

2. Introduction to differential equations (Notes on Diffy Qs, 0.2)

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The textbook: <https://www.jirka.org/diffyqs/>

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It is a *first order differential equation*.

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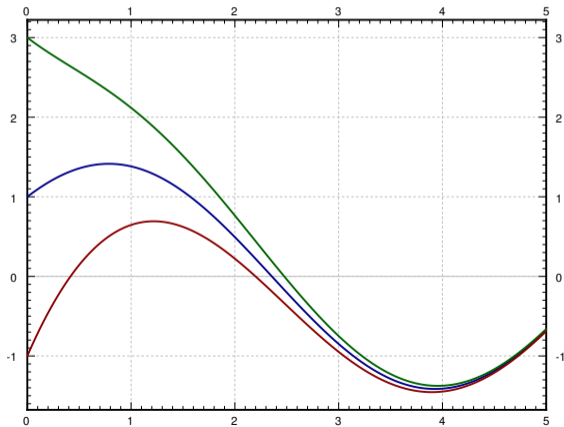
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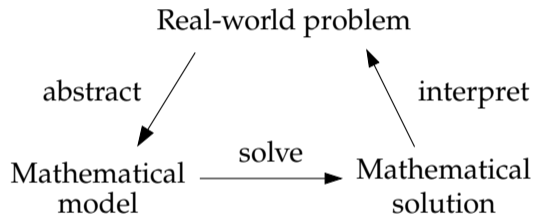
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For complicated cases we may have to be satisfied with approximate, numerical solutions.

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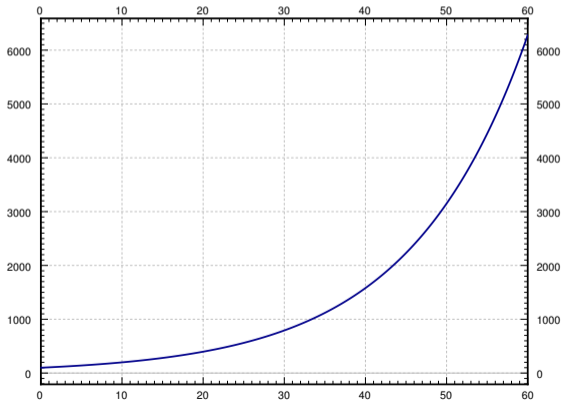
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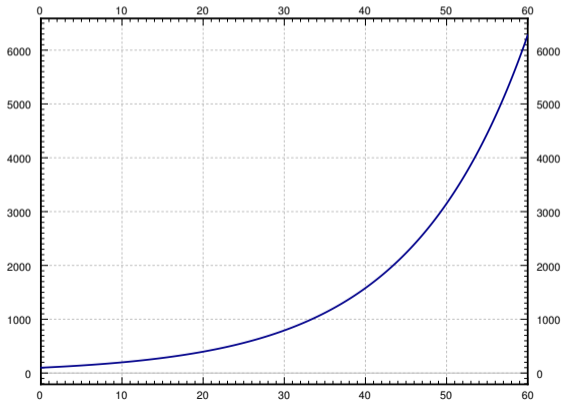
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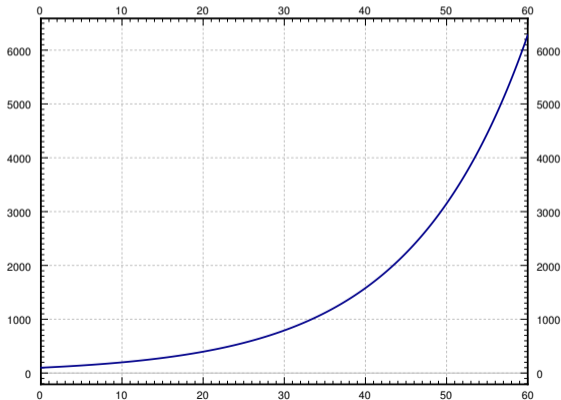
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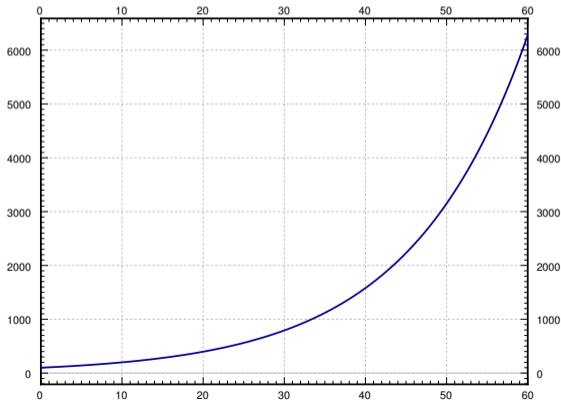
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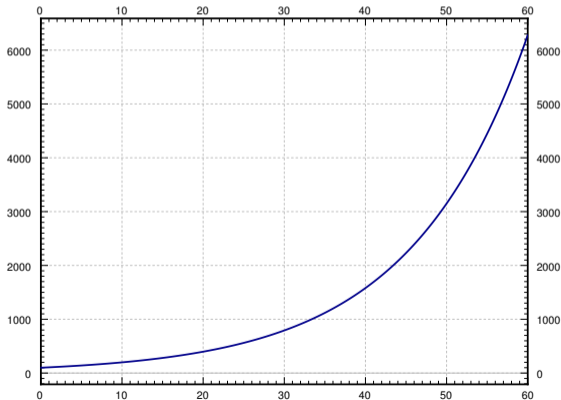
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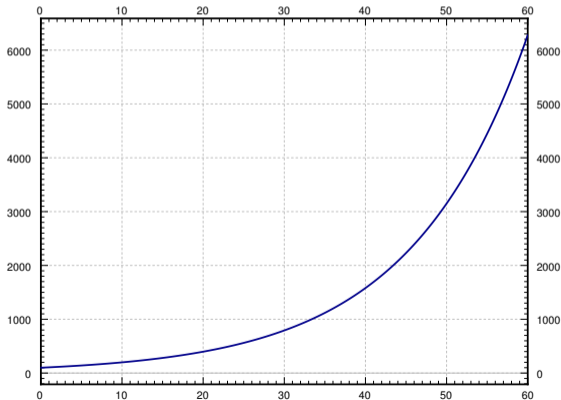
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Obviously this is approximate.



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For those that have not seen \sinh and \cosh , the *hyperbolic sine* and *hyperbolic cosine*:

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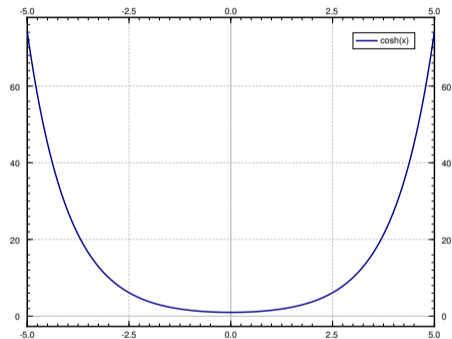
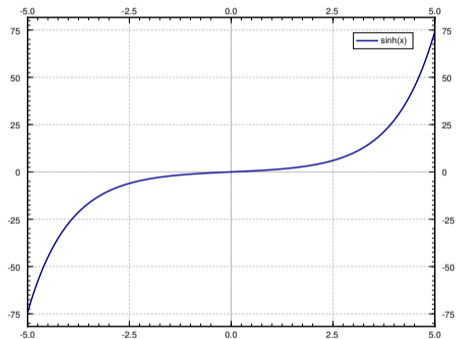
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$$\cosh 0 = 1, \quad \sinh 0 = 0, \quad \frac{d}{dx} \cosh x = \sinh x, \quad \frac{d}{dx} \sinh x = \cosh x.$$

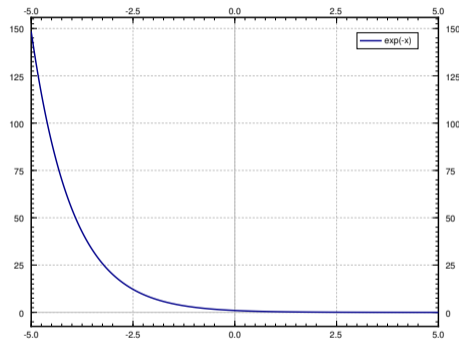
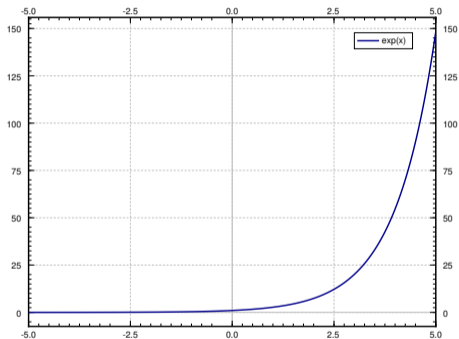
Remark: The shape of the graph of \cosh is called a *catenary*. The arch in Saint Louis is an inverted \cosh :

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

Here are the graphs of \sinh and \cosh :



Compare with the graphs of exponential growth e^x and exponential decay e^{-x}



Just for completeness here are the graphs of sin and cos:

