10. Numerical methods: Euler's method (Notes on Diffy Qs, 1.7)

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The textbook: `https://www.jirka.org/diffyqs/`

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We will cover the simplest method of all: Euler's method.

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Essentially, keep repeating : $\quad x_{i+1} = x_i + h, \quad y_{i+1} = y_i + hf(x_i, y_i).$

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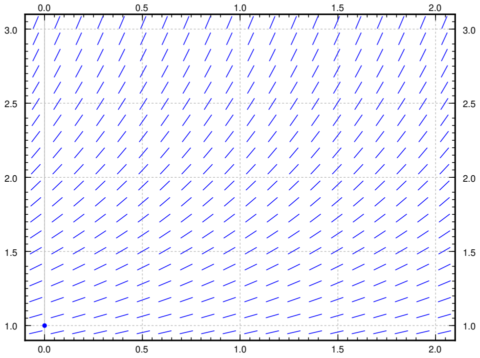

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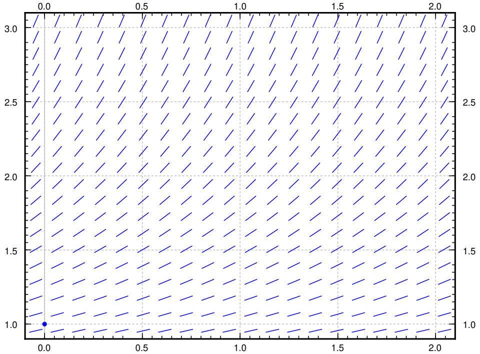

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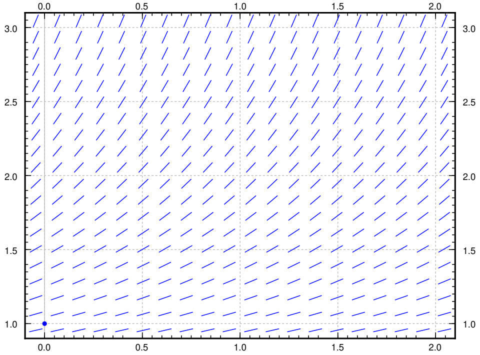

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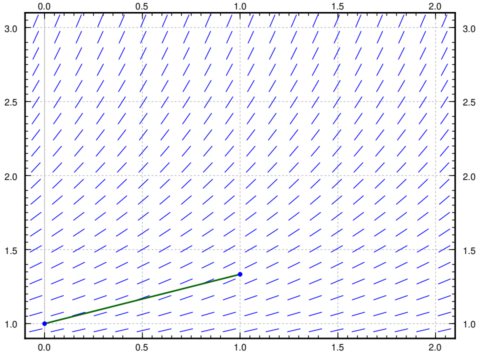

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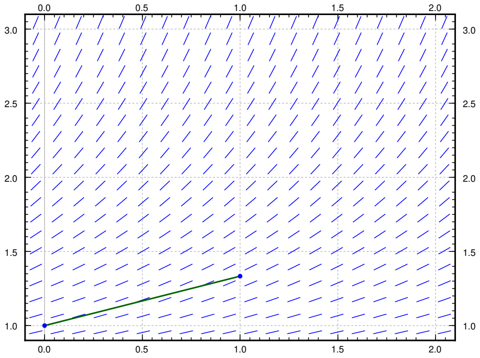

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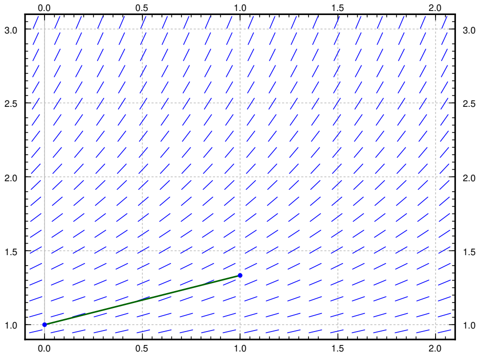

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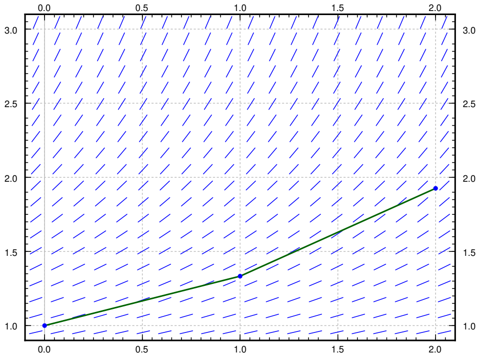

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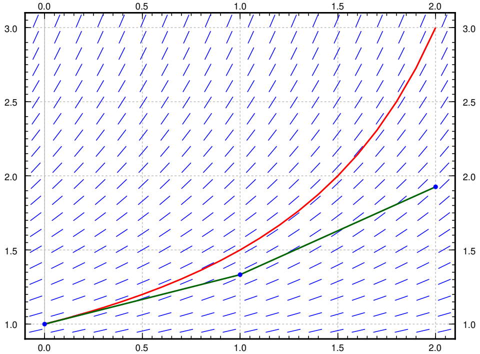
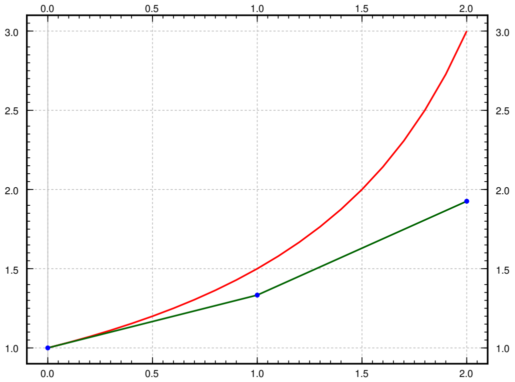


Let's do a few more *h*s.



$h = 1$

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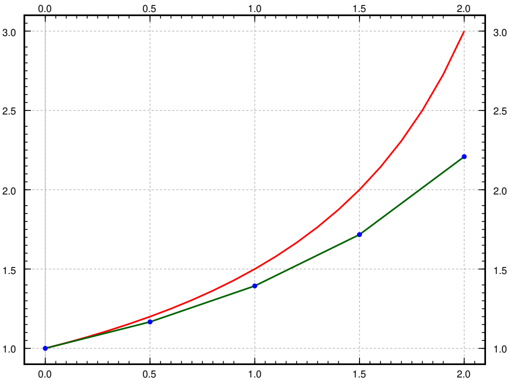
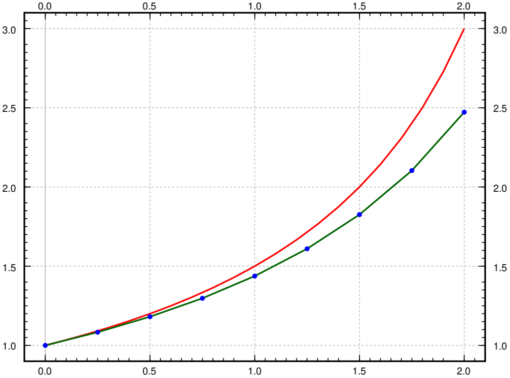


$$h = {}^1\!/_2 = 0.5$$

Let's do a few more hs.



$$h = {}^{1}/_{4} = 0.25$$

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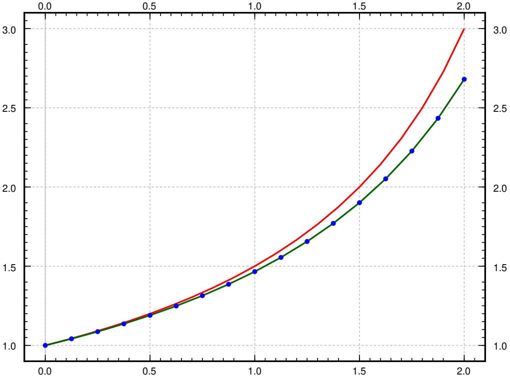
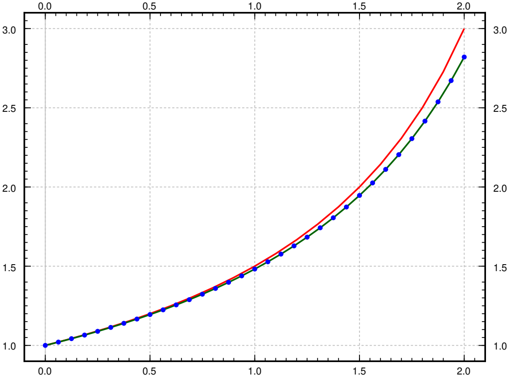


$$h = 1/8 = 0.125$$

Let’s do a few more hs.



$$h = {}^{1}\!/_{16} = 0.0625$$

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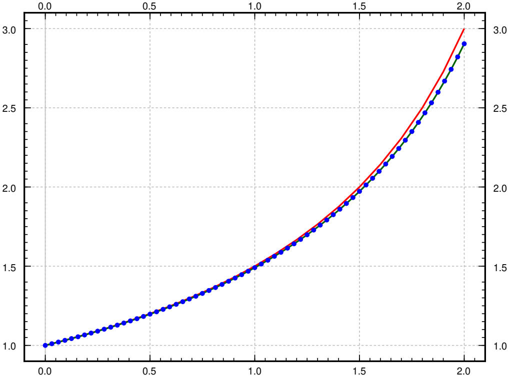

$$h = {}^{1}\!/_{32} = 0.03125$$

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h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
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Note: It seems like error halves when we halve h.

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Exercise: Take the last two approximate values, and (assume they are less than $y(2)$) assuming that the error halves each time, approximate $y(2)$, and hence the error. Is it close?

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By successive halvings, and assuming it halves each time we can estimate the error.

Exercise: Take the last two approximate values, and (assume they are less than $y(2)$) assuming that the error halves each time, approximate $y(2)$, and hence the error. Is it close?

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

The trick in practice is to estimate the error so that we can pick the right h.

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Suppose we attempt to find $y(3)$:

h	Approximate $y(3)$
1	3.16232
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0.25	6.86079
0.125	10.80321
0.0625	17.59893
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0.015625	50.40121
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What is going on? $y(3)$ does not exist, $y \longrightarrow \infty$ as $x \longrightarrow 3$.

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Numerical methods are still a current research topic.

The simplest method used in practice is the *Runge–Kutta method*. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h.

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The idea is the same as Euler except each step looks like

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f\big(x_i + {}^{h}\!/_{2}, y_i + k_1({}^{h}\!/_{2})\big), \\ k_3 &= f\big(x_i + {}^{h}\!/_{2}, y_i + k_2({}^{h}\!/_{2})\big), \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned} \qquad \begin{aligned} x_{i+1} &= x_i + h, \\ y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\, h, \end{aligned}$$