14. Constant coefficient second order linear ODEs (part 1) (Notes on Diffy Qs, 2.2)

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The textbook: `https://www.jirka.org/diffyqs/`

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Exercise: Check that y_1 and y_2 are solutions.

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So the solution is: $\quad y = -4e^{2x} + 5e^{3x}$

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Suppose that r_1 and r_2 are the roots of the characteristic equation.

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(ii) *If $r_1 = r_2$ (when $b^2 - 4ac = 0$), the general solution is*

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We have a double root $r_1 = r_2 = 4$. The general solution is

$$y = (C_1 + C_2 x)\, e^{4x} = C_1 e^{4x} + C_2 x e^{4x}.$$

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