20. Nonhomogeneous equations, part 2: variation of parameters (Notes on Diffy Qs, 2.5)

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The textbook: `https://www.jirka.org/diffyqs/`

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No luck ...

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We look for a solution of $Ly = f(x)$ of the form

$$y = u_1 y_1 + u_2 y_2,$$

where u_1, u_2 are functions.

Example: Solve $y'' + y = \tan x$.

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So

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In other words

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Soooo ... for y to satisfy $Ly = f(x)$, we must have $f(x) = u_1' y_1' + u_2' y_2'$.

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To summarize we need to solve the two imposed conditions:

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}}$$

In our case

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(Forget about constants of integration, we want a particular solution)

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The general solution to $y'' + y = \tan x$ is

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Note: Undetermined coefficients are usually less tedious if applicable.