7. Linear equations and the integrating factor (Notes on Diffy Qs, 1.4)

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The textbook: `https://www.jirka.org/diffyqs/`

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We have an explicit formula for the solution:

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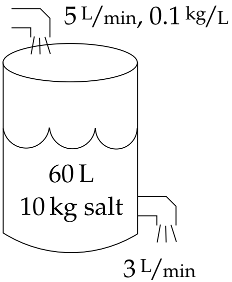

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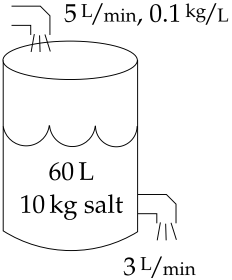

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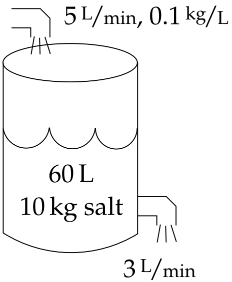

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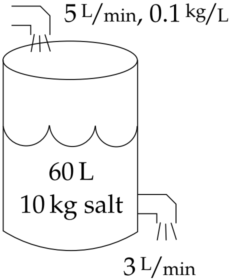

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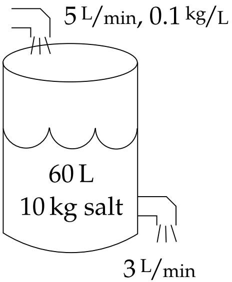

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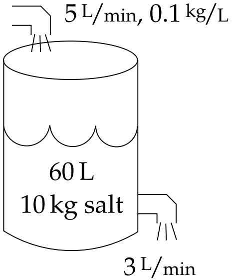

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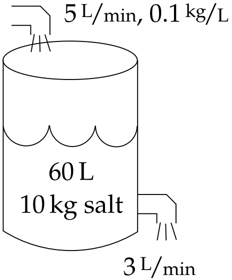

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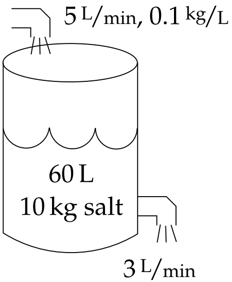

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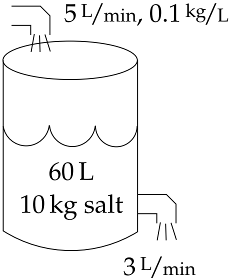

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Here rate in = 5, concentration in = 0.1, rate out = 3, concentration out = $\frac{x}{\text{volume}} = \frac{x}{60+(5-3)t}$

So

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3\frac{x}{60+2t}\right) \qquad \text{or} \qquad \frac{dx}{dt} + \frac{3}{60+2t}x = 0.5$$

For $\frac{dx}{dt} + \frac{3}{60+2t}x = 0.5$, the integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60 + 2t}dt\right)$$

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Compute

$$(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x = 0.5(60+2t)^{3/2}$$

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Compute

$$\begin{aligned}(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2} \\ \frac{d}{dt}\left[(60+2t)^{3/2}x\right] &= 0.5(60+2t)^{3/2} \\ (60+2t)^{3/2}x &= \int 0.5(60+2t)^{3/2}dt + C\end{aligned}$$

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(60+2t)^{3/2}x &= \int 0.5(60+2t)^{3/2}dt + C \\
x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}dt + C(60+2t)^{-3/2}
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x &= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2}
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x &= \frac{60+2t}{10} + C(60+2t)^{-3/2}
\end{aligned}$$

At $t = 0$, $x = 10$: $\quad 10 = x(0)$

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At $t = 0$, $x = 10$: $\quad 10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2} \quad \Rightarrow$

At $t = 0$, $x = 10$: $\quad 10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2} \quad \Longrightarrow \quad C = 4(60^{3/2}) \approx 1859.03.$

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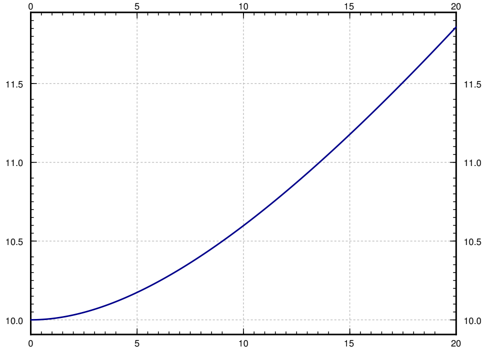
Here's a graph:



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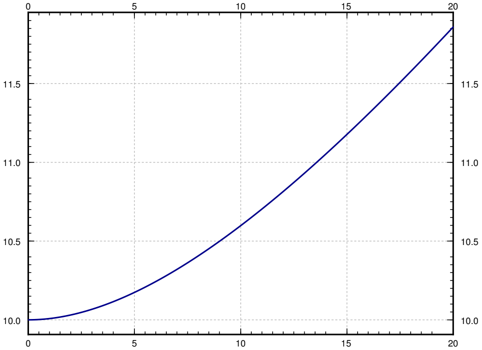

So when is the tank full?

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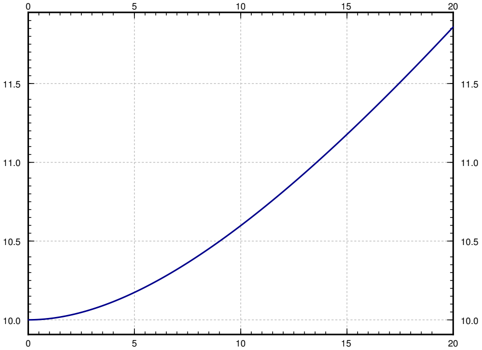

So when is the tank full?
The tank is full when $60 + 2t = 100$, or $t = 20$.

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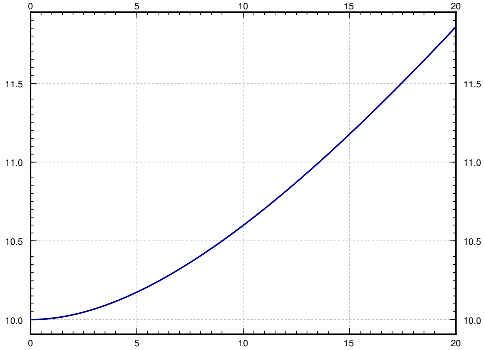

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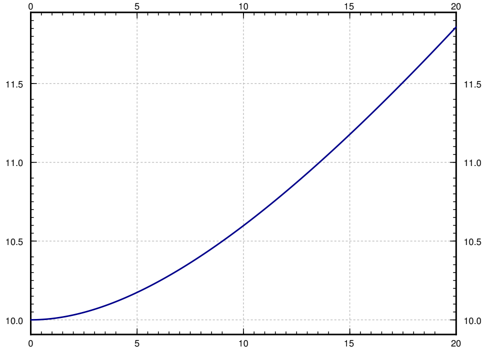

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$$x(20) = \frac{60 + 40}{10} + C(60 + 40)^{-3/2}$$

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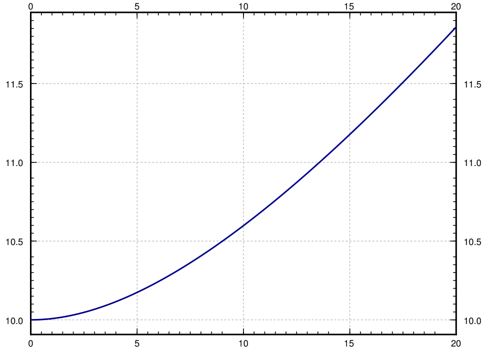

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$$x(20) = \frac{60 + 40}{10} + C(60 + 40)^{-3/2}$$
$$\approx 10 + 1859.03(100)^{-3/2} \approx 11.86$$

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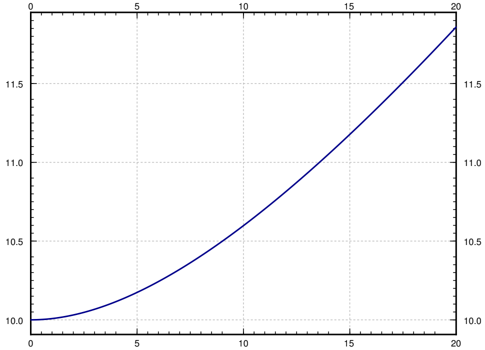

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$$
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x(20) &= \frac{60+40}{10} + C(60+40)^{-3/2} \\
&\approx 10 + 1859.03(100)^{-3/2} \approx 11.86
\end{aligned}
$$

The concentration when the tank is full is approx. $^{11.86}/_{100} = 0.1186\,^{\text{kg}}/_{\text{liter}}$.

(we started with $^1/_6$ or $0.1667\,^{\text{kg}}/_{\text{liter}}$.)