19. Nonhomogeneous equations, part 1: undetermined coefficients (Notes on Diffy Qs, 2.5)

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The textbook: `https://www.jirka.org/diffyqs/`

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Theorem: If y_c is the general solution to $Ly_c = 0$ and y_p is any particular solution to $Ly_p = f(x)$, then

$$y = y_c + y_p \qquad \text{is the general solution to } Ly = f(x).$$

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Note: Do not forget lower degree terms even if they don't appear on the right hand side:

E.g., for $\quad Ly = x^3 + 1, \quad$ try $y_p = Ax^3 + Bx^2 + Cx + D$.

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$\Rightarrow \quad -4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0 \quad \Rightarrow \quad -2A + 4B = 1$ and $2A + B = 0$

$\Rightarrow \quad A = {}^{-1}/_{10}$ and $B = {}^{1}/_{5}$.

Sines, cosines and exponentials are similar.

Example: Find a particular solution to $\quad y'' + 2y' + 2y = \cos(2x)$.

Linear combinations of derivatives of $\cos(2x)$ will be combinations of $\cos(2x)$ and $\sin(2x)$.

Try: $\quad y_p = A\cos(2x) + B\sin(2x)$.

$$\underbrace{-4A\cos(2x) - 4B\sin(2x)}_{y_p''} + 2\underbrace{\big(-2A\sin(2x) + 2B\cos(2x)\big)}_{y_p'}$$

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So $\quad y_p = A\cos(2x) + B\sin(2x) = \dfrac{-\cos(2x) + 2\sin(2x)}{10}$.

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Plug in, then solve for A, B, C, D, E, and F.

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The general solution is

$$y = y_c + y_p = C_1 e^{-3x} + C_2 e^{3x} + \frac{1}{6} x e^{3x}.$$

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So keep multiplying by x until duplication is gone but no more.

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$y = u + v \quad$ solves $\quad Ly = e^{2x} + \cos x$:

$$Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x.$$