18. Mechanical vibrations, part 2: free damped motion (Notes on Diffy Qs, 2.4)

Jiří Lebl

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The textbook: `https://www.jirka.org/diffyqs/`

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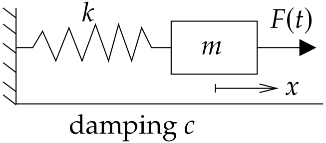

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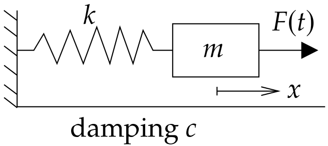

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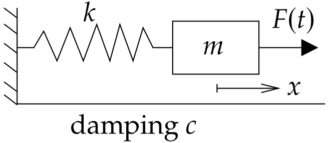

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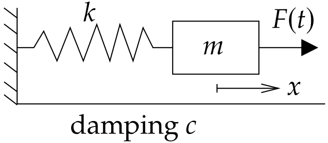

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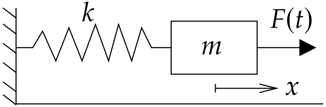

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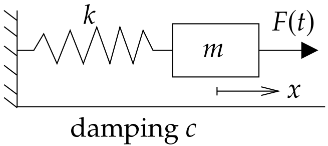

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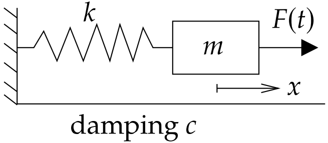

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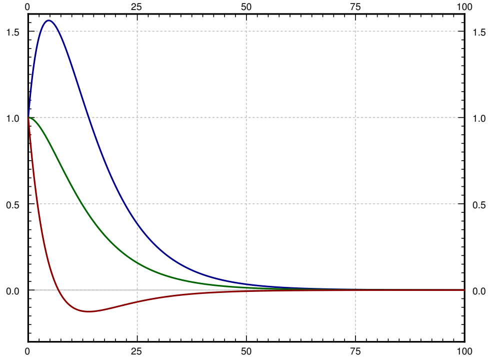

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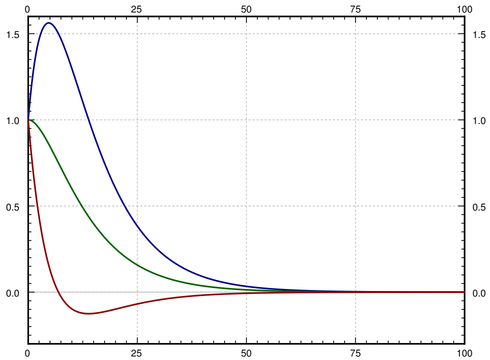

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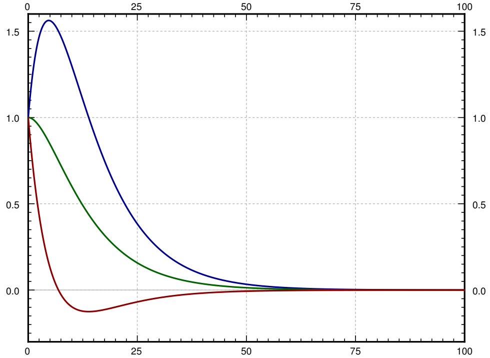

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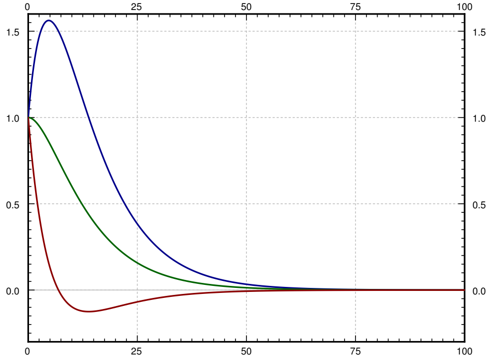

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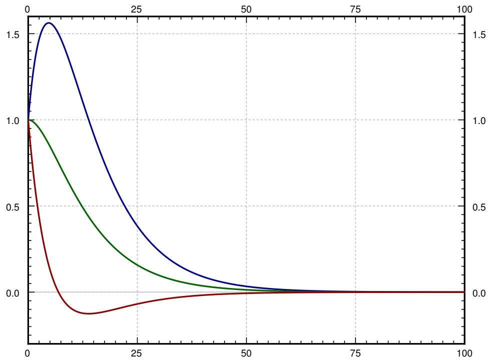

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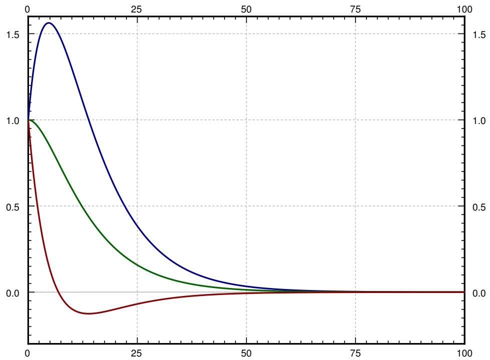

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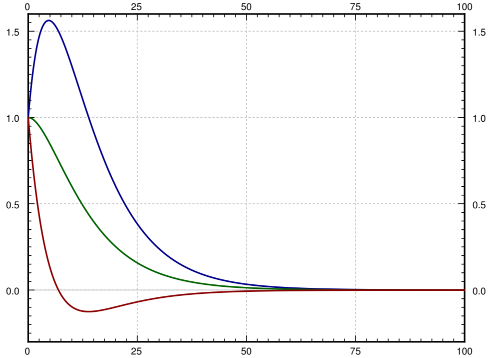

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Example: The mass is released from rest at position x_0: $x(0) = x_0$ and $x'(0) = 0$.

$$x(t) = \frac{x_0}{r_1 - r_2} \left(r_1 e^{r_2 t} - r_2 e^{r_1 t} \right).$$

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Everything is an approximation of reality, so best not to dwell on this edge case.

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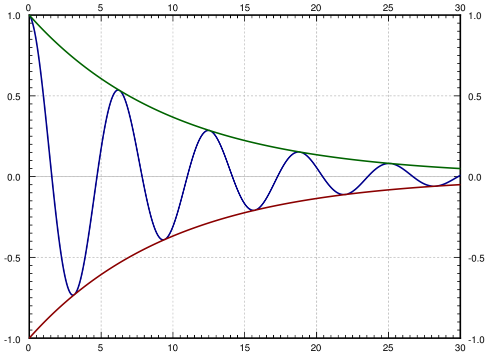

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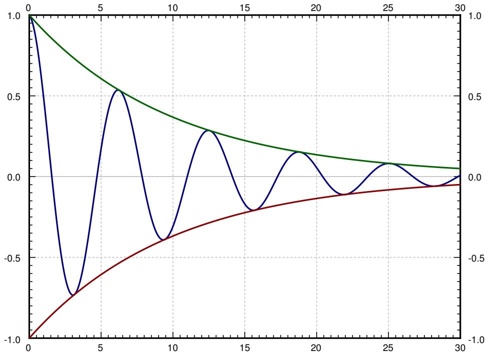

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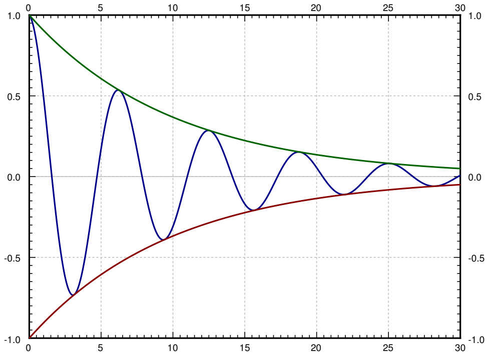

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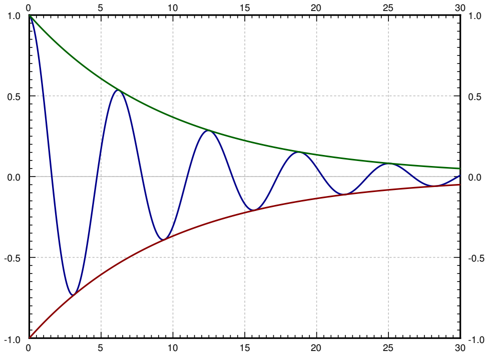

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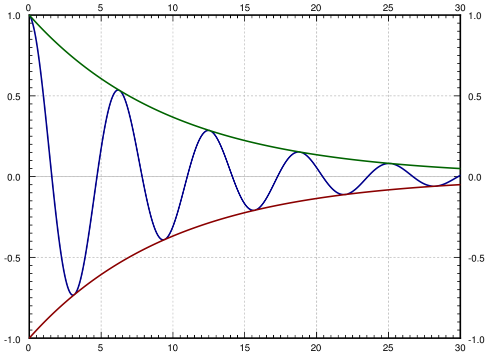

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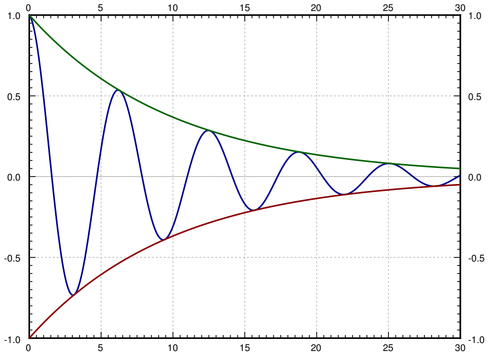

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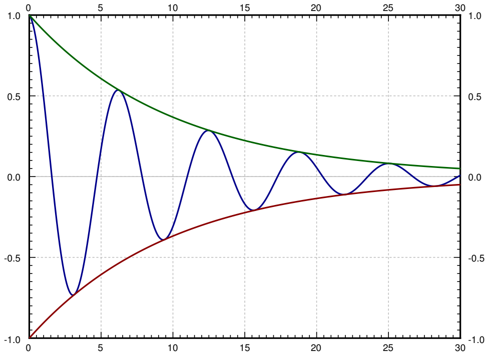

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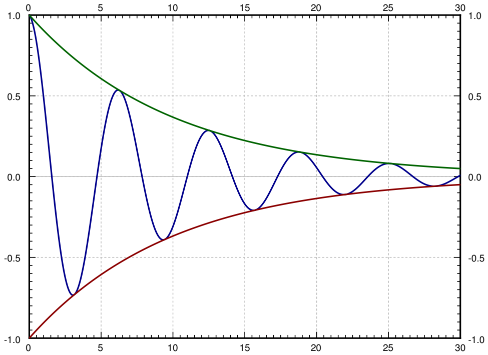

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