

3. Classification of differential equations (Notes on Diffy Qs, 0.3)

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The textbook: <https://www.jirka.org/diffyqs/>

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- *Ordinary differential equations* (ODE):
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- *Ordinary differential equations* (ODE):
Only one independent variable.
- *Partial differential equations* (PDE):
Several independent variables, using partial derivatives.

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(Heat equation)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

(Wave equation in 2 dimensions)

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Maxwell's equations for electromagnetics are a *system of partial differential equations* (system of PDE):

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho, & \nabla \cdot \vec{B} &= 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, & \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t}. \end{aligned}$$

(Note: divergence $\nabla \cdot$ and curl $\nabla \times$ are written in partial derivatives in x, y, z .)

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Remark: The most common equations in physics are first and second order.

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An ODE of order n can be put into the form:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = b(x).$$

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E.g., linear 2nd order ODE:

$$e^x \frac{d^2 y}{dx^2} + \sin(x) \frac{dy}{dx} + x^2 y = \frac{1}{x}$$

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Remark: Nonlinear equations are notoriously difficult to handle.

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We solve an inhomogeneous equation using the solution to the corresponding homogeneous equation.

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Autonomous equations often appear when the setup is independent of time.

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First order, linear, inhomogeneous, constant coefficient PDE system.