

## 5. Slope fields (Notes on Diffy Qs, 1.2)

Jiří Lebl

Oklahoma State University

The textbook: <https://www.jirka.org/diffyqs/>

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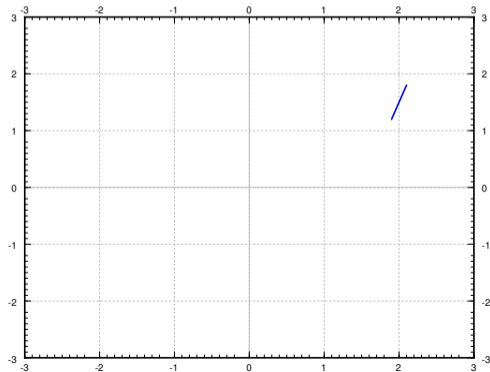
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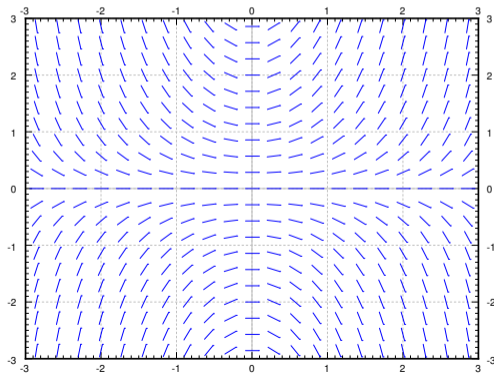
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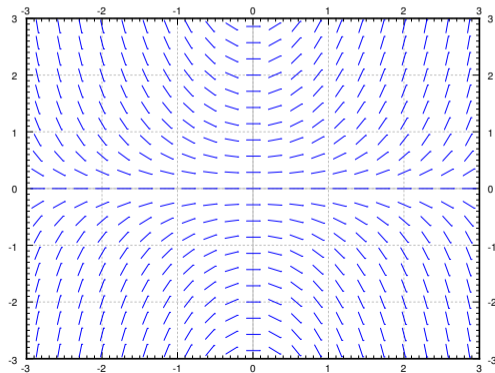
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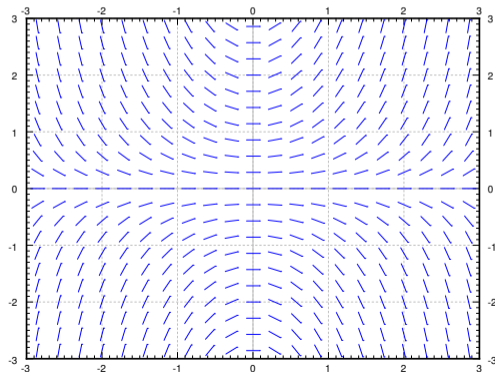
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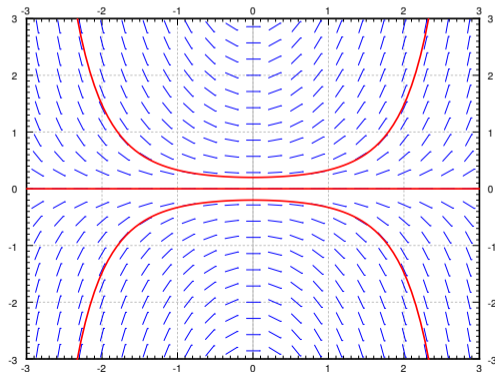
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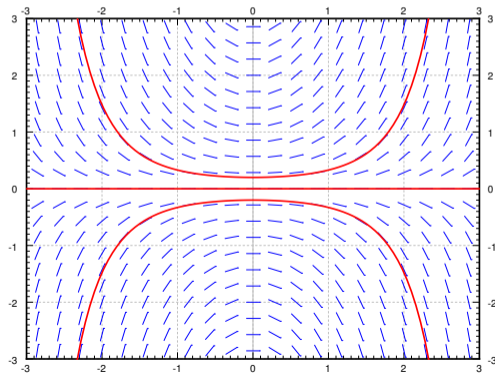
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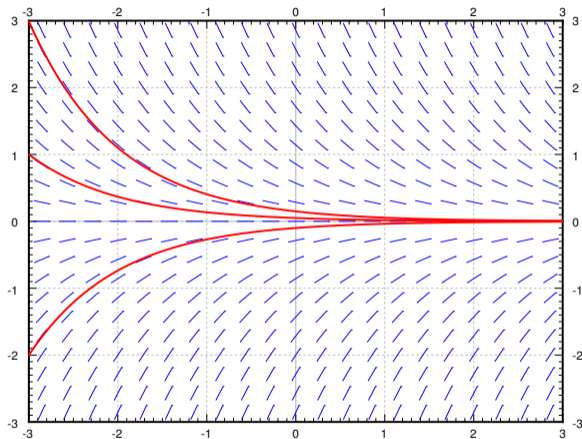
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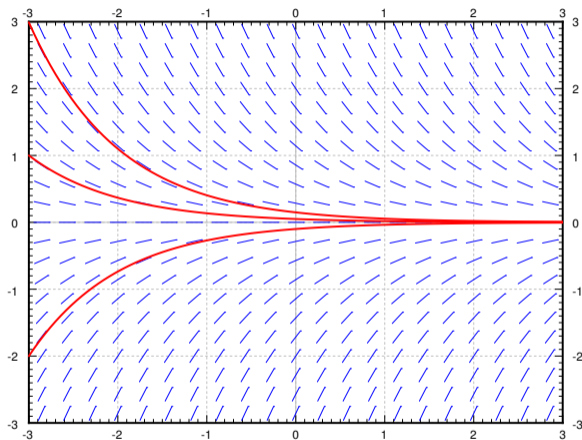
We can tell just by the slope field that  $y(0) > 0$   
leads to very different behavior from  $y(0) < 0$ .



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Note how in this case we can tell from the slope field that all solutions just tend to  $y = 0$ .

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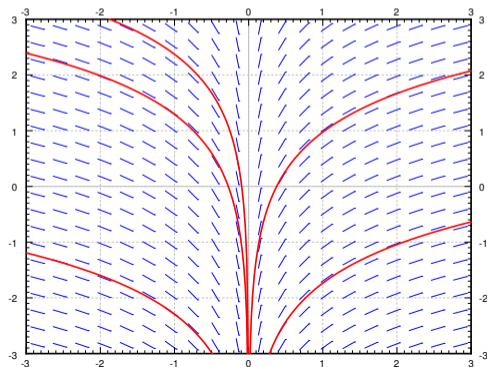
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There is no solution at  $x = 0$ .



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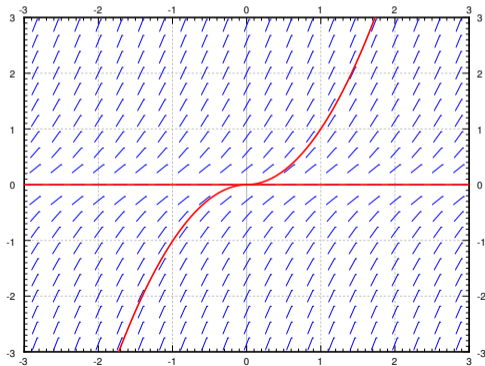
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$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$



So answer to (ii) can also be false.

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$A = y(0) = \frac{1}{C-0} = 1/C \Rightarrow C = 1/A \Rightarrow$  The solution is  $y = \frac{1}{1/A - x}$ .

If  $A = 1$ , the solution blows up when  $x = 1$ . The solution exists for  $x < 1$ .

If  $A = 100$ , the solution blows up when  $x = \frac{1}{100} = 0.01$ . The solution exists for  $x < 0.01$ .