25. Transforms of derivatives and ODEs, part 1 (Notes on Diffy Qs, 6.2)

Jiří Lebl

Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

To solve ODEs with Laplace, we need to know how derivatives transform.

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \leq Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \leq Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

$$\mathcal{L}\{g'(t)\}$$

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \leq Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt$$

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \leq Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt = \Big[e^{-st}g(t)\Big]_{t=0}^{\infty} - \int_0^\infty (-s)\,e^{-st}g(t)\,dt$$

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \leq Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt = \Big[e^{-st}g(t)\Big]_{t=0}^{\infty} - \int_0^\infty (-s)\,e^{-st}g(t)\,dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \le Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt = \left[e^{-st}g(t)\right]_{t=0}^\infty - \int_0^\infty (-s)\,e^{-st}g(t)\,dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

Rinse and repeat for higher derivatives:

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

To solve ODEs with Laplace, we need to know how derivatives transform.

Suppose $g(t)$ is a differentiable function of exponential order:

$|g(t)| \leq Me^{ct}$ for large t, so $\mathcal{L}\{g(t)\}$ exists and $\lim_{t\to\infty} e^{-st}g(t) = 0$ when $s > c$.

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st}g'(t)\,dt = \Big[e^{-st}g(t)\Big]_{t=0}^\infty - \int_0^\infty (-s)\,e^{-st}g(t)\,dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

Rinse and repeat for higher derivatives:

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Notice: $G(s)$ is not differentiated, it is multiplied by s.

Example: Consider $$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

Example: Consider $\quad x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Laplace the equation and write $X(s)$ for $\mathcal{L}\{x(t)\}$:

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\}$$

Example: Consider $\qquad x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Laplace the equation and write $X(s)$ for $\mathcal{L}\{x(t)\}$:

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\} \quad \Rightarrow \quad \underbrace{s^2X(s) - sx(0) - x'(0)}_{\mathcal{L}\{x''(t)\}} + \underbrace{X(s)}_{\mathcal{L}\{x(t)\}} = \frac{s}{s^2+4}.$$

Example: Consider $x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Laplace the equation and write $X(s)$ for $\mathcal{L}\{x(t)\}$:

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\} \quad \Rightarrow \quad \underbrace{s^2X(s) - sx(0) - x'(0)}_{\mathcal{L}\{x''(t)\}} + \underbrace{X(s)}_{\mathcal{L}\{x(t)\}} = \frac{s}{s^2+4}.$$

Plug in $x(0) = 0$ and $x'(0) = 1$:

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

Example: Consider $\quad x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Laplace the equation and write $X(s)$ for $\mathcal{L}\{x(t)\}$:

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\} \quad \Rightarrow \quad \underbrace{s^2X(s) - sx(0) - x'(0)}_{\mathcal{L}\{x''(t)\}} + \underbrace{X(s)}_{\mathcal{L}\{x(t)\}} = \frac{s}{s^2+4}.$$

Plug in $x(0) = 0$ and $x'(0) = 1$:

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

Solve for $X(s)$:

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

Example: Consider $\quad x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Laplace the equation and write $X(s)$ for $\mathcal{L}\{x(t)\}$:

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\} \quad \Rightarrow \quad \underbrace{s^2X(s) - sx(0) - x'(0)}_{\mathcal{L}\{x''(t)\}} + \underbrace{X(s)}_{\mathcal{L}\{x(t)\}} = \frac{s}{s^2+4}.$$

Plug in $x(0) = 0$ and $x'(0) = 1$:

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

Solve for $X(s)$:

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

Use partial fractions (exercise):

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.$$

Example: Consider $\quad x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$

Laplace the equation and write $X(s)$ for $\mathcal{L}\{x(t)\}$:

$$\mathcal{L}\{x''(t) + x(t)\} = \mathcal{L}\{\cos(2t)\} \quad \Rightarrow \quad \underbrace{s^2X(s) - sx(0) - x'(0)}_{\mathcal{L}\{x''(t)\}} + \underbrace{X(s)}_{\mathcal{L}\{x(t)\}} = \frac{s}{s^2+4}.$$

Plug in $x(0) = 0$ and $x'(0) = 1$:

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}$$

Solve for $X(s)$:

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

Use partial fractions (exercise):

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.$$

Take the inverse Laplace transform:

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

So the procedure for constant coefficient linear ODEs is simple:

So the procedure for constant coefficient linear ODEs is simple:

1. Transform the equation (the equation becomes algebraic):

$$\begin{aligned} x(t) &\quad\longrightarrow\quad X(s), \\ x'(t) &\quad\longrightarrow\quad sX(s) - x(0), \\ x''(t) &\quad\longrightarrow\quad s^2X(s) - sx(0) - x'(0), \end{aligned}$$

   etc.

So the procedure for constant coefficient linear ODEs is simple:

1. Transform the equation (the equation becomes algebraic):
   $$\begin{aligned} x(t) &\quad\longrightarrow\quad X(s), \\ x'(t) &\quad\longrightarrow\quad sX(s) - x(0), \\ x''(t) &\quad\longrightarrow\quad s^2X(s) - sx(0) - x'(0), \end{aligned}$$
   etc.
2. Plug in the initial conditions.

So the procedure for constant coefficient linear ODEs is simple:

1. Transform the equation (the equation becomes algebraic):
   $$\begin{aligned} x(t) &\quad\longrightarrow\quad X(s), \\ x'(t) &\quad\longrightarrow\quad sX(s) - x(0), \\ x''(t) &\quad\longrightarrow\quad s^2X(s) - sx(0) - x'(0), \end{aligned}$$
   etc.
2. Plug in the initial conditions.
3. Solve for $X(s)$.

So the procedure for constant coefficient linear ODEs is simple:

1. Transform the equation (the equation becomes algebraic):
$$\begin{aligned} x(t) &\quad\longrightarrow\quad X(s), \\ x'(t) &\quad\longrightarrow\quad sX(s) - x(0), \\ x''(t) &\quad\longrightarrow\quad s^2X(s) - sx(0) - x'(0), \end{aligned}$$
   etc.
2. Plug in the initial conditions.
3. Solve for $X(s)$.
4. Compute $x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$.

So the procedure for constant coefficient linear ODEs is simple:

1. Transform the equation (the equation becomes algebraic):
   $$\begin{array}{lcl} x(t) & \longrightarrow & X(s), \\ x'(t) & \longrightarrow & sX(s) - x(0), \\ x''(t) & \longrightarrow & s^2X(s) - sx(0) - x'(0), \end{array}$$
   etc.
2. Plug in the initial conditions.
3. Solve for $X(s)$.
4. Compute $x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$.

Remark 1: To be useful, everything in sight (namely the right hand side, the input) must have a Laplace transform. E.g., won't work for $x'' + x = \tan(t)$.

So the procedure for constant coefficient linear ODEs is simple:

1. Transform the equation (the equation becomes algebraic):

$$\begin{array}{lcl} x(t) & \longrightarrow & X(s), \\ x'(t) & \longrightarrow & sX(s) - x(0), \\ x''(t) & \longrightarrow & s^2X(s) - sx(0) - x'(0), \end{array}$$

   etc.

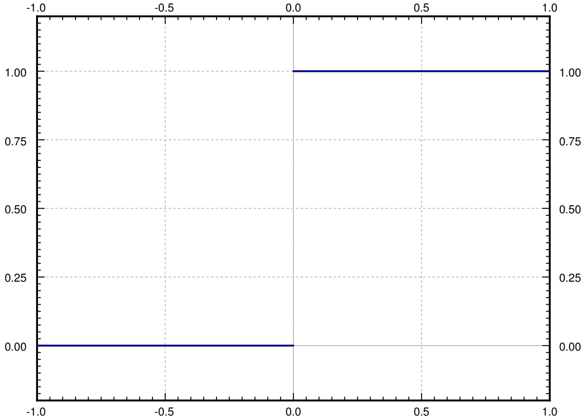
2. Plug in the initial conditions.
3. Solve for $X(s)$.
4. Compute $x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$.

Remark 1: To be useful, everything in sight (namely the right hand side, the input) must have a Laplace transform. E.g., won't work for $x'' + x = \tan(t)$.

Remark 2: On the other hand, Laplace can solve equations with many right hand sides (inputs) that the other techniques have no chance of handling.

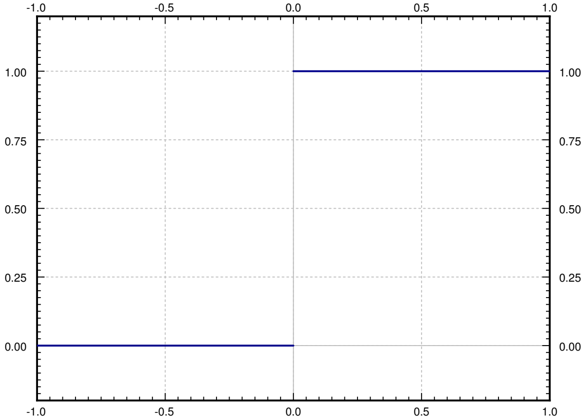
One type of function Laplace is very good at handling are functions defined piecewise. Here's where Heaviside comes in:

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$



One type of function Laplace is very good at handling are functions defined piecewise. Here's where Heaviside comes in:

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

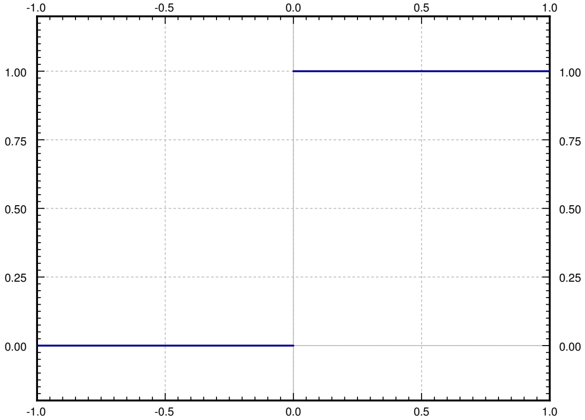


Most commonly used as

$$u(t-a) = \begin{cases} 0 & \text{if } t < a, \\ 1 & \text{if } t \geq a. \end{cases}$$

One type of function Laplace is very good at handling are functions defined piecewise. Here's where Heaviside comes in:

$$u(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ 1 & \text{if } \ t \geq 0. \end{cases}$$



Most commonly used as

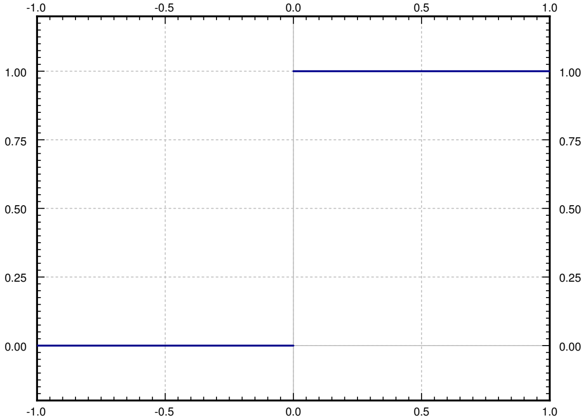
$$u(t-a) = \begin{cases} 0 & \text{if } \ t < a, \\ 1 & \text{if } \ t \geq a. \end{cases}$$

Example: Suppose

$$f(t) = \begin{cases} 0 & \text{if } \ t < \pi, \\ \sin t & \text{if } \ t \geq \pi. \end{cases}$$

One type of function Laplace is very good at handling are functions defined piecewise. Here's where Heaviside comes in:

$$u(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ 1 & \text{if } \ t \geq 0. \end{cases}$$



Most commonly used as

$$u(t-a) = \begin{cases} 0 & \text{if } \ t < a, \\ 1 & \text{if } \ t \geq a. \end{cases}$$

Example: Suppose

$$f(t) = \begin{cases} 0 & \text{if } \ t < \pi, \\ \sin t & \text{if } \ t \geq \pi. \end{cases}$$

Using Heaviside: $\quad f(t) = u(t-\pi)\,\sin t.$

A step function that is 0 everywhere except on $[1, 2)$ where it is 1 can be written as

$$u(t-1) - u(t-2).$$

A step function that is 0 everywhere except on $[1,2)$ where it is 1 can be written as

$$u(t-1)-u(t-2).$$

Useful for piecewise defined functions. E.g.,

$$f(t)=\begin{cases} t & \text{if } t\in[0,1),\\ -t+2 & \text{if } t\in[1,2),\\ 0 & \text{otherwise,}\end{cases}$$

becomes

$$f(t)=t\,\big(u(t)-u(t-1)\big)+(-t+2)\,\big(u(t-1)-u(t-2)\big).$$

A step function that is 0 everywhere except on $[1,2)$ where it is 1 can be written as

$$u(t-1)-u(t-2).$$

Useful for piecewise defined functions. E.g.,

$$f(t)=\begin{cases} t & \text{if } \ t\in[0,1), \\ -t+2 & \text{if } \ t\in[1,2), \\ 0 & \text{otherwise}, \end{cases}$$

becomes

$$f(t)=t\left(u(t)-u(t-1)\right)+(-t+2)\left(u(t-1)-u(t-2)\right).$$

We Laplace transform such functions using the *second shifting property*:

$$\mathcal{L}\left\{f(t-a)\,u(t-a)\right\}=e^{-as}\mathcal{L}\left\{f(t)\right\}.$$

A step function that is 0 everywhere except on $[1,2)$ where it is 1 can be written as

$$u(t-1) - u(t-2).$$

Useful for piecewise defined functions. E.g.,

$$f(t) = \begin{cases} t & \text{if } \ t \in [0,1), \\ -t+2 & \text{if } \ t \in [1,2), \\ 0 & \text{otherwise,} \end{cases}$$

becomes

$$f(t) = t\,\big(u(t) - u(t-1)\big) + (-t+2)\,\big(u(t-1) - u(t-2)\big).$$

We Laplace transform such functions using the *second shifting property*:

$$\mathcal{L}\big\{f(t-a)\,u(t-a)\big\} = e^{-as}\mathcal{L}\big\{f(t)\big\}.$$

Example:

$$\mathcal{L}\big\{t\,u(t-1)\big\}$$

A step function that is 0 everywhere except on $[1, 2)$ where it is 1 can be written as

$$u(t-1) - u(t-2).$$

Useful for piecewise defined functions. E.g.,

$$f(t) = \begin{cases} t & \text{if } \ t \in [0, 1), \\ -t + 2 & \text{if } \ t \in [1, 2), \\ 0 & \text{otherwise,} \end{cases}$$

becomes

$$f(t) = t\left(u(t) - u(t-1)\right) + (-t+2)\left(u(t-1) - u(t-2)\right).$$

We Laplace transform such functions using the *second shifting property*:

$$\mathcal{L}\left\{f(t-a)\,u(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t)\right\}.$$

Example:

$$\mathcal{L}\left\{t\,u(t-1)\right\} = \mathcal{L}\left\{\left((t-1)+1\right)u(t-1)\right\}$$

A step function that is 0 everywhere except on $[1, 2)$ where it is 1 can be written as

$$u(t-1) - u(t-2).$$

Useful for piecewise defined functions. E.g.,

$$f(t) = \begin{cases} t & \text{if } t \in [0,1), \\ -t+2 & \text{if } t \in [1,2), \\ 0 & \text{otherwise,} \end{cases}$$

becomes

$$f(t) = t\left(u(t) - u(t-1)\right) + (-t+2)\left(u(t-1) - u(t-2)\right).$$

We Laplace transform such functions using the *second shifting property*:

$$\mathcal{L}\left\{f(t-a)\,u(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t)\right\}.$$

Example:

$$\mathcal{L}\left\{t\,u(t-1)\right\} = \mathcal{L}\left\{\left((t-1)+1\right)u(t-1)\right\} = e^{-s}\mathcal{L}\left\{t+1\right\}$$

A step function that is 0 everywhere except on $[1, 2)$ where it is 1 can be written as

$$u(t-1) - u(t-2).$$

Useful for piecewise defined functions. E.g.,

$$f(t) = \begin{cases} t & \text{if } \ t \in [0, 1), \\ -t+2 & \text{if } \ t \in [1, 2), \\ 0 & \text{otherwise,} \end{cases}$$

becomes

$$f(t) = t\left(u(t) - u(t-1)\right) + (-t+2)\left(u(t-1) - u(t-2)\right).$$

We Laplace transform such functions using the *second shifting property*:

$$\mathcal{L}\left\{f(t-a)\,u(t-a)\right\} = e^{-as}\mathcal{L}\left\{f(t)\right\}.$$

Example:

$$\mathcal{L}\left\{t\,u(t-1)\right\} = \mathcal{L}\left\{\left((t-1)+1\right)u(t-1)\right\} = e^{-s}\mathcal{L}\left\{t+1\right\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

Example: Consider the mass-spring setup

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise.

Example: Consider the mass-spring setup

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise.

E.g., a rocket motor attached to the mass is fired for 4 seconds starting at $t = 1$.

Example: Consider the mass-spring setup

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise.

E.g., a rocket motor attached to the mass is fired for 4 seconds starting at $t = 1$. Or an RLC circuit where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$ held steady again starting at $t = 5$.

Example: Consider the mass-spring setup

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \le t < 5$ and zero otherwise.

E.g., a rocket motor attached to the mass is fired for 4 seconds starting at $t = 1$. Or an RLC circuit where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$ held steady again starting at $t = 5$.

Write $f(t) = u(t-1) - u(t-5)$ and transform the equation

$$x''(t) + x'(t) = u(t-1) - u(t-5) \qquad \Rightarrow \qquad s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

Example: Consider the mass-spring setup

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \leq t < 5$ and zero otherwise.

E.g., a rocket motor attached to the mass is fired for 4 seconds starting at $t = 1$. Or an RLC circuit where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$ held steady again starting at $t = 5$.

Write $f(t) = u(t-1) - u(t-5)$ and transform the equation

$$x''(t) + x'(t) = u(t-1) - u(t-5) \qquad \Rightarrow \qquad s^2 X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

Solve for $X(s)$:

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

Example: Consider the mass-spring setup

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \le t < 5$ and zero otherwise.

E.g., a rocket motor attached to the mass is fired for 4 seconds starting at $t = 1$. Or an RLC circuit where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$ held steady again starting at $t = 5$.

Write $f(t) = u(t-1) - u(t-5)$ and transform the equation

$$x''(t) + x'(t) = u(t-1) - u(t-5) \qquad \Rightarrow \qquad s^2 X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

Solve for $X(s)$:

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

Exercise (need to use partial fractions)**:**

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

$\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$ and the second shifting property (in reverse) says

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\}$$

$\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$ and the second shifting property (in reverse) says

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{1 - \cos t\}\right\}$$

$\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$ and the second shifting property (in reverse) says

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\big\{e^{-s}\mathcal{L}\{1 - \cos t\}\big\} = \big(1 - \cos(t-1)\big)\, u(t-1).$$

$\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$ and the second shifting property (in reverse) says

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{1 - \cos t\}\right\} = \big(1 - \cos(t-1)\big)\, u(t-1).$$

Similarly,

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\mathcal{L}\{1 - \cos t\}\right\} = \big(1 - \cos(t-5)\big)\, u(t-5).$$

$\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2+1)}$ and the second shifting property (in reverse) says

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{1 - \cos t\}\right\} = \big(1 - \cos(t-1)\big)\,u(t-1).$$

Similarly,

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\mathcal{L}\{1 - \cos t\}\right\} = \big(1 - \cos(t-5)\big)\,u(t-5).$$

Hence, as

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}\,,$$

$\mathcal{L}\{1-\cos t\} = \frac{1}{s(s^2+1)}$ and the second shifting property (in reverse) says

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{1-\cos t\}\right\} = \big(1-\cos(t-1)\big)\,u(t-1).$$

Similarly,

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\mathcal{L}\{1-\cos t\}\right\} = \big(1-\cos(t-5)\big)\,u(t-5).$$

Hence, as

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)},$$

we have

$$\begin{aligned} x(t) &= \big(1-\cos(t-1)\big)\,u(t-1) \\ &\quad - \big(1-\cos(t-5)\big)\,u(t-5). \end{aligned}$$