

11. Exact equations (part 1)

(Notes on Diffy Qs, 1.8)

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The textbook: <https://www.jirka.org/diffyqs/>

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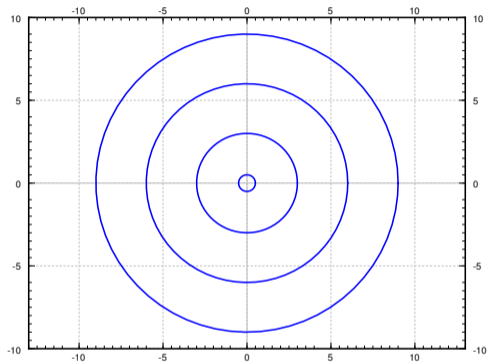
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Such equations come up when there is some conservation law at play.
(e.g., conservation of energy)

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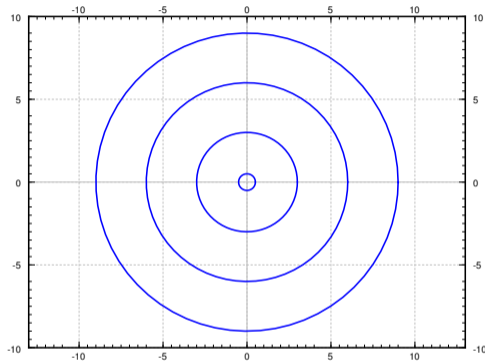


Solutions to $F(x, y) = x^2 + y^2 = C$.

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$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = F_x dx + F_y dy.$$



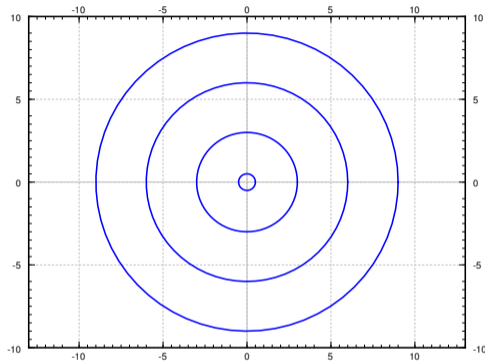
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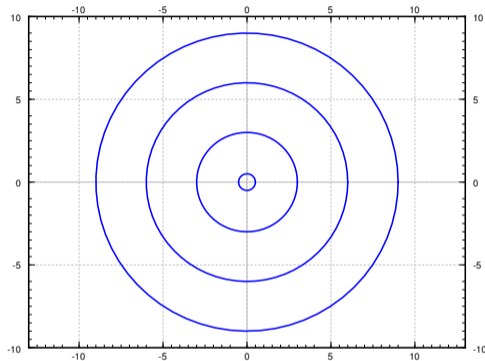
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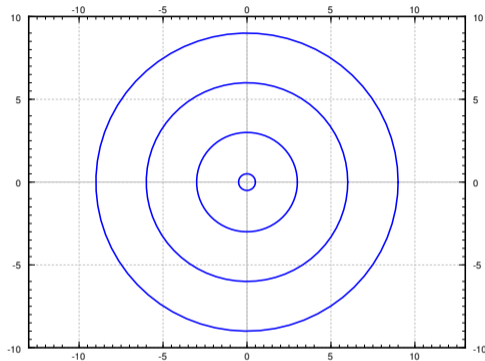
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In this case,

$$2x dx + 2y dy = 0 \quad \text{or} \quad 2x + 2y \frac{dy}{dx} = 0$$



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Normally, we start with $M dx + N dy = 0$ and we wish to find the unknown $F(x, y)$.

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Let γ be a path in the plane from (x_1, y_1) and ending at (x_2, y_2) and think of \vec{v} as force.

The work required to move along γ is
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Exact equations are conservative vector fields, and their implicit solutions are given by the potential functions.

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If $F(x, y)$ works, $F(x, y) + 3$ or $F(x, y) - 8$ also work.

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Roles of x and y (and so M and N) can be reversed.

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The equation is not exact! F does not exist.

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If M and N are continuously differentiable functions of (x, y) , and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

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$M_y = 1 = N_x$ so the equation is exact.

Integrate M in x : $F(x, y) = x^2 + xy + A(y)$

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Finally, solve $x^2 + xy - y = -1$ for y :

$$y = \frac{-x^2 - 1}{x - 1}.$$

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As $y(1) = 2$, we get $\tan(C) = 2$,

so $y = 2x$ is the solution (only for $x > 0$).

