# 8. Substitution
# (Notes on Diffy Qs, 1.5)

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The textbook: `https://www.jirka.org/diffyqs/`

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Note: $D = 0$ gives $y = x + 2$, but no $D$ gives $y = x$.

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Nothing wrong with making many guesses.

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We just need to solve this linear equation.

$v' - \frac{4(x+1)}{x}v = 4 \quad (v' + P(x)v = F(x) \text{ where } P(x) = -\frac{4(x+1)}{x} \text{ and } F(x) = 4)$

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In the solution, we assume $x > 0$ or $x < 0$ depending on the initial condition.

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**Example:** Solve $x^2y' = y^2 + xy, \quad y(1) = 1.$

Write $y' = (y/x)^2 + y/x,$ so $F(v) = v^2 + v$

Assume $x > 0$ due to the initial condition.

Substituting $v = y/x$ as above gets $xv' = v^2 + v - v = v^2$

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