

23. The Laplace transform, part 1 (Notes on Diffy Qs, 6.1)

Jiří Lebl

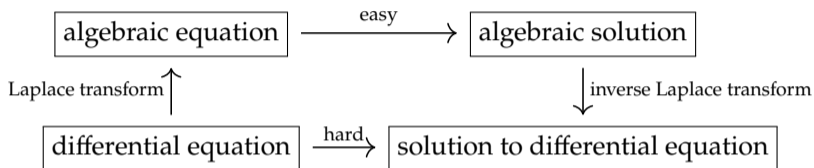
Oklahoma State University

The textbook: <https://www.jirka.org/diffyqs/>

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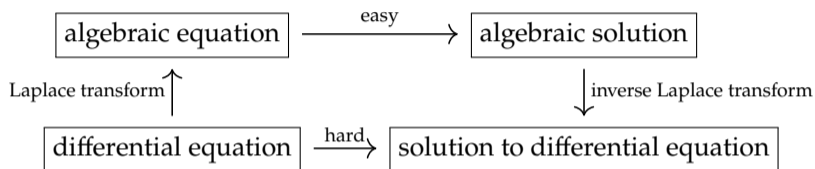
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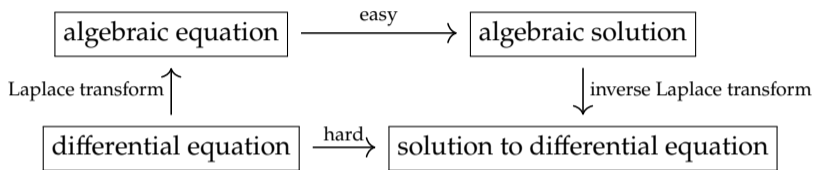
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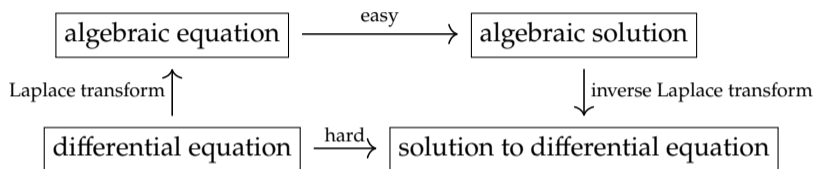


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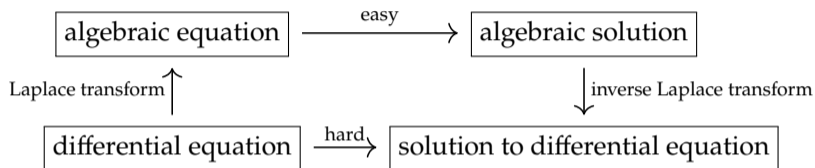
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What it does: Laplace takes a function of time t and gives a function of “frequency” s :

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Notation: We will use capital letters to represent the Laplace transforms.

In particular, consider

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

Think of t as time, $f(t)$ as the input signal, and $x(t)$ as the output.

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We transform the input $f(t)$ into $F(s)$.

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Then we invert the transform of $X(s)$ to find the output $x(t)$.

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Here are some useful Laplace transforms (C , ω , and a are constants):

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	
C	$\frac{C}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	
t^2	$\frac{2}{s^3}$	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$	
t^3	$\frac{6}{s^4}$	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$	
t^n	$\frac{n!}{s^{n+1}}$	$u(t - a)$	$\frac{e^{-as}}{s}$	$(a \geq 0)$
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t^n	$\frac{n!}{s^{n+1}}$	$u(t - a)$	$\frac{e^{-as}}{s} \quad (a \geq 0)$
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Exercise: Verify the table.

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Remark: Not all functions have a Laplace transform. E.g., $\frac{1}{t}$, $\tan t$, or e^{t^2}