21. Forced oscillations and resonance, part 1: Undamped forced motion and resonance (Notes on Diffy Qs, 2.6)

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Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

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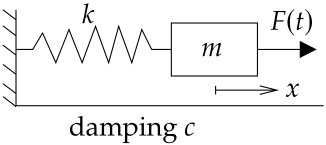

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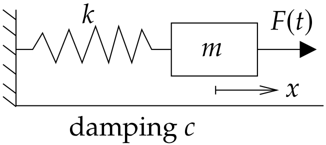

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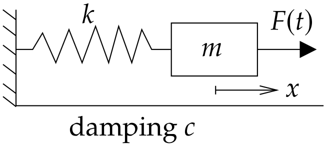

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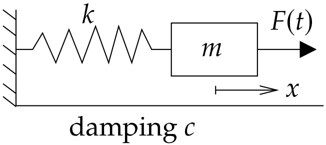

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Note: Using Fourier series, all periodic functions can be understood via this simple case.

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$$x = C \cos(\omega_0 t - \gamma) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

A superposition of two phase shifted cosine waves at different frequencies.

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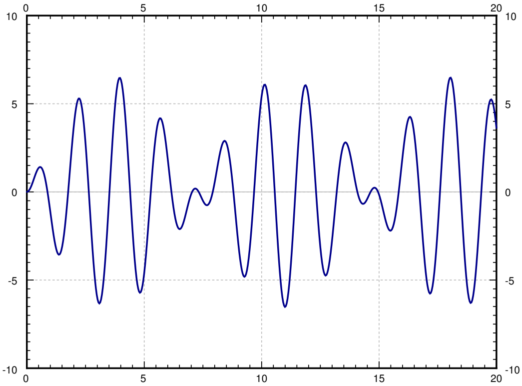

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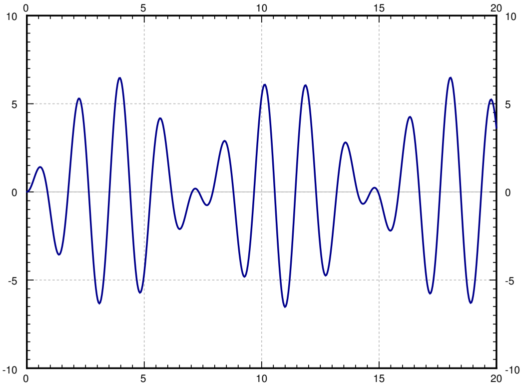

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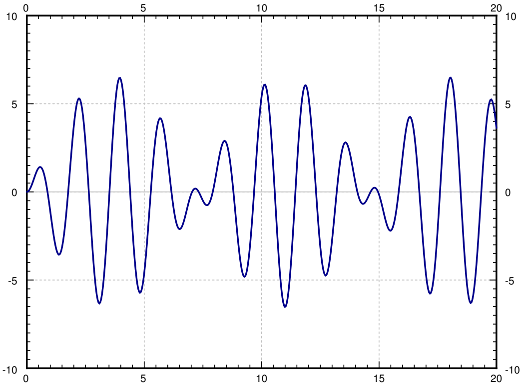

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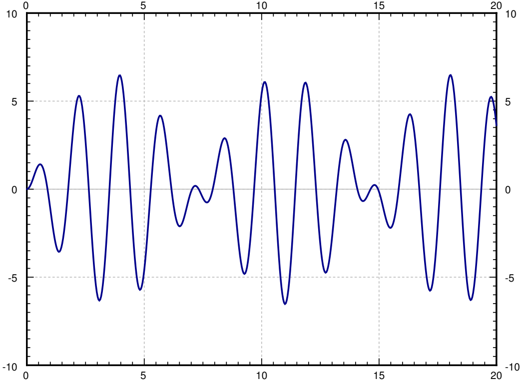

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The solution is a high frequency wave modulated by a low frequency wave.

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x_p oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$.
x_c only oscillates between $\pm\sqrt{C_1^2 + C_2^2}$.

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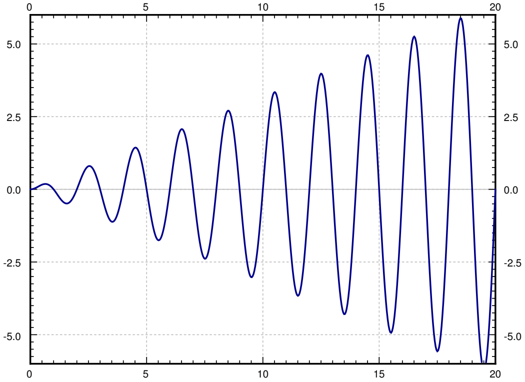

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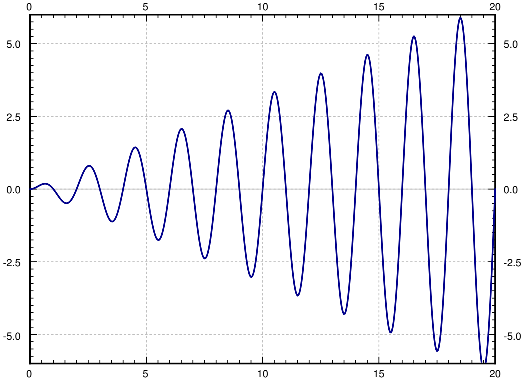

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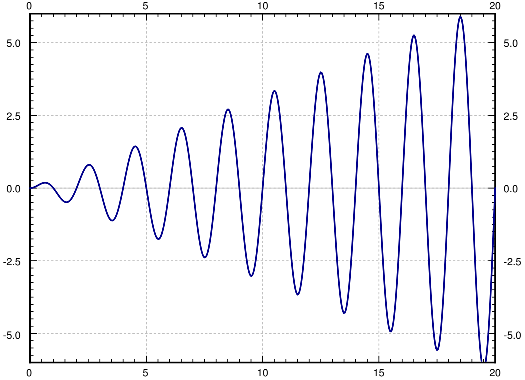

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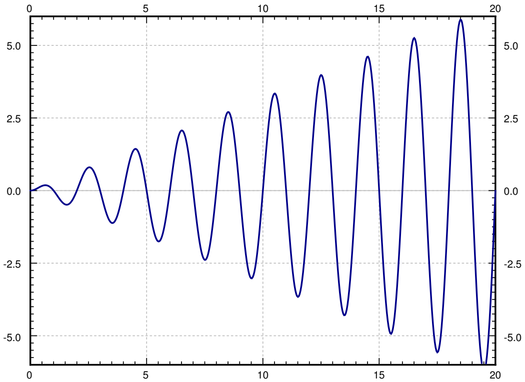

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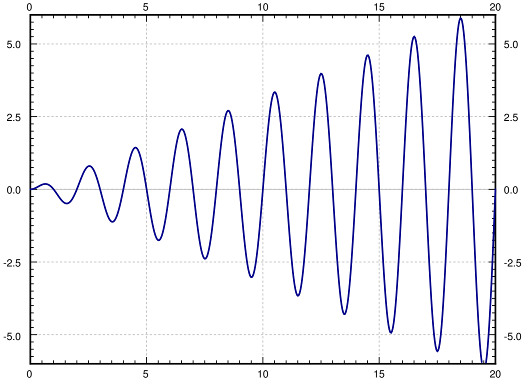

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Resonance can be bad:
Examples: Earthquakes, vibrations in engines, soldiers marching on a bridge, . . .