9. Autonomous equations (Notes on Diffy Qs, 1.6)

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Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

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Example: The cooling coffee problem

$$\frac{dx}{dt} = k(A - x) \qquad \text{(Newton's law of cooling)}$$

x=temperature of coffee, t=time,
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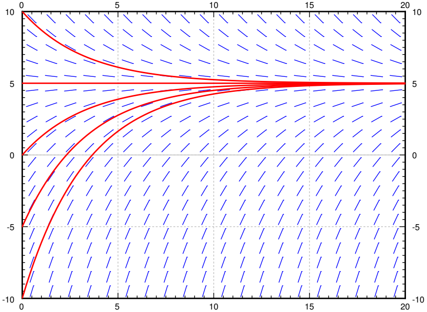

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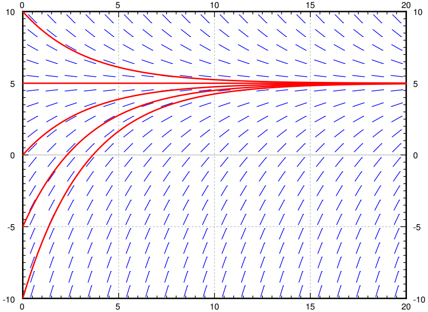

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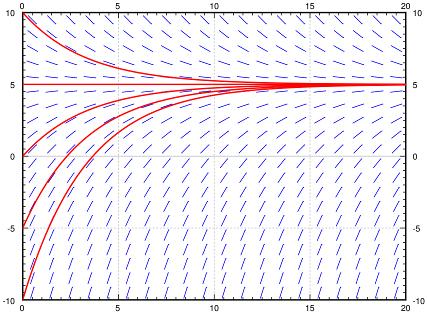

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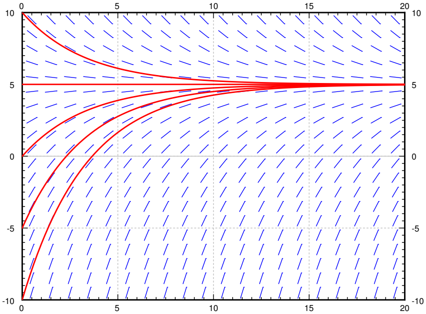

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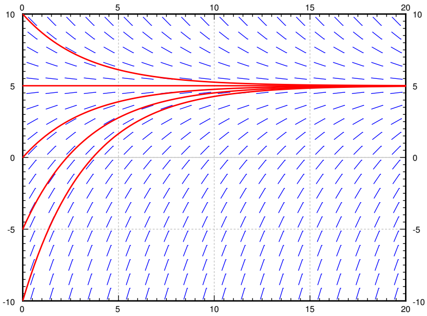

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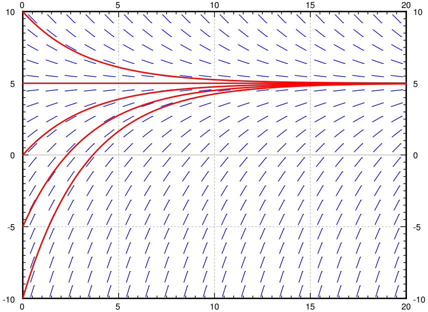

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A critical point that is not stable is called *unstable*.

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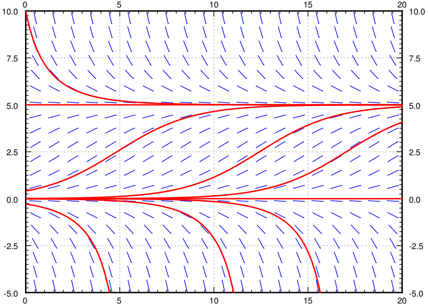

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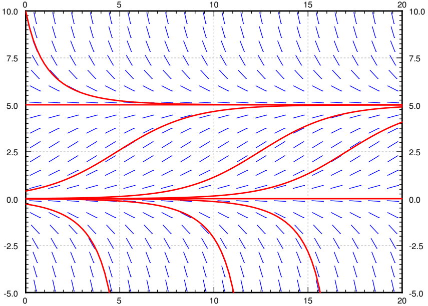

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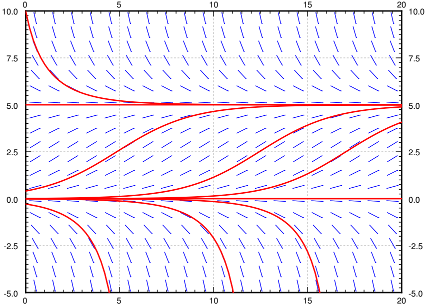

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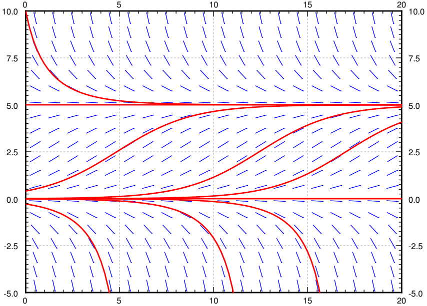

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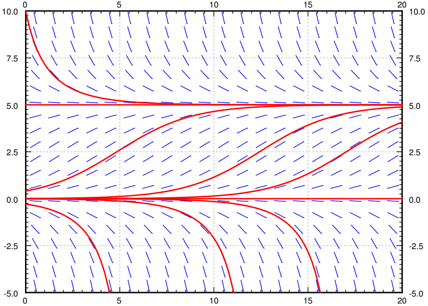

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By inspection,

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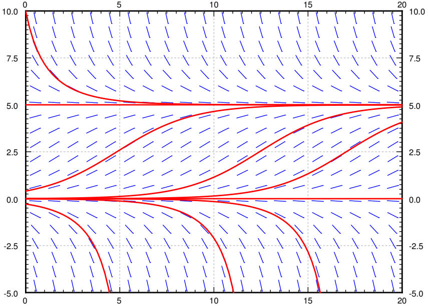

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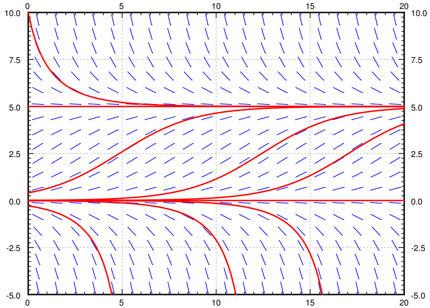

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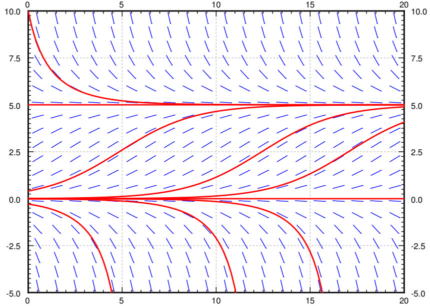

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So to find long term behavior, $x(t)$ for very large t, we don't need to solve.

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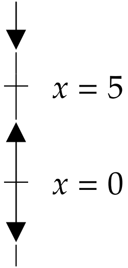

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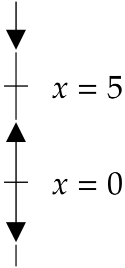

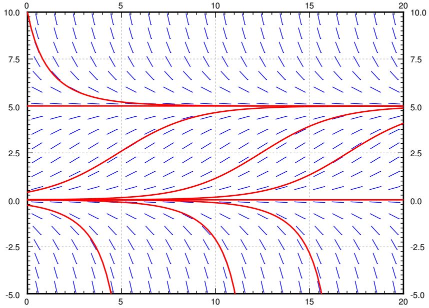

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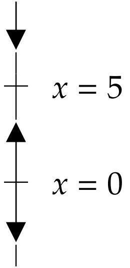

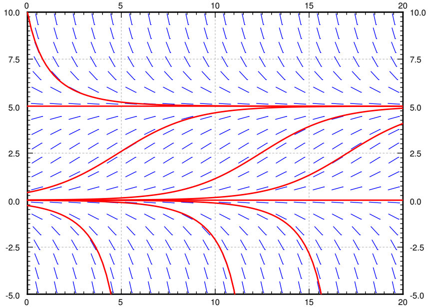

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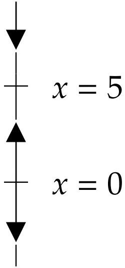

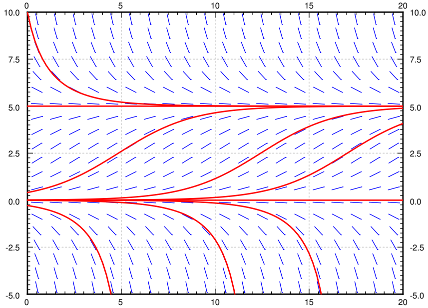

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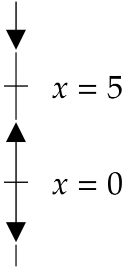

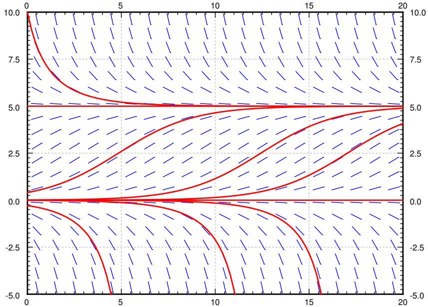

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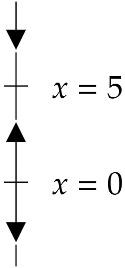

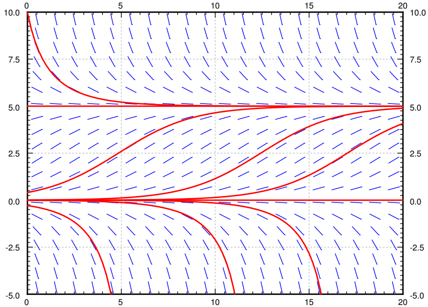

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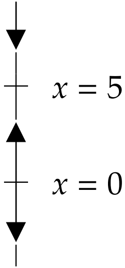

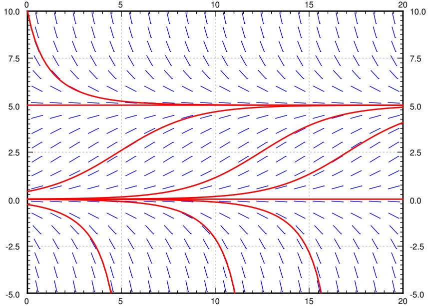

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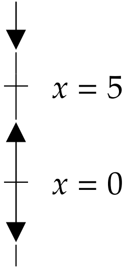

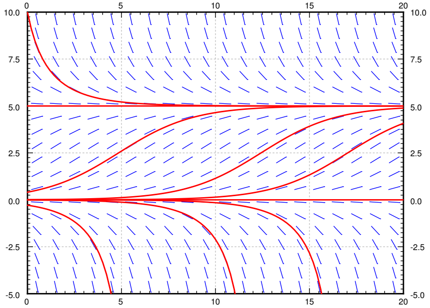

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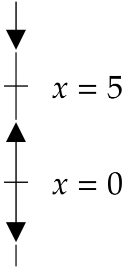

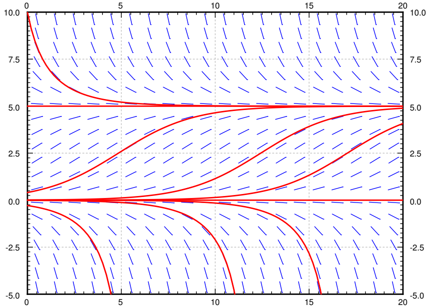

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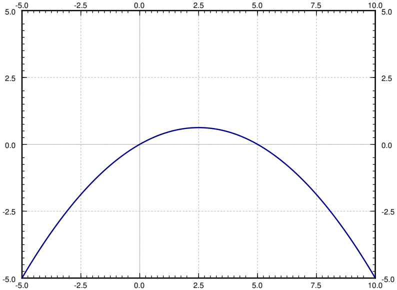
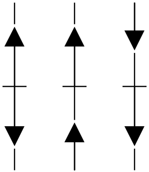


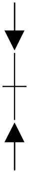
Graph of $y = f(x) = 0.1x(5 - x)$

Armed with a phase diagram, easy to sketch solutions.

Classifying critical points is also easy:



unstable



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Remark: If one arrow points in and one out, sometimes called *semistable*.

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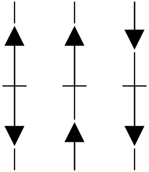

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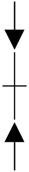

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Unstable points tend to be bad news:

Classifying critical points is also easy:

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Small changes in initial conditions lead to different outcomes.

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3 possibilities: $A > B$, or $A = B$, or A and B both complex (i.e. no real solutions).

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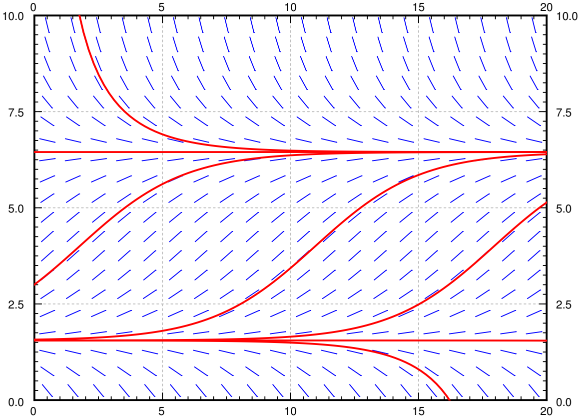

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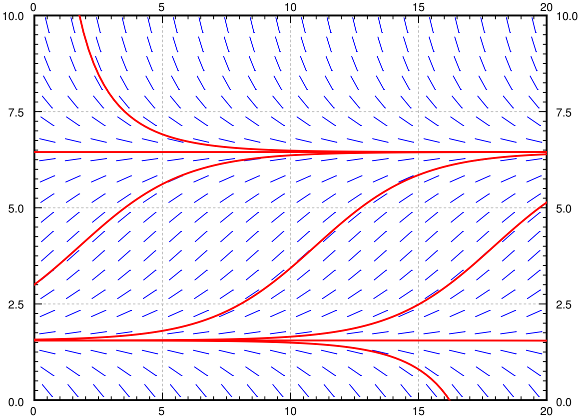

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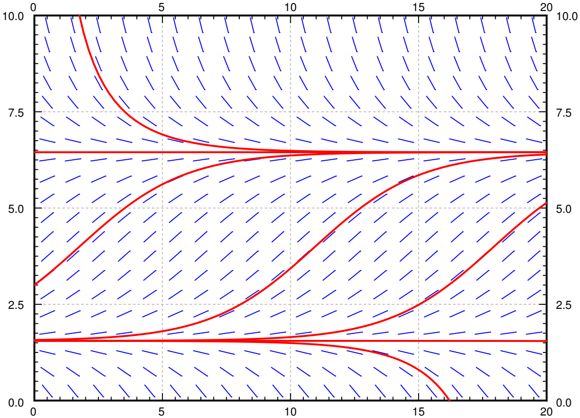

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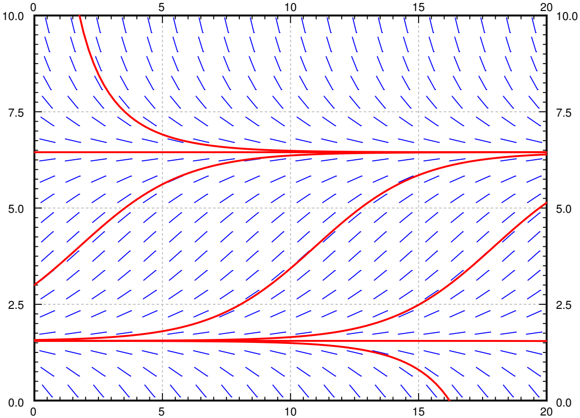

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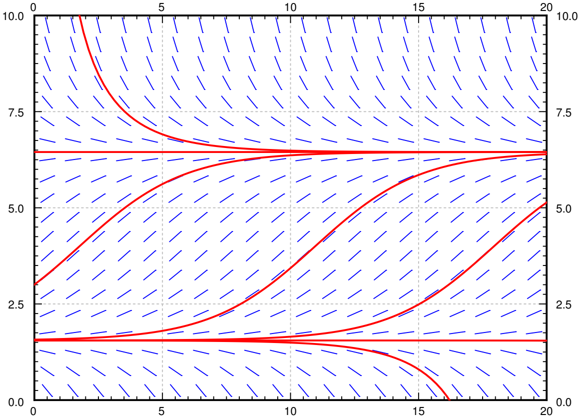

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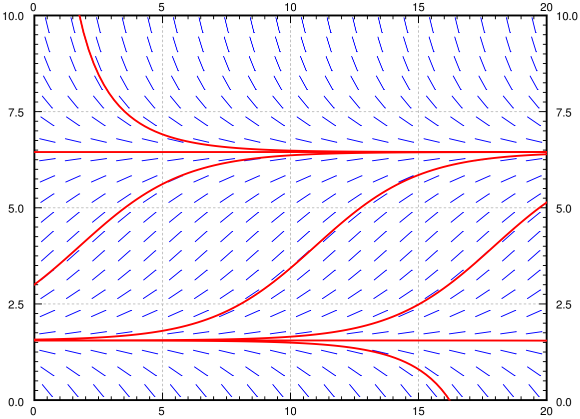

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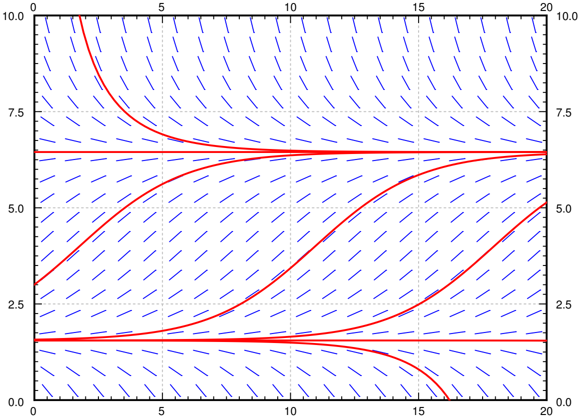

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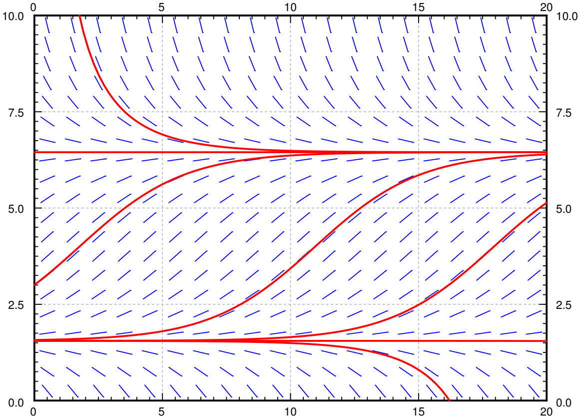

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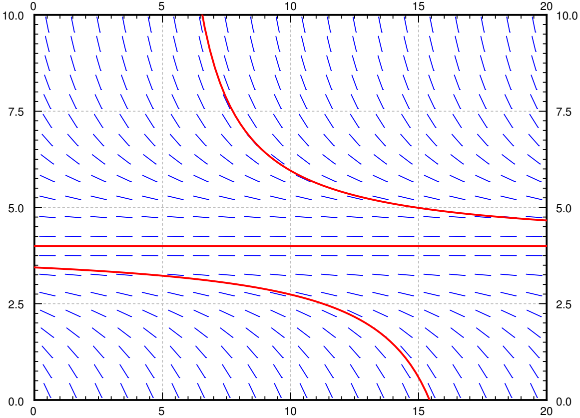
If there is an earthquake and population drops below B,
the alien race becomes vegetarian. (not good for us either)

Now harvest $h = 1.6$:



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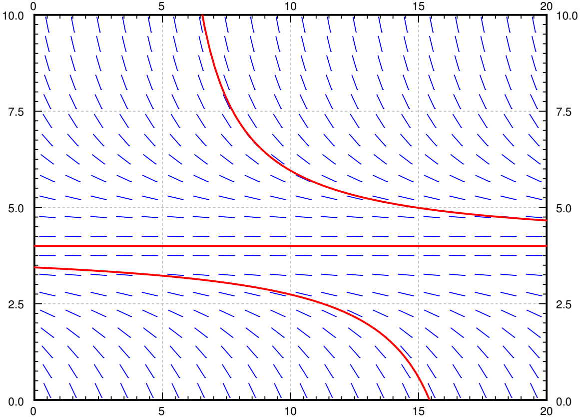

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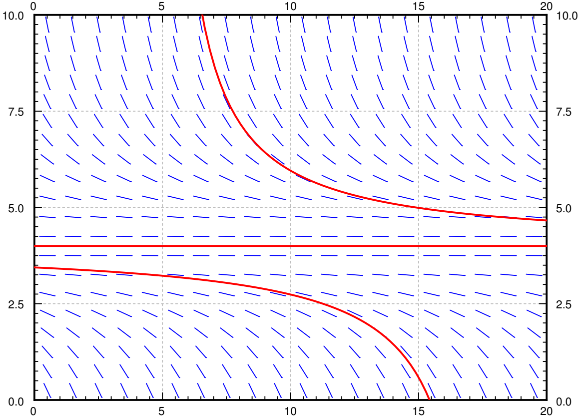

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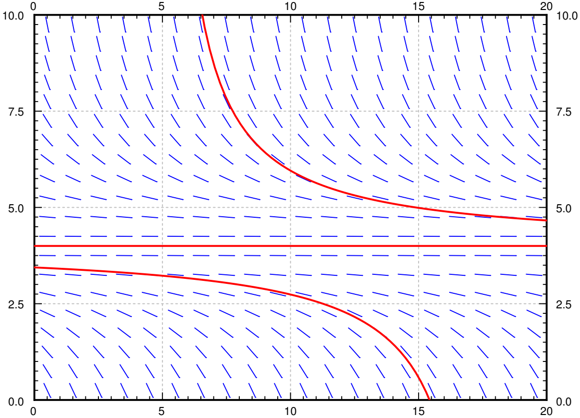

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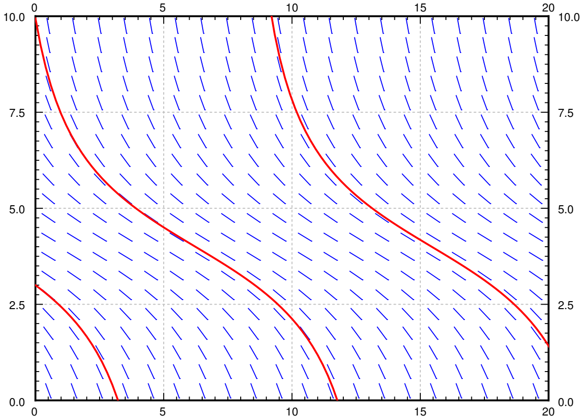

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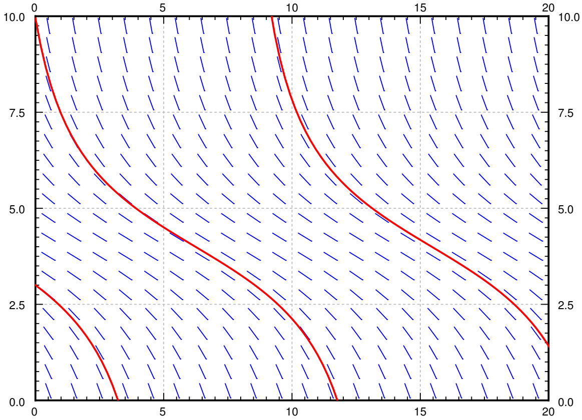

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No critical points.

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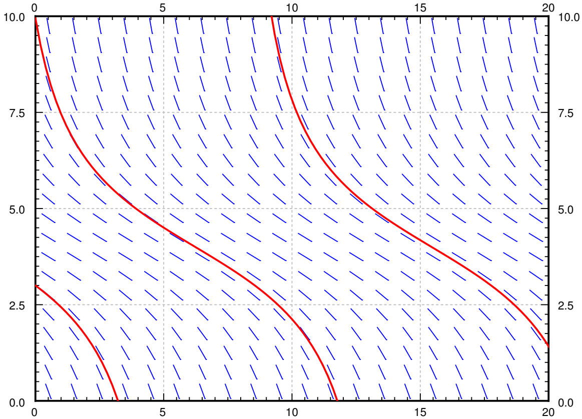

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No critical points. Population always goes to 0.