

16. Higher order linear ODEs (Notes on Diffy Qs, 2.3)

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The textbook: <https://www.jirka.org/diffyqs/>

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Theorem (Superposition)

If y_1, y_2, \dots, y_n are solutions of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = 0,$$

then the linear combination

$$y(x) = C_1y_1(x) + C_2y_2(x) + \dots + C_ny_n(x)$$

is also a solution for arbitrary constants C_1, C_2, \dots, C_n .

Theorem (Existence and uniqueness)

Suppose p_0 through p_{n-1} , and f are continuous functions on some interval I , a is a number in I , and b_0, b_1, \dots, b_{n-1} are constants. Then the equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1}.$$

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$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad c_1 e + c_2 e^2 + c_3 e^3 = 0 \quad \text{and} \quad c_1 e^2 + c_2 e^4 + c_3 e^6 = 0.$$

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Example: Solve $y''' - 3y'' - y' + 3y = 0$ subject to $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$.

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So the particular solution to our problem is:

$$y = \frac{-1}{4} e^{-x} + e^x + \frac{1}{4} e^{3x}.$$

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So $3 = (-1)(1)(-r_3)$ or $r_3 = 3$.

If a real root r is repeated k times, then we have solutions

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The general solution is

$$y = \underbrace{(C_1 + C_2x + C_3x^2)e^x}_{\text{terms coming from } r=1} + \underbrace{C_4}_{\text{from } r=0}.$$

Complex roots come in pairs $r = \alpha \pm i\beta$.

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If we have such a pair each repeated k times, the corresponding solution is

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