

## 6. Separable equations (Notes on Diffy Qs, 1.3)

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The textbook: <https://www.jirka.org/diffyqs/>

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But if the equation is so-called “separable,” then we can still integrate.

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Solve for  $y$  (if you can).

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Substitution formula from calculus says  $\int \frac{1}{g(y)} dy = \int f(x) dx + C$ . ✓

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Solutions such as  $y = 0$  are sometimes called *singular solutions*.

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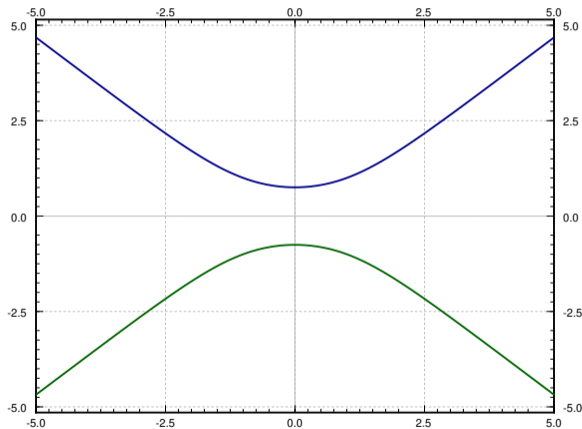
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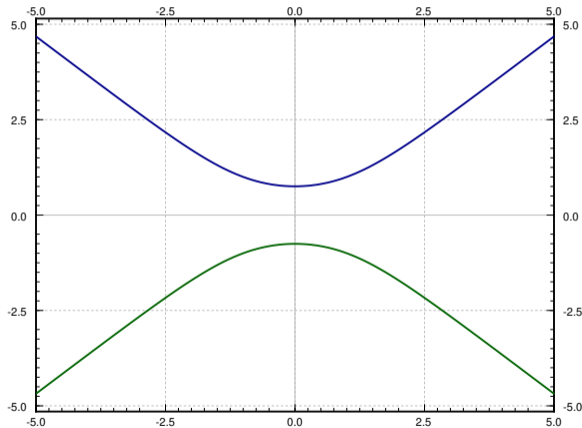
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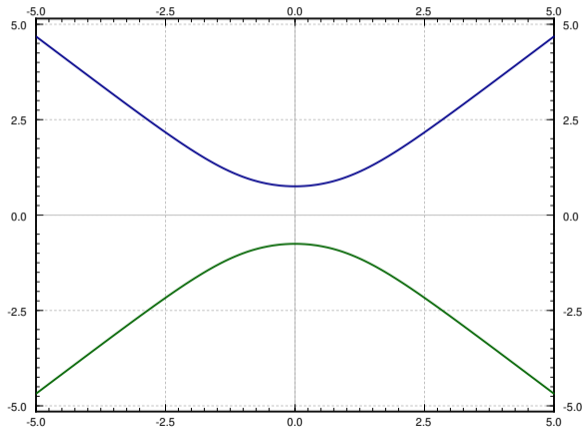


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A couple more examples of separable equations:

**Example:** Solve  $x^2y' = 1 - x^2 + y^2 - x^2y^2$ ,  $y(1) = 0$ .

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The particular solution we seek is  $y = \tan\left(\frac{-1}{x} - x + 2\right)$ .

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Second condition  $85 = T(1)$

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First condition:  $89 = T(0) = 22 + D \Rightarrow D = 67 \Rightarrow T = 22 + 67 e^{-kt}$

Second condition  $85 = T(1) = 22 + 67 e^{-k} \Rightarrow k = -\ln \frac{85-22}{67} \approx 0.0616$

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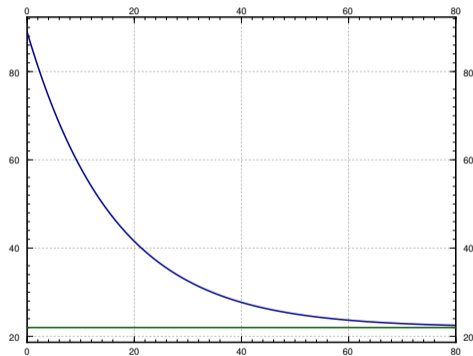
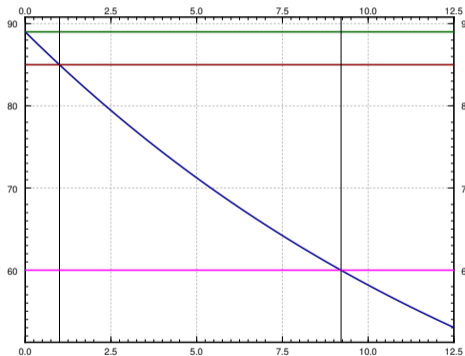
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Solve  $T = 60$  for time  $t$ :  $60 = 22 + 67e^{-0.0616t} \Rightarrow t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$  minutes.

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Solve  $T = 60$  for time  $t$ :  $60 = 22 + 67e^{-0.0616t} \Rightarrow t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$  minutes.

Bob can start drinking a little over 9 minutes from when the coffee was made.



Graphs of the coffee temperature function  $T(t)$  over different time frames.

89, 85, 60 lines are marked on the left, and  $T = A$  line is marked on the right.

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The general solution is

$$y = \frac{6}{x^2 + 2C} \quad \text{and} \quad y = 0.$$