22. Forced oscillations and resonance, part 2: Damped forced motion and practical resonance (Notes on Diffy Qs, 2.6)

Jiří Lebl

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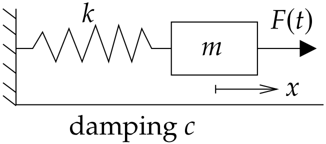
The textbook: `https://www.jirka.org/diffyqs/`

Recall our setup:



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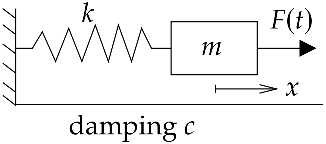

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$$mx'' + cx' + kx = F(t),$$

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k is the spring constant, and
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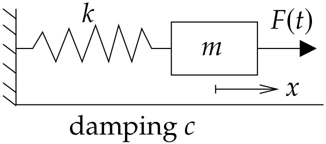

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Roots of the characteristic equation are $-p \pm \sqrt{p^2 - \omega_0^2}$, so the complementary solution is

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } c^2 > 4km, \\ C_1 e^{-pt} + C_2 t e^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt}\big(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\big) & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$.

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where $\omega_1 = \sqrt{\omega_0^2 - p^2}$.

Note that $x_c(t) \to 0$ as $t \to \infty$.

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$$\big((\omega_0^2 - \omega^2)B - 2\omega pA\big)\sin(\omega t) + \big((\omega_0^2 - \omega^2)A + 2\omega pB\big)\cos(\omega t) = \frac{F_0}{m}\cos(\omega t).$$

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Solve for A and B:

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}, \qquad B = \frac{2\omega pF_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}.$$

Compute the amplitude of x_p:

$$C = \sqrt{A^2 + B^2} = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}.$$

So our particular solution is

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x_p &= A\cos(\omega t) + B\sin(\omega t) \\
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Remark: γ can be computed as before, e.g. if $A \neq 0$ ($\omega \neq \omega_0$), then $\tan\gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}$. If $A = 0$, then $B = C = \frac{F_0}{2m\omega p}$, and $\gamma = \pi/2$.

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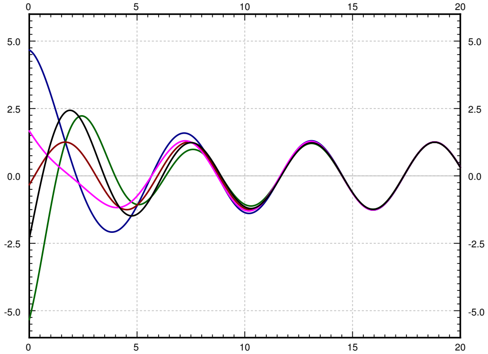
$x_{tr} \to 0$ as $t \to \infty$, so for large t, x_{tr} is negligible. Let's ignore x_{tr}.



$k = 1$, $m = 1$, $F_0 = 1$, $c = 0.7$, and $\omega = 1.1$.

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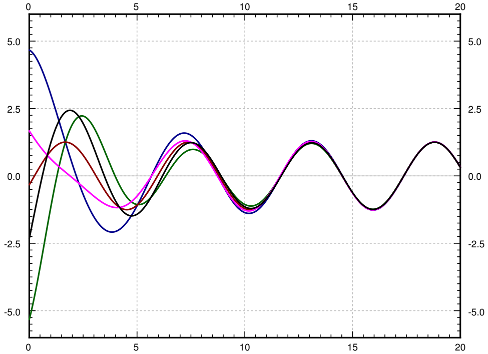
$\Rightarrow \quad$ smaller "transient region."



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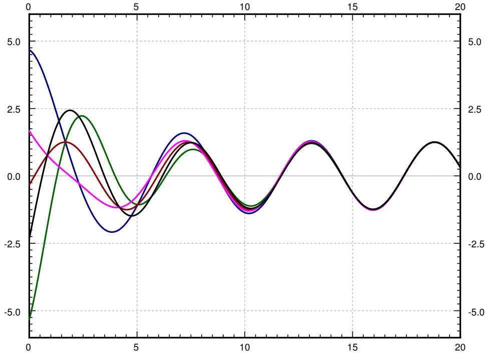

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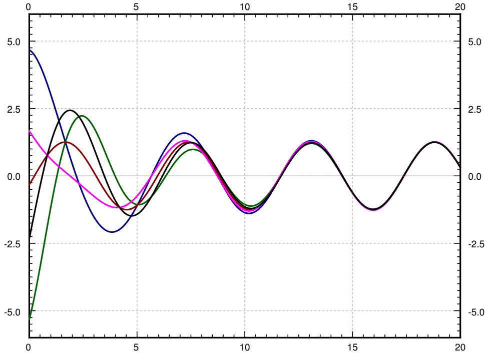

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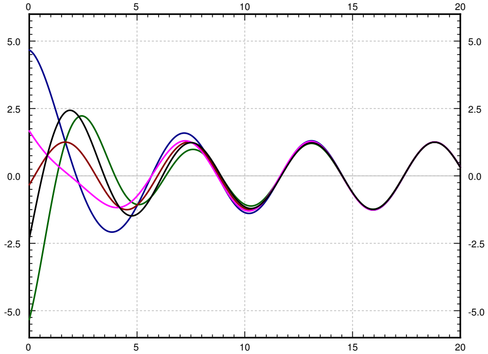

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Top line: $c = 0.4$, middle line: $c = 0.8$,
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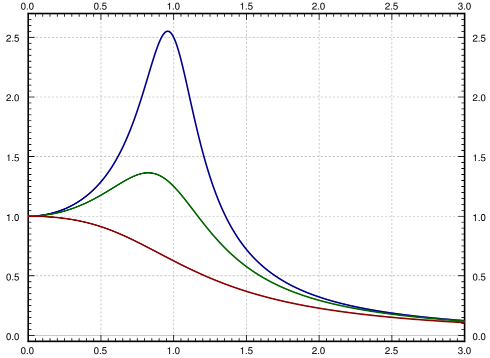

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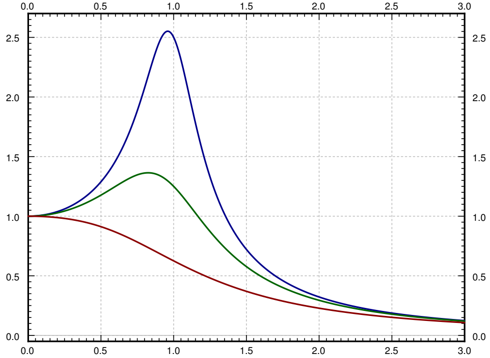

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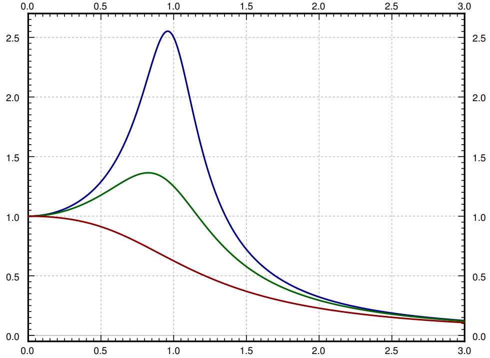

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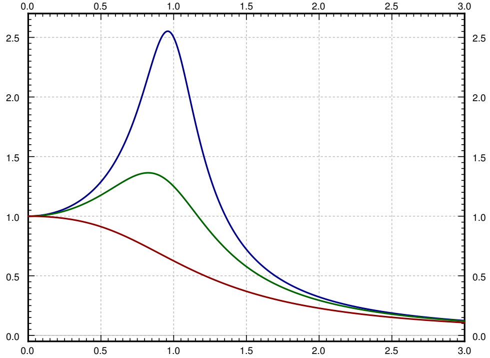

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We call this *practical resonance*.

Practical resonance gets bigger if c gets smaller, and disappears when damping is too large.

To find the max of $$C(\omega) = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}},$$

find where $$0 = C'(\omega) = \frac{-2\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m\left((2\omega p)^2 + (\omega_0^2 - \omega^2)^2\right)^{3/2}}.$$

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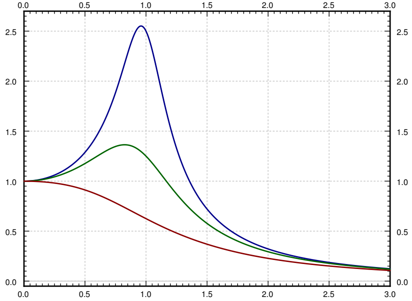

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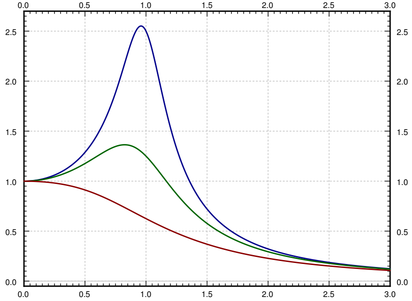

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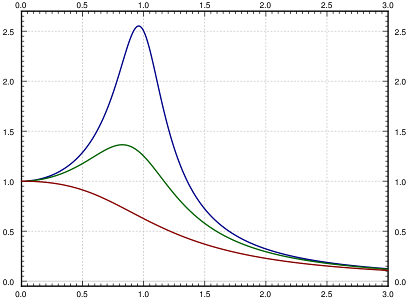

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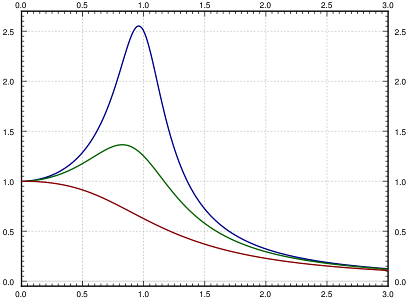

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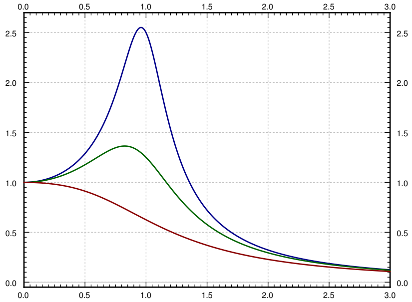

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(Indeed, if $c = 0$, ω_0 is the (pure) resonance frequency.)

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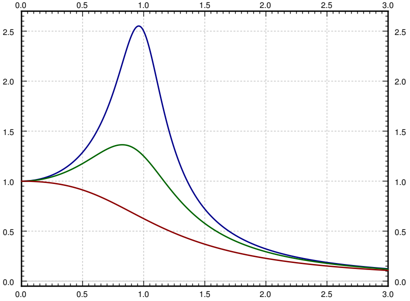

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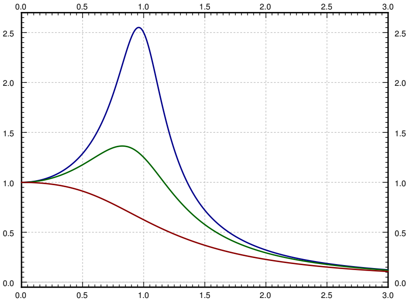

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Remark: $C(\omega) \to 0$ as $\omega \to \infty$. $\quad\Rightarrow\quad$ Large forcing frequency means small amplitude.