

12. Exact equations (part 2) (Notes on Diffy Qs, 1.8)

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The textbook: <https://www.jirka.org/diffyqs/>

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(Or reverse the roles of x and y and M and N).

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$$y^2 = \frac{C - (1/3)x^3}{x+1}, \quad \text{so} \quad y = \pm \sqrt{\frac{C - (1/3)x^3}{x+1}}.$$

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One could solve for x in terms of y for any initial condition (messy).

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$$\text{Let } Q(y) = \frac{M_y - N_x}{M} \Rightarrow u(y) = e^{-\int Q(y) dy}.$$

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$$\text{Notice } \frac{M_y - N_x}{N} = \frac{2y}{x + 1} \frac{1}{2y} = \frac{1}{x + 1} = P(x) \text{ is a function of } x \text{ alone.}$$

$$\text{Compute } u(x) = e^{\int P(x) dx} = e^{\ln(x+1)} = x + 1.$$

$$\text{The equation becomes: } x^2 + y^2 + 2y(x + 1) \frac{dy}{dx} = 0.$$

This is exact and we solved it a moment ago:

$$y = \pm \sqrt{\frac{C - (1/3)x^3}{x + 1}}$$

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