad xw' + 3w = 0 \quad \Rightarrow \quad w = Cx^{-3} \quad \Rightarrow \quad v = \frac{-C}{2x^2} \quad \Rightarrow \quad y_2 = y_1v = \frac{-C}{2x}$$

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \underbrace{\cancel{\left(y_1'' + p(x)y_1' + q(x)y_1\right)}}_{0}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0.$

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

$$\Rightarrow \quad xw' + 3w = 0 \quad \Rightarrow \quad w = Cx^{-3} \quad \Rightarrow \quad v = \frac{-C}{2x^2} \quad \Rightarrow \quad y_2 = y_1v = \frac{-C}{2x}$$

Any C works, e.g., $C = -2$ for $y_2 = \dfrac{1}{x}$.

If you have one solution we can find the second via the *reduction of order method*.

Suppose y_1 solves $y'' + p(x)y' + q(x)y = 0$.
Try to find a second solution of the form $y_2(x) = y_1(x)v(x)$. Plug in:

$$0 = y_2'' + p(x)y_2' + q(x)y_2 = \underbrace{y_1''v + 2y_1'v' + y_1v''}_{y_2''} + p(x)\underbrace{(y_1'v + y_1v')}_{y_2'} + q(x)\underbrace{y_1v}_{y_2}$$

$$= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\big(y_1'' + p(x)y_1' + q(x)y_1\big)}v.$$

So $y_1v'' + (2y_1' + p(x)y_1)v' = 0$.

Write $w = v'$ and we have a first order ODE $\quad y_1w' + (2y_1' + p(x)y_1)w = 0$.

Solve for w, and find v by antidifferentiating.

Example: $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$, let's find y_2.

$$\Rightarrow \quad xw' + 3w = 0 \quad \Rightarrow \quad w = Cx^{-3} \quad \Rightarrow \quad v = \frac{-C}{2x^2} \quad \Rightarrow \quad y_2 = y_1v = \frac{-C}{2x}$$

Any C works, e.g., $C = -2$ for $y_2 = \dfrac{1}{x}$. $\quad \Rightarrow \quad$ The general solution is $\quad y = C_1x + C_2\dfrac{1}{x}$.

We can even just write down a formula

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\big(y_1(x)\big)^2}\,dx$$

A useful warm-up for next time:

Exercise: For $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution.

A useful warm-up for next time:

Exercise: For $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution.
Hint: Try $y = x^r$.

A useful warm-up for next time:

Exercise: For $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution.
Hint: Try $y = x^r$.

Equations of the form $ax^2y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy–Euler equations*.