24. The Laplace transform, part 2
(Notes on Diffy Qs, 6.1)

Jiří Lebl

Oklahoma State University

The textbook: `https://www.jirka.org/diffyqs/`

As we said, not all functions have Laplace transforms. But many do.

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \leq Me^{ct}, \qquad \text{for sufficiently large } t.$$

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \leq Me^{ct}, \qquad \text{for sufficiently large } t.$$

One way to check this condition is to compute

$$\lim_{t \to \infty} \frac{f(t)}{e^{ct}}.$$

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \leq Me^{ct}, \qquad \text{for sufficiently large } t.$$

One way to check this condition is to compute

$$\lim_{t \to \infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \leq Me^{ct}, \qquad \text{for sufficiently large } t.$$

One way to check this condition is to compute

$$\lim_{t \to \infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

A sum of exponential order functions is of exponential order.

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \le Me^{ct}, \qquad \text{for sufficiently large } t.$$

One way to check this condition is to compute

$$\lim_{t \to \infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

A sum of exponential order functions is of exponential order.

Every polynomial is of exponential order.

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \leq Me^{ct}, \qquad \text{for sufficiently large } t.$$

One way to check this condition is to compute

$$\lim_{t \to \infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

A sum of exponential order functions is of exponential order.

Every polynomial is of exponential order.

e^{at} is of exponential order for any a.

As we said, not all functions have Laplace transforms. But many do.

A function $f(t)$ is of *exponential order* as $t \to \infty$ if there are constants M and c such that

$$|f(t)| \le Me^{ct}, \qquad \text{for sufficiently large } t.$$

One way to check this condition is to compute

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

A sum of exponential order functions is of exponential order.

Every polynomial is of exponential order.

e^{at} is of exponential order for any a.

e^{t^2} is **not** of exponential order for any a.

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The theorem is not difficult to see:

Suppose $|f(t)| \leq Me^{ct}$ for all $t > 0$. Let $s > c$, or in other words $(s - c) > 0$.

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The theorem is not difficult to see:

Suppose $|f(t)| \leq Me^{ct}$ for all $t > 0$. Let $s > c$, or in other words $(s - c) > 0$.

By the comparison theorem from calculus, the improper integral $\mathcal{L}\{f(t)\}$ exists because the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})\,dt$$

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The theorem is not difficult to see:

Suppose $|f(t)| \leq Me^{ct}$ for all $t > 0$. Let $s > c$, or in other words $(s - c) > 0$.

By the comparison theorem from calculus, the improper integral $\mathcal{L}\{f(t)\}$ exists because the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})\, dt = M \int_0^\infty e^{-(s-c)t}\, dt$$

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The theorem is not difficult to see:

Suppose $|f(t)| \leq Me^{ct}$ for all $t > 0$. Let $s > c$, or in other words $(s - c) > 0$.

By the comparison theorem from calculus, the improper integral $\mathcal{L}\{f(t)\}$ exists because the following integral exists

$$\int_0^{\infty} e^{-st}(Me^{ct})\,dt = M\int_0^{\infty} e^{-(s-c)t}\,dt = M\left[\frac{e^{-(s-c)t}}{-(s-c)}\right]_{t=0}^{\infty}$$

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The theorem is not difficult to see:

Suppose $|f(t)| \le Me^{ct}$ for all $t > 0$. Let $s > c$, or in other words $(s - c) > 0$.

By the comparison theorem from calculus, the improper integral $\mathcal{L}\{f(t)\}$ exists because the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})\,dt = M\int_0^\infty e^{-(s-c)t}\,dt = M\left[\frac{e^{-(s-c)t}}{-(s-c)}\right]_{t=0}^{\infty} = \frac{M}{s-c}.$$

Theorem (Existence)**:** Let $f(t)$ be continuous on the interval $[0, \infty)$ and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The theorem is not difficult to see:

Suppose $|f(t)| \leq Me^{ct}$ for all $t > 0$. Let $s > c$, or in other words $(s - c) > 0$.

By the comparison theorem from calculus, the improper integral $\mathcal{L}\{f(t)\}$ exists because the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})\,dt = M\int_0^\infty e^{-(s-c)t}\,dt = M\left[\frac{e^{-(s-c)t}}{-(s-c)}\right]_{t=0}^\infty = \frac{M}{s-c}.$$

Moreover, if f is of exponential order, then

$$\lim_{s\to\infty} F(s) = 0.$$

Theorem (Uniqueness)**:** Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

Theorem (Uniqueness)**:** Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

The theorems also hold for piecewise continuous functions: functions with a discrete set of jump discontinuities like the Heaviside.

However, for uniqueness, we can only conclude $f(t) = g(t)$ outside of discontinuities.

Theorem (Uniqueness)**:** Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

The theorems also hold for piecewise continuous functions: functions with a discrete set of jump discontinuities like the Heaviside.

However, for uniqueness, we can only conclude $f(t) = g(t)$ outside of discontinuities.

Uniqueness says it makes sense to invert the transform: Given $F(s)$, find the unique $f(t)$.

Theorem (Uniqueness)**:** Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

The theorems also hold for piecewise continuous functions: functions with a discrete set of jump discontinuities like the Heaviside.

However, for uniqueness, we can only conclude $f(t) = g(t)$ outside of discontinuities.

Uniqueness says it makes sense to invert the transform: Given $F(s)$, find the unique $f(t)$.

Suppose $F(s) = \mathcal{L}\{f(t)\}$. Define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \stackrel{\text{def}}{=} f(t).$$

Theorem (Uniqueness)**:** Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

The theorems also hold for piecewise continuous functions: functions with a discrete set of jump discontinuities like the Heaviside.

However, for uniqueness, we can only conclude $f(t) = g(t)$ outside of discontinuities.

Uniqueness says it makes sense to invert the transform: Given $F(s)$, find the unique $f(t)$.

Suppose $F(s) = \mathcal{L}\left\{f(t)\right\}$. Define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\left\{F(s)\right\} \stackrel{\text{def}}{=} f(t).$$

There is an integral formula for the inverse, but it is complicated.
We will be satisfied with using the table of transforms we computed.

Theorem (Uniqueness)**:** Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

The theorems also hold for piecewise continuous functions: functions with a discrete set of jump discontinuities like the Heaviside.

However, for uniqueness, we can only conclude $f(t) = g(t)$ outside of discontinuities.

Uniqueness says it makes sense to invert the transform: Given $F(s)$, find the unique $f(t)$.

Suppose $F(s) = \mathcal{L}\{f(t)\}$. Define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \overset{\text{def}}{=} f(t).$$

There is an integral formula for the inverse, but it is complicated.
We will be satisfied with using the table of transforms we computed.

Example: Via the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s)+BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2 + s + 1}{s^3 + s}$$

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{s^2 + s + 1}{s(s^2 + 1)}$$

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{s^2 + s + 1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

The inverse is also linear:

$$\mathcal{L}^{-1}\left\{AF(s) + BG(s)\right\} = A\mathcal{L}^{-1}\left\{F(s)\right\} + B\mathcal{L}^{-1}\left\{G(s)\right\}.$$

Of course, also $\mathcal{L}^{-1}\left\{AF(s)\right\} = A\mathcal{L}^{-1}\left\{F(s)\right\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2+s+1}{s^3+s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Clear denominators to find $s^2 + s + 1 = A(s^2+1) + s(Bs+C)$.

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{s^2 + s + 1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Clear denominators to find $s^2 + s + 1 = A(s^2 + 1) + s(Bs + C)$. Expand and equate coefficients to find $A + B = 1$, $C = 1$, $A = 1$, and $B = 0$.

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2+s+1}{s^3+s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Clear denominators to find $s^2+s+1 = A(s^2+1) + s(Bs+C)$. Expand and equate coefficients to find $A+B=1$, $C=1$, $A=1$, and $B=0$. That is,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2+s+1}{s^3+s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}.$$

Clear denominators to find $s^2+s+1 = A(s^2+1) + s(Bs+C)$. Expand and equate coefficients to find $A+B=1$, $C=1$, $A=1$, and $B=0$. That is,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

By linearity,

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\}$$

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{s^2 + s + 1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Clear denominators to find $s^2 + s + 1 = A(s^2 + 1) + s(Bs + C)$. Expand and equate coefficients to find $A + B = 1$, $C = 1$, $A = 1$, and $B = 0$. That is,

$$F(s) = \frac{s^2 + s + 1}{s^3 + s} = \frac{1}{s} + \frac{1}{s^2 + 1}.$$

By linearity,

$$\mathcal{L}^{-1}\left\{\frac{s^2 + s + 1}{s^3 + s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

The inverse is also linear:

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, also $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$.

Example: Let us find the inverse Laplace transform of $F(s) = \dfrac{s^2 + s + 1}{s^3 + s}$.

Use the *method of partial fractions* to split up $F(s)$:

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{s^2 + s + 1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Clear denominators to find $s^2 + s + 1 = A(s^2 + 1) + s(Bs + C)$. Expand and equate coefficients to find $A + B = 1$, $C = 1$, $A = 1$, and $B = 0$. That is,

$$F(s) = \frac{s^2 + s + 1}{s^3 + s} = \frac{1}{s} + \frac{1}{s^2 + 1}.$$

By linearity,

$$\mathcal{L}^{-1}\left\{\frac{s^2 + s + 1}{s^3 + s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = 1 + \sin t.$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

Next find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

Next find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

Next find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t).$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

Next find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t).$$

Now put it together with the shifting property

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at} f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

Next find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t).$$

Now put it together with the shifting property

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\}$$

A useful property is the *shifting property* (or *first shifting property*). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a).$$

This property is useful if the denominator is more complicated, e.g., an irreducible quadratic. Then complete the square $(s+a)^2 + b$ and use the shifting property.

Example: Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$.

Complete the square in the denominator: $s^2 + 4s + 8 = (s+2)^2 + 4$.

Next find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t).$$

Now put it together with the shifting property

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+4}\right\} = \frac{1}{2}e^{-2t}\sin(2t).$$

We often try to apply the inverse Laplace transform to rational functions:

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials.

We often try to apply the inverse Laplace transform to rational functions:

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials.

If $\frac{F(s)}{G(s)}$ is the Laplace transform of an exponential order function, it goes to zero as $s \to \infty$:

We often try to apply the inverse Laplace transform to rational functions:

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials.

If $\frac{F(s)}{G(s)}$ is the Laplace transform of an exponential order function, it goes to zero as $s \longrightarrow \infty$:
So the degree of $F(s)$ is smaller than that of $G(s)$. That is, $\frac{F(s)}{G(s)}$ is a *proper rational function*.

We often try to apply the inverse Laplace transform to rational functions:

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials.

If $\frac{F(s)}{G(s)}$ is the Laplace transform of an exponential order function, it goes to zero as $s \longrightarrow \infty$:
So the degree of $F(s)$ is smaller than that of $G(s)$. That is, $\frac{F(s)}{G(s)}$ is a *proper rational function*.

For proper rational functions, partial fractions method applies without polynomial division, and the techniques above always work.

We often try to apply the inverse Laplace transform to rational functions:

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials.

If $\frac{F(s)}{G(s)}$ is the Laplace transform of an exponential order function, it goes to zero as $s \longrightarrow \infty$:
So the degree of $F(s)$ is smaller than that of $G(s)$. That is, $\frac{F(s)}{G(s)}$ is a *proper rational function*.

For proper rational functions, partial fractions method applies without polynomial division, and the techniques above always work.

Though for partial fractions, you still need to factor the denominator, which can be hard (finding roots).