23. The Laplace transform, part 1
(Notes on Diffy Qs, 6.1)

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The textbook: `https://www.jirka.org/diffyqs/`

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Notation: We will use capital letters to represent the Laplace transforms.

In particular, consider

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

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We transform the input $f(t)$ into $F(s)$.

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Then we invert the transform of $X(s)$ to find the output $x(t)$.

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Here are some useful Laplace transforms (C, ω, and a are constants):

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C	$\frac{C}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	
t^2	$\frac{2}{s^3}$	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$	
t^3	$\frac{6}{s^4}$	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$	
t^n	$\frac{n!}{s^{n+1}}$	$u(t-a)$	$\frac{e^{-as}}{s}$	$(a \geq 0)$
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Exercise: Verify the table.

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Remark: Not all functions have a Laplace transform. E.g., $\frac{1}{t}$, $\tan t$, or e^{t^2}