

# Notes on Diffy Qs

---

*Differential Equations for Engineers*

by Jiří Lebl

October 24, 2011

Typeset in L<sup>A</sup>T<sub>E</sub>X.

Copyright ©2008–2011 Jiří Lebl



This work is licensed under the Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/3.0/us/> or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.

You can use, print, duplicate, share these notes as much as you want. You can base your own notes on these and reuse parts if you keep the license the same. If you plan to use these commercially (sell them for more than just duplicating cost), then you need to contact me and we will work something out. If you are printing a course pack for your students, then it is fine if the duplication service is charging a fee for printing and selling the printed copy. I consider that duplicating cost.

During the writing of these notes, the author was in part supported by NSF grant DMS-0900885.

See <http://www.jirka.org/diffyqs/> for more information (including contact information).

# Contents

<b>Introduction</b>	<b>5</b>
0.1 Notes about these notes . . . . .	5
0.2 Introduction to differential equations . . . . .	7
<b>1 First order ODEs</b>	<b>13</b>
1.1 Integrals as solutions . . . . .	13
1.2 Slope fields . . . . .	18
1.3 Separable equations . . . . .	22
1.4 Linear equations and the integrating factor . . . . .	27
1.5 Substitution . . . . .	32
1.6 Autonomous equations . . . . .	37
1.7 Numerical methods: Euler's method . . . . .	42
<b>2 Higher order linear ODEs</b>	<b>47</b>
2.1 Second order linear ODEs . . . . .	47
2.2 Constant coefficient second order linear ODEs . . . . .	51
2.3 Higher order linear ODEs . . . . .	57
2.4 Mechanical vibrations . . . . .	62
2.5 Nonhomogeneous equations . . . . .	70
2.6 Forced oscillations and resonance . . . . .	77
<b>3 Systems of ODEs</b>	<b>85</b>
3.1 Introduction to systems of ODEs . . . . .	85
3.2 Matrices and linear systems . . . . .	90
3.3 Linear systems of ODEs . . . . .	99
3.4 Eigenvalue method . . . . .	103
3.5 Two dimensional systems and their vector fields . . . . .	110
3.6 Second order systems and applications . . . . .	115
3.7 Multiple eigenvalues . . . . .	124
3.8 Matrix exponentials . . . . .	129
3.9 Nonhomogeneous systems . . . . .	136

<b>4</b>	<b>Fourier series and PDEs</b>	<b>149</b>
4.1	Boundary value problems . . . . .	149
4.2	The trigonometric series . . . . .	157
4.3	More on the Fourier series . . . . .	167
4.4	Sine and cosine series . . . . .	176
4.5	Applications of Fourier series . . . . .	184
4.6	PDEs, separation of variables, and the heat equation . . . . .	190
4.7	One dimensional wave equation . . . . .	201
4.8	D'Alembert solution of the wave equation . . . . .	208
4.9	Steady state temperature and the Laplacian . . . . .	214
<b>5</b>	<b>Eigenvalue problems</b>	<b>221</b>
5.1	Sturm-Liouville problems . . . . .	221
5.2	Application of eigenfunction series . . . . .	229
5.3	Steady periodic solutions . . . . .	233
<b>6</b>	<b>The Laplace transform</b>	<b>241</b>
6.1	The Laplace transform . . . . .	241
6.2	Transforms of derivatives and ODEs . . . . .	248
6.3	Convolution . . . . .	256
<b>7</b>	<b>Power series methods</b>	<b>261</b>
7.1	Power series . . . . .	261
7.2	Series solutions of linear second order ODEs . . . . .	269
	<b>Further Reading</b>	<b>277</b>
	<b>Solutions to Selected Exercises</b>	<b>279</b>
	<b>Index</b>	<b>287</b>

# Introduction

## 0.1 Notes about these notes

This book originated from my class notes for teaching Math 286, differential equations, at the University of Illinois at Urbana-Champaign in fall 2008 and spring 2009. It is a first course on differential equations for engineers. I have also taught Math 285 at UIUC and Math 20D at UCSD using a subset of this book. The standard book for the UIUC course is Edwards and Penney, *Differential Equations and Boundary Value Problems* [EP], fourth edition. Some examples and applications are taken more or less from this book, though they also appear in many other sources, of course. Among other books I have used as sources of information and inspiration are E.L. Ince's classic (and inexpensive) *Ordinary Differential Equations* [I], and also my undergraduate textbooks, Stanley Farlow's *Differential Equations and Their Applications* [F], which is now available from Dover, Berg and McGregor's *Elementary Partial Differential Equations* [BM], and Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems* [BD]. See the Further Reading chapter at the end of the book.

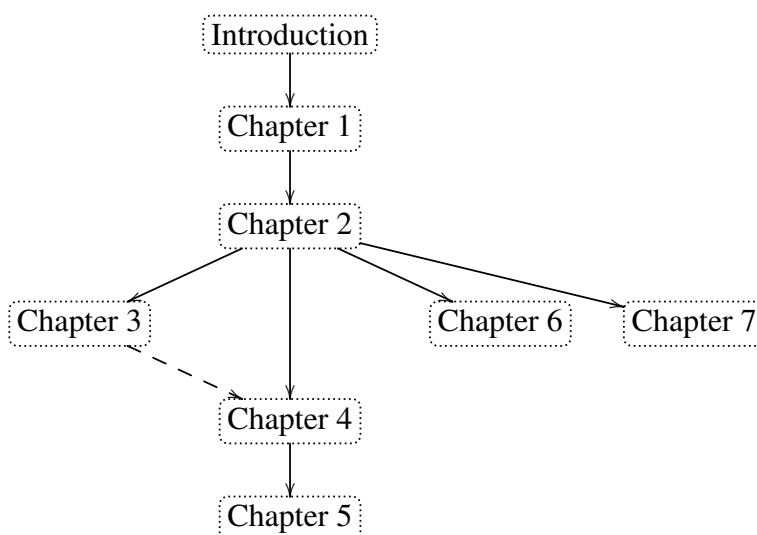
I taught the UIUC courses with the IODE software (<http://www.math.uiuc.edu/iode/>). IODE is a free software package that is used either with Matlab (proprietary) or Octave (free software). Projects and labs from the IODE website are referenced throughout the notes. They need not be used for this course, but I recommend using them. The graphs in the notes were made with the Genius software (see <http://www.jirka.org/genius.html>). I have used Genius in class to show these (and other) graphs.

These notes are available from <http://www.jirka.org/diffyqs/>. Check there for any possible updates or errata. The  $\text{\LaTeX}$  source is also available from the same site for possible modification and customization.

I would like to acknowledge Rick Laugesen. I have used his handwritten class notes the first time I taught Math 286. My organization of this book, and the choice of material covered, is heavily influenced by his class notes. Many examples and computations are taken from his notes. I am also heavily indebted to Rick for all the advice he has given me, not just on teaching Math 286. For spotting errors and other suggestions, I would also like to acknowledge (in no particular order): John P. D'Angelo, Sean Raleigh, Jessica Robinson, Michael Angelini, Leonardo Gomes, Jeff Winegar, Ian Simon, Thomas Wicklund, Eliot Brenner, Sean Robinson, Jannett Susberry, Dana Al-Quadi, Cesar Alvarez, Cem Bagdatlioglu, Nathan Wong, Alison Shive, Shawn White, Wing Yip Ho, Joanne

Shin, Gladys Cruz, Jonathan Gomez, Janelle Louie, Navid Froutan, Grace Victorine, Paul Pearson, Jared Teague, Ziad Adwan, Martin Weilandt, and probably others I have forgotten. Finally I would like to acknowledge NSF grant DMS-0900885.

The organization of these notes to some degree requires that they be done in order. Later chapters can be dropped. The dependence of the material covered is roughly given in the following diagram:



There are some references in chapters 4 and 5 to material from chapter 3 (some linear algebra), but these references are not absolutely essential and can be skimmed over, so chapter 3 can safely be dropped, while still covering chapters 4 and 5. The notes are done for two types of courses. Either at 4 hours a week for a semester (Math 286 at UIUC):

Introduction, chapter 1, chapter 2, chapter 3, chapter 4, chapter 5 (or 6 or 7).

Or a shorter version (Math 285 at UIUC) of the course at 3 hours a week for a semester:

Introduction, chapter 1, chapter 2, chapter 4, (and maybe chapter 5, 6, or 7).

The schedule assumes you spend two class periods in the computer lab with IODE. IODE need not be used for either version. If IODE is not used, some additional material (such as chapter 7) may need to be covered instead.

The lengths of the chapter on Laplace transform (chapter 6) and the chapter on Sturm-Liouville (chapter 5) are approximately the same and are interchangeable time-wise. Laplace transform is not normally covered at UIUC 285/286. I think it is essential that any notes on differential equations at least mention the Laplace transform. Power series (chapter 7) is a shorter chapter that may be easier to fit in if time is short.

## 0.2 Introduction to differential equations

*Note: more than 1 lecture, §1.1 in [EP], chapter 1 in [BD]*

### 0.2.1 Differential equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose that all your classes from now on were given in Swahili. Then it would be important to first learn Swahili, otherwise you will have a very tough time getting a good grade in your other classes.

You have already seen many differential equations without perhaps knowing about it. And you have even solved simple differential equations when you were taking calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2 \cos t. \quad (1)$$

Here  $x$  is the *dependent variable* and  $t$  is the *independent variable*. Equation (1) is a basic example of a *differential equation*. In fact it is an example of a *first order differential equation*, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

### 0.2.2 Solutions of differential equations

Solving the differential equation means finding  $x$  in terms of  $t$ . That is, we want to find a function of  $t$ , which we will call  $x$ , such that when we plug  $x$ ,  $t$ , and  $\frac{dx}{dt}$  into (1), the equation holds. It is the same idea as it would be for a normal (algebraic) equation of just  $x$  and  $t$ . We claim that

$$x = x(t) = \cos t + \sin t$$

is a *solution*. How do we check? We simply plug  $x$  into equation (1)! First we need to compute  $\frac{dx}{dt}$ . We find that  $\frac{dx}{dt} = -\sin t + \cos t$ . Now let us compute the left hand side of (1).

$$\frac{dx}{dt} + x = (-\sin t + \cos t) + (\cos t + \sin t) = 2 \cos t.$$

Yay! We got precisely the right hand side. But there is more! We claim  $x = \cos t + \sin t + e^{-t}$  is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

Again plugging into the left hand side of (1)

$$\frac{dx}{dt} + x = (-\sin t + \cos t - e^{-t}) + (\cos t + \sin t + e^{-t}) = 2 \cos t.$$

And it works yet again!

So there can be many different solutions. In fact, for this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t}$$

for some constant  $C$ . See Figure 1 for the graph of a few of these solutions. We will see how we can find these solutions a few lectures from now.

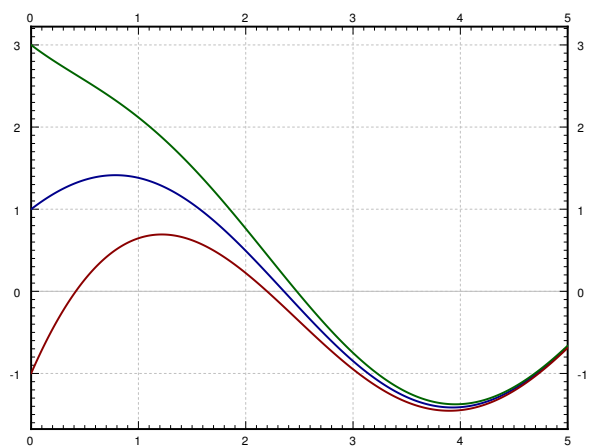


Figure 1: Few solutions of  $\frac{dx}{dt} + x = 2 \cos t$ .

It turns out that solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions.

For most of the course we will look at *ordinary differential equations* or ODEs, by which we mean that there is only one independent variable and derivatives are only with respect to this one variable. If there are several independent variables, we will get *partial differential equations* or PDEs. We will briefly see these near the end of the course.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. It is important to know when it is easy to find solutions and how to do so. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you have never seen before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

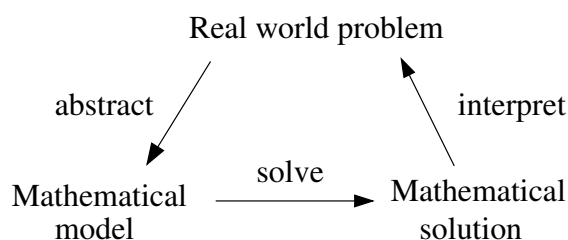
### 0.2.3 Differential equations in practice

So how do we use differential equations in science and engineering? First, we have some *real world problem* that we wish to understand. We make some simplifying assumptions and create a



*mathematical model*. That is, we translate the real world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real world problem we started with.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course we will focus mostly on the mathematical analysis. Sometimes we will work with simple real world examples, so that we have some intuition and motivation about what we are doing.



Let us look at an example of this process. One of the most basic differential equations is the standard *exponential growth model*. Let  $P$  denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria will be proportional to the population. I.e. a large population grows quicker. Let  $t$  denote time (say in seconds) and  $P$  the population. Our model will be

$$\frac{dP}{dt} = kP,$$

for some positive constant  $k > 0$ .

**Example 0.2.1:** Suppose there are 100 bacteria at time 0 and 200 bacteria at time 10s. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

First we have to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where  $C$  is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, so what now? We do not know  $C$  and we do not know  $k$ . But we know something. We know that  $P(0) = 100$ , and we also know that  $P(10) = 200$ . Let us plug these conditions in and see what happens.

$$\begin{aligned} 100 &= P(0) = Ce^{k0} = C, \\ 200 &= P(10) = 100e^{k10}. \end{aligned}$$

Therefore,  $2 = e^{10k}$  or  $\frac{\ln 2}{10} = k \approx 0.069$ . So we know that

$$P(t) = 100e^{(\ln 2)t/10} \approx 100e^{0.069t}.$$

At one minute,  $t = 60$ , the population is  $P(60) = 6400$ . See Figure 2 on the next page.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We have made assumptions that might not be exactly true. But if our assumptions are reasonable, then there will be approximately 6400 bacteria. Also note that in real life  $P$  is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds,  $P(61) \approx 6859.35$ .

Normally, the  $k$  in  $P' = kP$  will be known, and you will want to solve the equation for different *initial conditions*. What does that mean? Suppose  $k = 1$  for simplicity. Now suppose we want to solve the equation  $\frac{dP}{dt} = P$  subject to  $P(0) = 1000$  (the initial condition). Then the solution turns out to be (exercise)

$$P(t) = 1000 e^t.$$

We will call  $P(t) = Ce^t$  the *general solution*, as every solution of the equation can be written in this form for some constant  $C$ . Then you will need an initial condition to find out what  $C$  is to find the *particular solution* we are looking for. Generally, when we say “particular solution,” we just mean some solution.

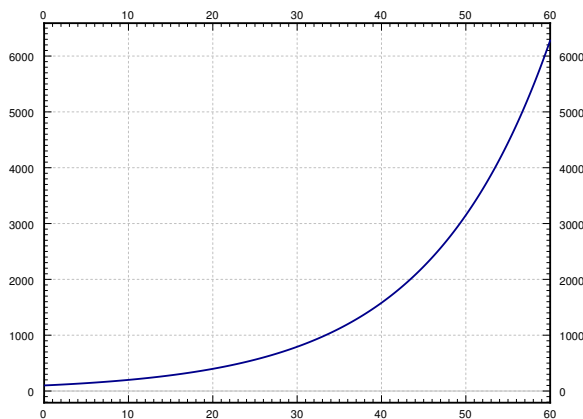


Figure 2: Bacteria growth in the first 60 seconds.

Let us get to what we will call the four fundamental equations. These equations appear very often and it is useful to just memorize what their solutions are. These solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. There is no need to wonder if you have remembered the solution correctly.

First such equation is,

$$\frac{dy}{dx} = ky,$$

for some constant  $k > 0$ . Here  $y$  is the dependent and  $x$  the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We have already seen that this function is a solution above with different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant  $k > 0$ . The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

**Exercise 0.2.1:** Check that the  $y$  given is really a solution to the equation.

Next, take the *second order differential equation*

$$\frac{d^2y}{dx^2} = -k^2y,$$

for some constant  $k > 0$ . The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Note that because we have a second order differential equation, we have two constants in our general solution.

**Exercise 0.2.2:** Check that the  $y$  given is really a solution to the equation.

And finally, take the second order differential equation

$$\frac{d^2y}{dx^2} = k^2y,$$

for some constant  $k > 0$ . The general solution for this equation is

$$y(x) = C_1 e^{kx} + C_2 e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know,  $\cosh$  and  $\sinh$  are defined by

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2}, \\ \sinh x &= \frac{e^x - e^{-x}}{2}.\end{aligned}$$

These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as  $\cosh 0 = 1$ ,  $\sinh 0 = 0$ , and  $\frac{d}{dx} \cosh x = \sinh x$  (no that is not a typo) and  $\frac{d}{dx} \sinh x = \cosh x$ .

**Exercise 0.2.3:** Check that both forms of the  $y$  given are really solutions to the equation.

An interesting note about  $\cosh$ : The graph of  $\cosh$  is the exact shape a hanging chain will make. This shape is called a *catenary*. Contrary to popular belief this is not a parabola. If you invert the graph of  $\cosh$  it is also the ideal arch for supporting its own weight. For example, the gateway arch in Saint Louis is an inverted graph of  $\cosh$  (if it were just a parabola it might fall down). The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

### 0.2.4 Exercises

**Exercise 0.2.4:** Show that  $x = e^{4t}$  is a solution to  $x''' - 12x'' + 48x' - 64x = 0$ .

**Exercise 0.2.5:** Show that  $x = e^t$  is not a solution to  $x''' - 12x'' + 48x' - 64x = 0$ .

**Exercise 0.2.6:** Is  $y = \sin t$  a solution to  $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$ ? Justify.

**Exercise 0.2.7:** Let  $y'' + 2y' - 8y = 0$ . Now try a solution of the form  $y = e^{rx}$  for some (unknown) constant  $r$ . Is this a solution for some  $r$ ? If so, find all such  $r$ .

**Exercise 0.2.8:** Verify that  $x = Ce^{-2t}$  is a solution to  $x' = -2x$ . Find  $C$  to solve for the initial condition  $x(0) = 100$ .

**Exercise 0.2.9:** Verify that  $x = C_1e^{-t} + C_2e^{2t}$  is a solution to  $x'' - x' - 2x = 0$ . Find  $C_1$  and  $C_2$  to solve for the initial conditions  $x(0) = 10$  and  $x'(0) = 0$ .

**Exercise 0.2.10:** Find a solution to  $(x')^2 + x^2 = 4$  using your knowledge of derivatives of functions that you know from basic calculus.

**Exercise 0.2.11:** Solve:

a)  $\frac{dA}{dt} = -10A, A(0) = 5.$

b)  $\frac{dH}{dx} = 3H, H(0) = 1.$

c)  $\frac{d^2y}{dx^2} = 4y, y(0) = 0, y'(0) = 1.$

d)  $\frac{d^2x}{dy^2} = -9x, x(0) = 1, x'(0) = 0.$

*Note: Exercises with numbers 101 and higher have solutions in the back*

**Exercise 0.2.101:** Show that  $x = e^{-2t}$  is a solution to  $x'' + 4x' + 4x = 0$ .

**Exercise 0.2.102:** Is  $y = x^2$  a solution to  $x^2y'' - 2y = 0$ ? Justify.

**Exercise 0.2.103:** Let  $xy'' - y' = 0$ . Try a solution of the form  $y = x^r$ . Is this a solution for some  $r$ ? If so, find all such  $r$ .

**Exercise 0.2.104:** Verify that  $x = C_1e^t + C_2$  is a solution to  $x'' - x' = 0$ . Find  $C_1$  and  $C_2$  so that  $x$  satisfies  $x(0) = 10$  and  $x'(0) = 100$ .

**Exercise 0.2.105:** Solve  $\frac{d\varphi}{ds} = 8\varphi$  and  $\varphi(0) = -9$ .

# Chapter 1

## First order ODEs

### 1.1 Integrals as solutions

*Note: 1 lecture (or less), §1.2 in [EP], covered in §1.2 and §2.1 in [BD]*

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y),$$

or just

$$y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that  $f$  is a function of  $x$  alone, that is, the equation is

$$y' = f(x). \tag{1.1}$$

We could just integrate (antidifferentiate) both sides with respect to  $x$ .

$$\int y'(x) dx = \int f(x) dx + C,$$

that is

$$y(x) = \int f(x) dx + C.$$

This  $y(x)$  is actually the general solution. So to solve (1.1), we find some antiderivative of  $f(x)$  and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite

integral. The indefinite integral is really the *antiderivative* (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write  $\int f(x) dx + C$  as

$$\int_{x_0}^x f(t) dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral.

**Example 1.1.1:** Find the general solution of  $y' = 3x^2$ .

Elementary calculus tells us that the general solution must be  $y = x^3 + C$ . Let us check:  $y' = 3x^2$ . We have gotten *precisely* our equation back.

Normally, we also have an initial condition such as  $y(x_0) = y_0$  for some two numbers  $x_0$  and  $y_0$  ( $x_0$  is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is  $y' = f(x)$ ,  $y(x_0) = y_0$ . Then the solution is

$$y(x) = \int_{x_0}^x f(s) ds + y_0. \quad (1.2)$$

Let us check! We compute  $y' = f(x)$  (by fundamental theorem of calculus) and by Jupiter,  $y$  is a solution. Is it the one satisfying the initial condition? Well,  $y(x_0) = \int_{x_0}^{x_0} f(x) dx + y_0 = y_0$ . It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (1.2) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

**Example 1.1.2:** Solve

$$y' = e^{-x^2}, \quad y(0) = 1.$$

By the preceding discussion, the solution must be

$$y(x) = \int_0^x e^{-s^2} ds + 1.$$

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y).$$

Let us write the equation in Leibniz notation.

$$\frac{dy}{dx} = f(y).$$

Now we use the inverse function theorem from calculus to switch the roles of  $x$  and  $y$  to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with  $dx$  and  $dy$ . It is tempting to just do algebra with  $dx$  and  $dy$  as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y) = \int \frac{1}{f(y)} dy + C.$$

Finally, we try to solve for  $y$ .

**Example 1.1.3:** Previously, we guessed  $y' = ky$  (for some  $k > 0$ ) has the solution  $y = Ce^{kx}$ . We can now find the solution without guessing. First we note that  $y = 0$  is a solution. Henceforth, we assume  $y \neq 0$ . We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln |y| + D,$$

where  $D$  is an arbitrary constant. Now we solve for  $y$  (actually for  $|y|$ ).

$$|y| = e^{kx-kD} = e^{-kD} e^{kx}.$$

If we replace  $e^{-kD}$  with an arbitrary constant  $C$  we can get rid of the absolute value bars (which we can do as  $D$  was arbitrary). In this way, we also incorporate the solution  $y = 0$ . We get the same general solution as we guessed before,  $y = Ce^{kx}$ .

**Example 1.1.4:** Find the general solution of  $y' = y^2$ .

First we note that  $y = 0$  is a solution. We can now assume that  $y \neq 0$ . Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for  $y = \frac{1}{C-x}$ . So the general solution is

$$y = \frac{1}{C-x} \quad \text{or} \quad y = 0.$$

Note the singularities of the solution. If for example  $C = 1$ , then the solution “blows up” as we approach  $x = 1$ . Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation  $y' = y^2$  is very nice and defined everywhere, but the solution is only defined on some interval  $(-\infty, C)$  or  $(C, \infty)$ .

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

**Example 1.1.5:** Suppose a car drives at a speed  $e^{t/2}$  meters per second, where  $t$  is time in seconds. How far did the car get in 2 seconds (starting at  $t = 0$ )? How far in 10 seconds?

Let  $x$  denote the distance the car traveled. The equation is

$$x' = e^{t/2}.$$

We can just integrate this equation to get that

$$x(t) = 2e^{t/2} + C.$$

We still need to figure out  $C$ . We know that when  $t = 0$ , then  $x = 0$ . That is,  $x(0) = 0$ . So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus  $C = -2$  and

$$x(t) = 2e^{t/2} - 2.$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \quad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}.$$

**Example 1.1.6:** Suppose that the car accelerates at a rate of  $t^2$  m/s<sup>2</sup>. At time  $t = 0$  the car is at the 1 meter mark and is traveling at 10 m/s. Where is the car at time  $t = 10$ .

Well this is actually a second order problem. If  $x$  is the distance traveled, then  $x'$  is the velocity, and  $x''$  is the acceleration. The equation with initial conditions is

$$x'' = t^2, \quad x(0) = 1, \quad x'(0) = 10.$$

What if we say  $x' = v$ . Then we have the problem

$$v' = t^2, \quad v(0) = 10.$$

Once we solve for  $v$ , we can integrate and find  $x$ .

**Exercise 1.1.1:** Solve for  $v$ , and then solve for  $x$ . Find  $x(10)$  to answer the question.



**1.1.1 Exercises**

**Exercise 1.1.2:** Solve  $\frac{dy}{dx} = x^2 + x$  for  $y(1) = 3$ .

**Exercise 1.1.3:** Solve  $\frac{dy}{dx} = \sin(5x)$  for  $y(0) = 2$ .

**Exercise 1.1.4:** Solve  $\frac{dy}{dx} = \frac{1}{x^2-1}$  for  $y(0) = 0$ .

**Exercise 1.1.5:** Solve  $y' = y^3$  for  $y(0) = 1$ .

**Exercise 1.1.6** (little harder): Solve  $y' = (y - 1)(y + 1)$  for  $y(0) = 3$ .

**Exercise 1.1.7:** Solve  $\frac{dy}{dx} = \frac{1}{y+1}$  for  $y(0) = 0$ .

**Exercise 1.1.8:** Solve  $y'' = \sin x$  for  $y(0) = 0$ .

**Exercise 1.1.9:** A spaceship is traveling at the speed  $2t^2 + 1$  km/s ( $t$  is time in seconds). It is pointing directly away from earth and at time  $t = 0$  it is 1000 kilometers from earth. How far from earth is it at one minute from time  $t = 0$ ?

**Exercise 1.1.10:** Solve  $\frac{dx}{dt} = \sin(t^2) + t$ ,  $x(0) = 20$ . It is OK to leave your answer as a definite integral.

**Exercise 1.1.101:** Solve  $\frac{dy}{dx} = e^x + x$  and  $y(0) = 10$ .

**Exercise 1.1.102:** Solve  $x' = \frac{1}{x^2}$ ,  $x(1) = 1$ .

**Exercise 1.1.103:** Solve  $x' = \frac{1}{\cos(x)}$ ,  $x(0) = \frac{\pi}{2}$ .

**Exercise 1.1.104:** Sid is in a car travelling at speed  $10t + 70$  miles per hour away from Las Vegas, where  $t$  is in hours. At  $t = 0$  the Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later.

## 1.2 Slope fields

*Note: 1 lecture, §1.3 in [EP], §1.1 in [BD]*

At this point it may be good to first try the Lab I and/or Project I from the IODE website: <http://www.math.uiuc.edu/iode/>.

As we said, the general first order equation we are studying looks like

$$y' = f(x, y).$$

In general, we cannot simply solve these kinds of equations explicitly. It would be good if we could at least figure out the shape and behavior of the solutions, or if we could even find approximate solutions for any equation.

### 1.2.1 Slope fields

As you have seen in IODE Lab I (if you did it), the equation  $y' = f(x, y)$  gives you a slope at each point in the  $(x, y)$ -plane. We can plot the slope at lots of points as a short line through the point  $(x, y)$  with the slope  $f(x, y)$ . See Figure 1.1.

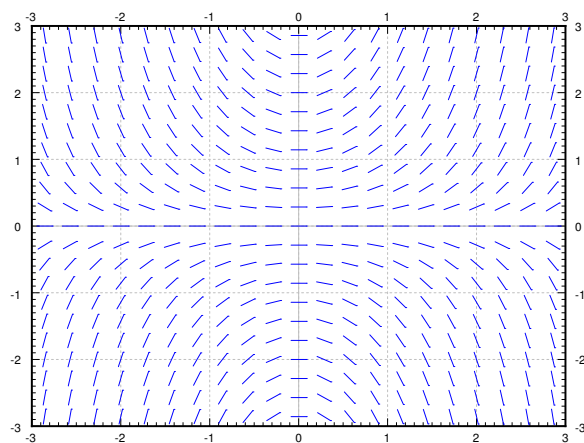


Figure 1.1: Slope field of  $y' = xy$ .

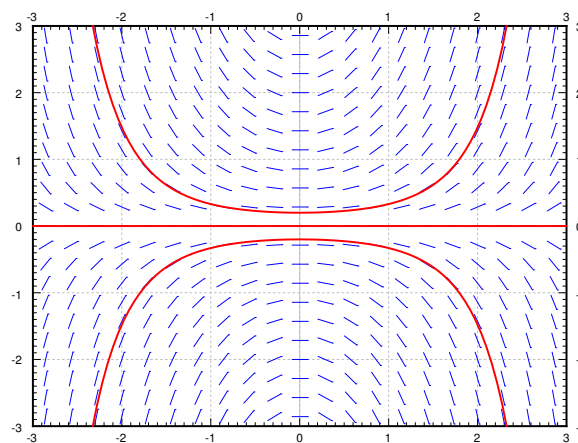


Figure 1.2: Slope field of  $y' = xy$  with a graph of solutions satisfying  $y(0) = 0.2$ ,  $y(0) = 0$ , and  $y(0) = -0.2$ .

We call this picture the *slope field* of the equation. If we are given a specific initial condition  $y(x_0) = y_0$ , we can look at the location  $(x_0, y_0)$  and follow the slopes. See Figure 1.2.

By looking at the slope field we can get a lot of information about the behavior of solutions. For example, in Figure 1.2 we can see what the solutions do when the initial conditions are  $y(0) > 0$ ,  $y(0) = 0$  and  $y(0) < 0$ . Note that a small change in the initial condition causes quite different

behavior. On the other hand, plotting a few solutions of the equation  $y' = -y$ , we see that no matter what  $y(0)$  is, all solutions tend to zero as  $x$  tends to infinity. See Figure 1.3.

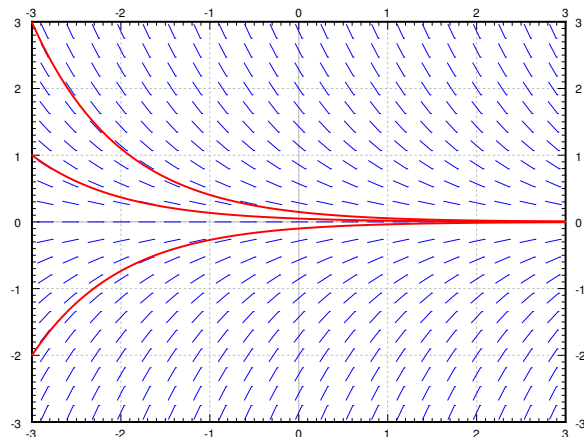


Figure 1.3: Slope field of  $y' = -y$  with a graph of a few solutions.

## 1.2.2 Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

- (i) Does a solution *exist*?
- (ii) Is the solution *unique* (if it exists)?

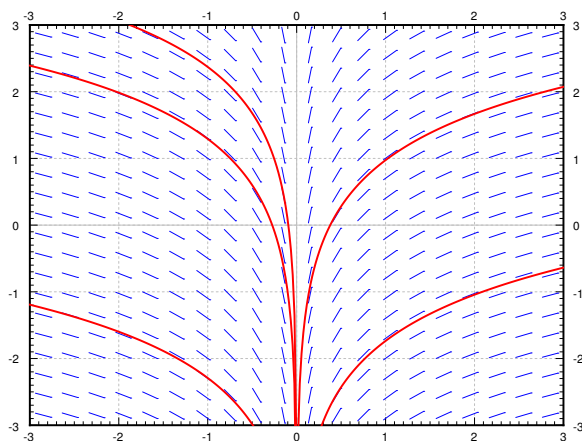
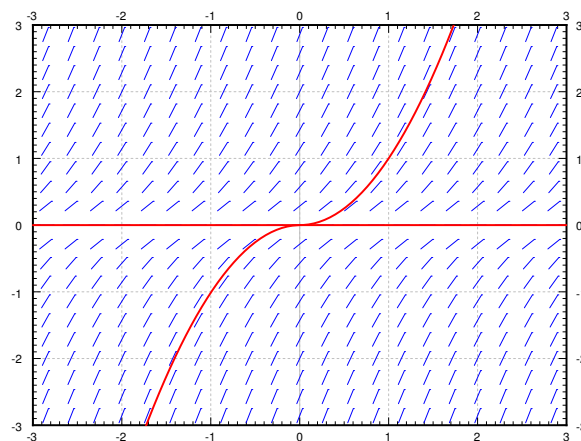
What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

**Example 1.2.1:** Attempt to solve:

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Integrate to find the general solution  $y = \ln|x| + C$ . Note that the solution does not exist at  $x = 0$ . See Figure 1.4 on the next page.

Figure 1.4: Slope field of  $y' = 1/x$ .Figure 1.5: Slope field of  $y' = 2\sqrt{|y|}$  with two solutions satisfying  $y(0) = 0$ .

**Example 1.2.2:** Solve:

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

See Figure 1.5. Note that  $y = 0$  is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

It is actually hard to tell by staring at the slope field that the solution will not be unique. Is there any hope? Of course there is. It turns out that the following theorem is true. It is known as Picard's theorem\*.

**Theorem 1.2.1** (Picard's theorem on existence and uniqueness). *If  $f(x, y)$  is continuous (as a function of two variables) and  $\frac{\partial f}{\partial y}$  exists and is continuous near some  $(x_0, y_0)$ , then a solution to*

$$y' = f(x, y), \quad y(x_0) = y_0,$$

*exists (at least for some small interval of  $x$ 's) and is unique.*

Note that the problems  $y' = 1/x$ ,  $y(0) = 0$  and  $y' = 2\sqrt{|y|}$ ,  $y(0) = 0$  do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

**Example 1.2.3:** For some constant  $A$ , solve:

$$y' = y^2, \quad y(0) = A.$$

---

\*Named after the French mathematician Charles Émile Picard (1856 – 1941)

We know how to solve this equation. First assume that  $A \neq 0$ , so  $y$  is not equal to zero at least for some  $x$  near 0. So  $x' = 1/y^2$ , so  $x = -1/y + C$ , so  $y = \frac{1}{C-x}$ . If  $y(0) = A$ , then  $C = 1/A$  so

$$y = \frac{1}{1/A - x}.$$

If  $A = 0$ , then  $y = 0$  is a solution.

For example, when  $A = 1$  the solution “blows up” at  $x = 1$ . Hence, the solution does not exist for all  $x$  even if the equation is nice everywhere. The equation  $y' = y^2$  certainly looks nice.

For the most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds “globally” unlike for the equation  $y' = y^2$ .

### 1.2.3 Exercises

**Exercise 1.2.1:** Sketch direction field for  $y' = e^{x-y}$ . How do the solutions behave as  $x$  grows? Can you guess a particular solution by looking at the direction field?

**Exercise 1.2.2:** Sketch direction field for  $y' = x^2$ .

**Exercise 1.2.3:** Sketch direction field for  $y' = y^2$ .

**Exercise 1.2.4:** Is it possible to solve the equation  $y' = \frac{xy}{\cos x}$  for  $y(0) = 1$ ? Justify.

**Exercise 1.2.5:** Is it possible to solve the equation  $y' = y\sqrt{|x|}$  for  $y(0) = 0$ ? Is the solution unique? Justify.

**Exercise 1.2.101:** Sketch the slope field of  $y' = y^3$ . Can you visually find the solution that satisfies  $y(0) = 0$ ?

**Exercise 1.2.102:** Is it possible to solve  $y' = xy$  for  $y(0) = 0$ ? Is the solution unique?

**Exercise 1.2.103:** Is it possible to solve  $y' = \frac{x}{x^2-1}$  for  $y(1) = 0$ ?

## 1.3 Separable equations

*Note: 1 lecture, §1.4 in [EP], §2.2 in [BD]*

When a differential equation is of the form  $y' = f(x)$ , we can just integrate:  $y = \int f(x) dx + C$ . Unfortunately this method no longer works for the general form of the equation  $y' = f(x, y)$ . Integrating both sides yields

$$y = \int f(x, y) dx + C.$$

Notice the dependence on  $y$  in the integral.

### 1.3.1 Separable equations

Let us suppose that the equation is *separable*. That is, let us consider

$$y' = f(x)g(y),$$

for some functions  $f(x)$  and  $g(y)$ . Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x) dx.$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x) dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for  $y$ .

**Example 1.3.1:** Take the equation

$$y' = xy.$$

First note that  $y = 0$  is a solution, so assume  $y \neq 0$  from now on. Write the equation as  $\frac{dy}{dx} = xy$ , then

$$\int \frac{dy}{y} = \int x dx + C.$$

We compute the antiderivatives to get

$$\ln |y| = \frac{x^2}{2} + C.$$

Or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} e^C = D e^{\frac{x^2}{2}},$$

where  $D > 0$  is some constant. Because  $y = 0$  is a solution and because of the absolute value we actually can write:

$$y = D e^{\frac{x^2}{2}},$$

for any number  $D$  (including zero or negative).

We check:

$$y' = D x e^{\frac{x^2}{2}} = x \left( D e^{\frac{x^2}{2}} \right) = xy.$$

Yay!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemed to be doing a different operation to each side. Let us work this method out more rigorously.

$$\frac{dy}{dx} = f(x)g(y)$$

We rewrite the equation as follows. Note that  $y = y(x)$  is a function of  $x$  and so is  $\frac{dy}{dx}$ !

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

We integrate both sides with respect to  $x$ .

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int f(x) dx + C.$$

We can use the change of variables formula.

$$\int \frac{1}{g(y)} dy = \int f(x) dx + C.$$

And we are done.

### 1.3.2 Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation

$$y' = \frac{xy}{y^2 + 1}.$$

We separate variables,

$$\frac{y^2 + 1}{y} dy = \left( y + \frac{1}{y} \right) dy = x dx.$$

We integrate to get

$$\frac{y^2}{2} + \ln |y| = \frac{x^2}{2} + C,$$

or perhaps the easier looking expression (where  $D = 2C$ )

$$y^2 + 2 \ln |y| = x^2 + D.$$

It is not easy to find the solution explicitly as it is hard to solve for  $y$ . We will, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that implicit solutions satisfy the differential equation. In this case, we differentiate to get

$$y' \left( 2y + \frac{2}{y} \right) = 2x.$$

It is simple to see that the differential equation holds. If you want to compute values for  $y$ , you might have to be tricky. For example, you can graph  $x$  as a function of  $y$ , and then flip your paper. Computers are also good at some of these tricks, but you have to be careful.

We note above that the equation also has a solution  $y = 0$ . In this case, it turns out that the general solution is  $y^2 + 2 \ln |y| = x^2 + C$  together with  $y = 0$ . These outlying solutions such as  $y = 0$  are sometimes called *singular solutions*.

### 1.3.3 Examples

**Example 1.3.2:** Solve  $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$ ,  $y(1) = 0$ .

First factor the right hand side to obtain

$$x^2 y' = (1 - x^2)(1 + y^2).$$

We separate variables, integrate and solve for  $y$

$$\begin{aligned} \frac{y'}{1 + y^2} &= \frac{1 - x^2}{x^2}, \\ \frac{y'}{1 + y^2} &= \frac{1}{x^2} - 1, \\ \arctan(y) &= \frac{-1}{x} - x + C, \\ y &= \tan \left( \frac{-1}{x} - x + C \right). \end{aligned}$$

Now solve for the initial condition,  $0 = \tan(-2 + C)$  to get  $C = 2$  (or  $2 + \pi$ , etc...). The solution we are seeking is, therefore,

$$y = \tan \left( \frac{-1}{x} - x + 2 \right).$$



**Example 1.3.3:** Suppose Bob made a cup of coffee, and the water was boiling (100 degrees Celsius) at time  $t = 0$  minutes. Suppose Bob likes to drink his coffee at 70 degrees. Let the ambient (room) temperature be 26 degrees. Furthermore, suppose Bob measured the temperature of the coffee at 1 minute and found that it dropped to 95 degrees. When should Bob start drinking?

Let  $T$  be the temperature of coffee, let  $A$  be the ambient (room) temperature. Then for some  $k$  the temperature of coffee is:

$$\frac{dT}{dt} = k(A - T).$$

For our setup  $A = 26$ ,  $T(0) = 100$ ,  $T(1) = 95$ . We separate variables and integrate ( $C$  and  $D$  will denote arbitrary constants)

$$\begin{aligned}\frac{1}{T - A} \frac{dT}{dt} &= -k, \\ \ln(T - A) &= -kt + C, \text{ (note that } T - A > 0\text{)} \\ T - A &= D e^{-kt}, \\ T &= D e^{-kt} + A.\end{aligned}$$

That is,  $T = 26 + D e^{-kt}$ . We plug in the first condition  $100 = T(0) = 26 + D$  and hence  $D = 74$ . We have  $T = 26 + 74 e^{-kt}$ . We plug in  $95 = T(1) = 26 + 74 e^{-k}$ . Solving for  $k$  we get  $k = -\ln \frac{95-26}{74} \approx 0.07$ . Now we solve for the time  $t$  that gives us a temperature of 70 degrees. That is, we solve  $70 = 26 + 74 e^{-0.07t}$  to get  $t = -\frac{\ln \frac{70-26}{74}}{0.07} \approx 7.43$  minutes. So Bob can begin to drink the coffee at about 7 and a half minutes from the time Bob made it. Probably about the amount of time it took us to calculate how long it would take.

**Example 1.3.4:** Find the general solution to  $y' = \frac{-xy^2}{3}$  (including singular solutions).

First note that  $y = 0$  is a solution (a singular solution). So assume that  $y \neq 0$  and write

$$\begin{aligned}\frac{-3}{y^2} y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}.\end{aligned}$$

### 1.3.4 Exercises

**Exercise 1.3.1:** Solve  $y' = x/y$ .

**Exercise 1.3.2:** Solve  $y' = x^2 y$ .

**Exercise 1.3.3:** Solve  $\frac{dx}{dt} = (x^2 - 1)t$ , for  $x(0) = 0$ .

**Exercise 1.3.4:** Solve  $\frac{dx}{dt} = x \sin(t)$ , for  $x(0) = 1$ .

**Exercise 1.3.5:** Solve  $\frac{dy}{dx} = xy + x + y + 1$ . Hint: Factor the right hand side.

**Exercise 1.3.6:** Solve  $xy' = y + 2x^2y$ , where  $y(1) = 1$ .

**Exercise 1.3.7:** Solve  $\frac{dy}{dx} = \frac{y^2 + 1}{x^2 + 1}$ , for  $y(0) = 1$ .

**Exercise 1.3.8:** Find an implicit solution for  $\frac{dy}{dx} = \frac{x^2 + 1}{y^2 + 1}$ , for  $y(0) = 1$ .

**Exercise 1.3.9:** Find explicit solution for  $y' = xe^{-y}$ ,  $y(0) = 1$ .

**Exercise 1.3.10:** Find explicit solution for  $xy' = e^{-y}$ , for  $y(1) = 1$ .

**Exercise 1.3.11:** Find explicit solution for  $y' = ye^{-x^2}$ ,  $y(0) = 1$ . It is alright to leave a definite integral in your answer.

**Exercise 1.3.12:** Suppose a cup of coffee is at 100 degrees Celsius at time  $t = 0$ , it is at 70 degrees at  $t = 10$  minutes, and it is at 50 degrees at  $t = 20$  minutes. Compute the ambient temperature.

**Exercise 1.3.101:** Solve  $y' = 2xy$ .

**Exercise 1.3.102:** Solve  $x' = 3xt^2 - 3t^2$ ,  $x(0) = 2$ .

**Exercise 1.3.103:** Find implicit solution for  $x' = \frac{1}{3x^2+1}$ ,  $x(0) = 1$ .

**Exercise 1.3.104:** Find explicit solution to  $xy' = y^2$ ,  $y(1) = 1$ .

## 1.4 Linear equations and the integrating factor

*Note: 1 lecture, §1.5 in [EP], §2.1 in [BD]*

One of the most important types of equations we will learn how to solve are the so-called *linear equations*. In fact, the majority of the course will focus on linear equations. In this lecture we will focus on the *first order linear equation*. A first order equation is linear if we can put it into the following form:

$$y' + p(x)y = f(x). \quad (1.3)$$

Here the word “linear” means linear in  $y$  and  $y'$ ; no higher powers nor functions of  $y$  or  $y'$  appear. The dependence on  $x$  can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever  $p(x)$  and  $f(x)$  are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

First we find a function  $r(x)$  such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}[r(x)y].$$

Then we can multiply both sides of (1.3) by  $r(x)$  to obtain

$$\frac{d}{dx}[r(x)y] = r(x)f(x).$$

Now we integrate both sides. The right hand side does not depend on  $y$  and the left hand side is written as a derivative of a function. Afterwards, we solve for  $y$ . The function  $r(x)$  is called the *integrating factor* and the method is called the *integrating factor method*.

We are looking for a function  $r(x)$ , such that if we differentiate it, we get the same function back multiplied by  $p(x)$ . That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)dx}.$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)dx} y' + e^{\int p(x)dx} p(x)y &= e^{\int p(x)dx} f(x), \\ \frac{d}{dx} [e^{\int p(x)dx} y] &= e^{\int p(x)dx} f(x), \\ e^{\int p(x)dx} y &= \int e^{\int p(x)dx} f(x) dx + C, \\ y &= e^{-\int p(x)dx} \left( \int e^{\int p(x)dx} f(x) dx + C \right). \end{aligned}$$

Of course, to get a closed form formula for  $y$ , we need to be able to find a closed form formula for the integrals appearing above.

**Example 1.4.1:** Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1.$$

First note that  $p(x) = 2x$  and  $f(x) = e^{x-x^2}$ . The integrating factor is  $r(x) = e^{\int p(x) dx} = e^{x^2}$ . We multiply both sides of the equation by  $r(x)$  to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx} [e^{x^2} y] &= e^x. \end{aligned}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^x + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned}$$

Next, we solve for the initial condition  $-1 = y(0) = 1 + C$ , so  $C = -2$ . The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing  $e^{\int p(x) dx}$ . You can always add a constant of integration, but those constants will not matter in the end.

**Exercise 1.4.1:** Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.

An advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$\boxed{y(x) = e^{-\int_{x_0}^x p(s) ds} \left( \int_{x_0}^x e^{\int_{x_0}^t p(s) ds} f(t) dt + y_0 \right).} \quad (1.4)$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

**Exercise 1.4.2:** Check that  $y(x_0) = y_0$  in formula (1.4).

**Exercise 1.4.3:** Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2-x}, \quad y(0) = 10.$$

**Remark 1.4.1:** Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem  $y' + p(x)y = f(x)$ ,  $y(x_0) = y_0$ , there is always an explicit formula (1.4) for the solution. Second, it follows from the formula (1.4) that if  $p(x)$  and  $f(x)$  are continuous on some interval  $(a, b)$ , then the solution  $y(x)$  exists and is differentiable on  $(a, b)$ . Compare with the simple nonlinear example we have seen previously,  $y' = y^2$ , and compare to Theorem 1.2.1.

**Example 1.4.2:** Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Let us come up with the equation. Let  $x$  denote the kilograms of salt in the tank, let  $t$  denote the time in minutes. Then for a small change  $\Delta t$  in time, the change in  $x$  (denoted  $\Delta x$ ) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by  $\Delta t$  and taking the limit  $\Delta t \rightarrow 0$  we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example, we have

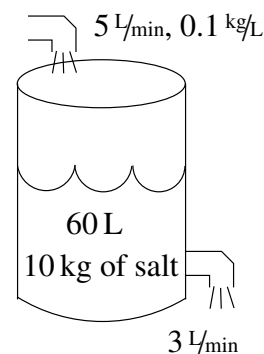
$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5 - 3)t}. \end{aligned}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60 + 2t}\right).$$

Or in the form (1.3)

$$\frac{dx}{dt} + \frac{3}{60 + 2t}x = 0.5.$$



Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t} dt\right) = \exp\left(\frac{3}{2} \ln(60+2t)\right) = (60+2t)^{3/2}.$$

We multiply both sides of the equation to get

$$\begin{aligned} (60+2t)^{3/2} \frac{dx}{dt} + (60+2t)^{3/2} \frac{3}{60+2t} x &= 0.5(60+2t)^{3/2}, \\ \frac{d}{dt} [(60+2t)^{3/2} x] &= 0.5(60+2t)^{3/2}, \\ (60+2t)^{3/2} x &= \int 0.5(60+2t)^{3/2} dt + C, \\ x &= (60+2t)^{-3/2} \int \frac{(60+2t)^{3/2}}{2} dt + C(60+2t)^{-3/2}, \\ x &= (60+2t)^{-3/2} \frac{1}{10} (60+2t)^{5/2} + C(60+2t)^{-3/2}, \\ x &= \frac{60+2t}{10} + C(60+2t)^{-3/2}. \end{aligned}$$

We need to find  $C$ . We know that at  $t = 0$ ,  $x = 10$ . So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2},$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in  $x$  when the tank is full. So we note that the tank is full when  $60+2t = 100$ , or when  $t = 20$ . So

$$x(20) = \frac{60+40}{10} + C(60+40)^{-3/2} \approx 10 + 1859.03(100)^{-3/2} \approx 11.86.$$

The concentration at the end is approximately  $0.1186 \text{ kg/liter}$  and we started with  $\frac{1}{6}$  or  $0.167 \text{ kg/liter}$ .

### 1.4.1 Exercises

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

**Exercise 1.4.4:** Solve  $y' + xy = x$ .

**Exercise 1.4.5:** Solve  $y' + 6y = e^x$ .

**Exercise 1.4.6:** Solve  $y' + 3x^2y = \sin(x)e^{-x^3}$ , with  $y(0) = 1$ .

**Exercise 1.4.7:** Solve  $y' + \cos(x)y = \cos(x)$ .

**Exercise 1.4.8:** Solve  $\frac{1}{x^2+1}y' + xy = 3$ , with  $y(0) = 0$ .

**Exercise 1.4.9:** Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream. a) Find the concentration of toxic substance as a function of time in both lakes. b) When will the concentration in the first lake be below 0.001 kg per liter. c) When will the concentration in the second lake be maximal.

**Exercise 1.4.10:** Newton's law of cooling states that  $\frac{dx}{dt} = -k(x - A)$  where  $x$  is the temperature,  $t$  is time,  $A$  is the ambient temperature, and  $k > 0$  is a constant. Suppose that  $A = A_0 \cos(\omega t)$  for some constants  $A_0$  and  $\omega$ . That is, the ambient temperature oscillates (for example night and day temperatures). a) Find the general solution. b) In the long term, will the initial conditions make much of a difference? Why or why not.

**Exercise 1.4.11:** Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

**Exercise 1.4.12:** Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

**Exercise 1.4.101:** Solve  $y' + 3x^2y = x^2$ .

**Exercise 1.4.102:** Solve  $y' + 2 \sin(2x)y = 2 \sin(2x)$ ,  $y(\pi/2) = 3$ .

**Exercise 1.4.103:** Suppose a water tank is being pumped out at  $3 \text{ } l_{\min}$ . The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at  $2 \text{ } l_{\min}$ , with concentration  $20t \text{ } g/L$  at time  $t$ . When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

**Exercise 1.4.104:** Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is suppose that  $\frac{dP}{dt} = (2 - 0.1t)P$ . If  $P(0) = 1000$ , find the population at  $t = 5$ .

## 1.5 Substitution

*Note: 1 lecture, §1.6 in [EP], not in [BD]*

Just like when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

### 1.5.1 Substitution

The equation

$$y' = (x - y + 1)^2$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We will use another variable  $v$ , which we will treat as a function of  $x$ . Let us try

$$v = x - y + 1.$$

We need to figure out  $y'$  in terms of  $v'$ ,  $v$  and  $x$ . We differentiate (in  $x$ ) to obtain  $v' = 1 - y'$ . So  $y' = 1 - v'$ . We plug this into the equation to get

$$1 - v' = v^2.$$

In other words,  $v' = 1 - v^2$ . Such an equation we know how to solve.

$$\frac{1}{1 - v^2} dv = dx.$$

So

$$\begin{aligned} \frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| &= x + C, \\ \left| \frac{v+1}{v-1} \right| &= e^{2x+2C}, \end{aligned}$$

or  $\frac{v+1}{v-1} = De^{2x}$  for some constant  $D$ . Note that  $v = 1$  and  $v = -1$  are also solutions.

Now we need to “unsubstitute” to obtain

$$\frac{x - y + 2}{x - y} = De^{2x},$$

and also the two solutions  $x - y + 1 = 1$  or  $y = x$ , and  $x - y + 1 = -1$  or  $y = x + 2$ . We solve the first equation for  $y$ .

$$\begin{aligned} x - y + 2 &= (x - y)De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned}$$



Note that  $D = 0$  gives  $y = x + 2$ , but no value of  $D$  gives the solution  $y = x$ .

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general things to look for. We summarize a few of these in a table.

When you see	Try substituting
$yy'$	$v = y^2$
$y^2y'$	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Usually you try to substitute in the “most complicated” part of the equation with the hopes of simplifying it. The above table is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

## 1.5.2 Bernoulli equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called *Bernoulli equation*<sup>†</sup>.

$$y' + p(x)y = q(x)y^n.$$

This equation looks a lot like a linear equation except for the  $y^n$ . If  $n = 0$  or  $n = 1$ , then the equation is linear and we can solve it. Otherwise, the substitution  $v = y^{1-n}$  transforms the Bernoulli equation into a linear equation. Note that  $n$  need not be an integer.

**Example 1.5.1:** Solve

$$xy' + y(x + 1) + xy^5 = 0, \quad y(1) = 1.$$

First, the equation is Bernoulli ( $p(x) = (x + 1)/x$  and  $q(x) = -1$ ). We substitute

$$v = y^{1-5} = y^{-4}, \quad v' = -4y^{-5}y'.$$

In other words,  $(-1/4)y^5v' = y'$ . So

$$\begin{aligned} xy' + y(x + 1) + xy^5 &= 0, \\ \frac{-xy^5}{4}v' + y(x + 1) + xy^5 &= 0, \\ \frac{-x}{4}v' + y^{-4}(x + 1) + x &= 0, \\ \frac{-x}{4}v' + v(x + 1) + x &= 0, \end{aligned}$$

<sup>†</sup>There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654 – 1705).

and finally

$$v' - \frac{4(x+1)}{x}v = 4.$$

Now the equation is linear. We can use the integrating factor method. In particular, we will use formula (1.4). Let us assume that  $x > 0$  so  $|x| = x$ . This assumption is OK, as our initial condition is  $x = 1$ . Let us compute the integrating factor. Here  $p(s)$  from formula (1.4) is  $\frac{-4(s+1)}{s}$ .

$$\begin{aligned} e^{\int_1^x p(s) ds} &= \exp\left(\int_1^x \frac{-4(s+1)}{s} ds\right) = e^{-4x-4\ln(x)+4} = e^{-4x+4} x^{-4} = \frac{e^{-4x+4}}{x^4}, \\ e^{-\int_1^x p(s) ds} &= e^{4x+4\ln(x)-4} = e^{4x-4} x^4. \end{aligned}$$

We now plug in to (1.4)

$$\begin{aligned} v(x) &= e^{-\int_1^x p(s) ds} \left( \int_1^x e^{\int_1^t p(s) ds} 4 dt + 1 \right) \\ &= e^{4x-4} x^4 \left( \int_1^x 4 \frac{e^{-4t+4}}{t^4} dt + 1 \right). \end{aligned}$$

Note that the integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now “unsubstitute”

$$\begin{aligned} y^{-4} &= e^{4x-4} x^4 \left( 4 \int_1^x \frac{e^{-4t+4}}{t^4} dt + 1 \right), \\ y &= \frac{e^{-x+1}}{x \left( 4 \int_1^x \frac{e^{-4t+4}}{t^4} dt + 1 \right)^{1/4}}. \end{aligned}$$

### 1.5.3 Homogeneous equations

Another type of equations we can solve by substitution are the so-called *homogeneous equations*. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right).$$

Here we try the substitutions

$$v = \frac{y}{x} \quad \text{and therefore} \quad y' = v + xv'.$$

We note that the equation is transformed into

$$v + xv' = F(v) \quad \text{or} \quad xv' = F(v) - v \quad \text{or} \quad \frac{v'}{F(v) - v} = \frac{1}{x}.$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v} dv = \ln |x| + C.$$

**Example 1.5.2:** Solve

$$x^2 y' = y^2 + xy, \quad y(1) = 1.$$

We put the equation into the form  $y' = (y/x)^2 + y/x$ . Now we substitute  $v = y/x$  to get the separable equation

$$xv' = v^2 + v - v = v^2,$$

which has a solution

$$\begin{aligned} \int \frac{1}{v^2} dv &= \ln |x| + C, \\ \frac{-1}{v} &= \ln |x| + C, \\ v &= \frac{-1}{\ln |x| + C}. \end{aligned}$$

We unsubststitute

$$\begin{aligned} \frac{y}{x} &= \frac{-1}{\ln |x| + C}, \\ y &= \frac{-x}{\ln |x| + C}. \end{aligned}$$

We want  $y(1) = 1$ , so

$$1 = y(1) = \frac{-1}{\ln |1| + C} = \frac{-1}{C}.$$

Thus  $C = -1$  and the solution we are looking for is

$$y = \frac{-x}{\ln |x| - 1}.$$

### 1.5.4 Exercises

Hint: Answers need not always be in closed form.

**Exercise 1.5.1:** Solve  $y' + y(x^2 - 1) + xy^6 = 0$ , with  $y(1) = 1$ .

**Exercise 1.5.2:** Solve  $2yy' + 1 = y^2 + x$ , with  $y(0) = 1$ .

**Exercise 1.5.3:** Solve  $y' + xy = y^4$ , with  $y(0) = 1$ .

**Exercise 1.5.4:** Solve  $yy' + x = \sqrt{x^2 + y^2}$ .

**Exercise 1.5.5:** Solve  $y' = (x + y - 1)^2$ .

**Exercise 1.5.6:** Solve  $y' = \frac{x^2 - y^2}{xy}$ , with  $y(1) = 2$ .

**Exercise 1.5.101:** Solve  $xy' + y + y^2 = 0$ ,  $y(1) = 2$ .

**Exercise 1.5.102:** Solve  $xy' + y + x = 0$ ,  $y(1) = 1$ .

**Exercise 1.5.103:** Solve  $y^2 y' = y^3 - 3x$ ,  $y(0) = 2$ .

**Exercise 1.5.104:** Solve  $2yy' = e^{y^2 - x^2} + 2x$ .

## 1.6 Autonomous equations

Note: 1 lecture, §2.2 in [EP], §2.5 in [BD]

Let us consider problems of the form

$$\frac{dx}{dt} = f(x),$$

where the derivative of solutions depends only on  $x$  (the dependent variable). Such equations are called *autonomous equations*. If we think of  $t$  as time, the naming comes from the fact that the equation is independent of time.

Let us come back to the cooling coffee problem (see Example 1.3.3). Newton's law of cooling says that

$$\frac{dx}{dt} = -k(x - A),$$

where  $x$  is the temperature,  $t$  is time,  $k$  is some constant, and  $A$  is the ambient temperature. See Figure 1.6 for an example with  $k = 0.3$  and  $A = 5$ .

Note the solution  $x = A$  (in the figure  $x = 5$ ). We call these constant solutions the *equilibrium solutions*. The points on the  $x$  axis where  $f(x) = 0$  are called *critical points*. The point  $x = A$  is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution  $x = A$  is “stable” in that small perturbations in  $x$  do not lead to substantially different solutions as  $t$  grows. If we change the initial condition a little bit, then as  $t \rightarrow \infty$  we get  $x \rightarrow A$ . We call such critical points *stable*. In this simple example it turns out that all solutions in fact go to  $A$  as  $t \rightarrow \infty$ . If a critical point is not stable we would say it is *unstable*.

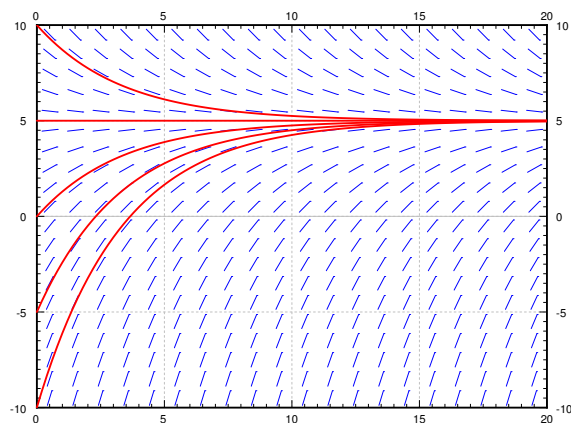


Figure 1.6: Slope field and some solutions of  $x' = -0.3(x - 5)$ .

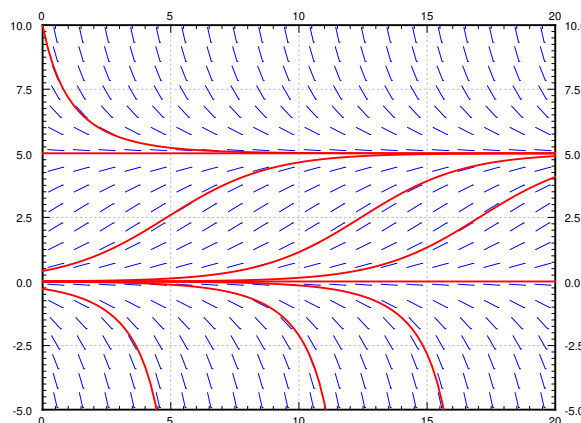


Figure 1.7: Slope field and some solutions of  $x' = 0.1x(5 - x)$ .

Let us consider the *logistic equation*

$$\frac{dx}{dt} = kx(M - x),$$

for some positive  $k$  and  $M$ . This equation is commonly used to model population if we know the limiting population  $M$ , that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than  $x' = kx$ . In the real world there is no such thing as negative population, but we will still consider negative  $x$  for the purposes of the math.

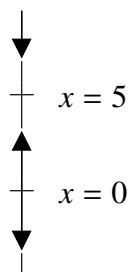
See Figure 1.7 on the preceding page for an example. Note two critical points,  $x = 0$  and  $x = 5$ . The critical point at  $x = 5$  is stable. On the other hand the critical point at  $x = 0$  is unstable.

It is not really necessary to find the exact solutions to talk about the long term behavior of the solutions. For example, from the above we can easily see that

$$\lim_{t \rightarrow \infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Where DNE means “does not exist.” From just looking at the slope field we cannot quite decide what happens if  $x(0) < 0$ . It could be that the solution does not exist for  $t$  all the way to  $\infty$ . Think of the equation  $x' = x^2$ , we have seen that it only exists for some finite period of time. Same can happen here. In our example equation above it will actually turn out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to  $-\infty$ , but it may get there rather quickly.

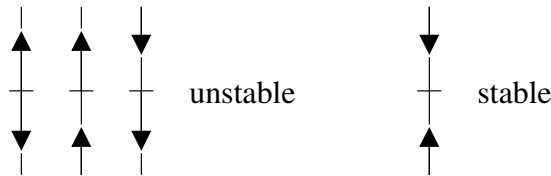
Often we are interested only in the long term behavior of the solution and we would be doing unnecessary work if we solved the equation exactly. It is easier to just look at the *phase diagram* or *phase portrait*, which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable  $x$ . We draw the  $x$  axis, we mark all the critical points, and then we draw arrows in between. If  $f(x) > 0$ , we draw an up arrow. If  $f(x) < 0$ , we draw a down arrow.



Armed with the phase diagram, it is easy to sketch the solutions approximately.

**Exercise 1.6.1:** Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.

Once we draw the phase diagram, we can easily classify critical points as stable or unstable<sup>‡</sup>.



Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of  $h$  million humans per year. Suppose  $x$  is the number of humans in millions on the planet and  $t$  is time in years. Let  $M$  be the limiting population when no harvesting is done.  $k > 0$  is some constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h.$$

We expand the right hand side and solve for critical points

$$\frac{dx}{dt} = -kx^2 + kMx - h.$$

Critical points  $A$  and  $B$  are

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k}, \quad B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}.$$

**Exercise 1.6.2:** Draw the phase diagram for different possibilities. Note that these possibilities are  $A > B$ , or  $A = B$ , or  $A$  and  $B$  both complex (i.e. no real solutions). Hint: Fix some simple  $k$  and  $M$  and then vary  $h$ .

For example, let  $M = 8$  and  $k = 0.1$ . When  $h = 1$ , then  $A$  and  $B$  are distinct and positive. The graph we will get is given in Figure 1.8 on the next page. As long as the population starts above  $B$ , which is approximately 1.55 million, then the population will not die out. It will in fact tend towards  $A \approx 6.45$  million. If ever some catastrophe happens and the population drops below  $B$ , humans will die out, and the fast food restaurant serving them will go out of business.

When  $h = 1.6$ , then  $A = B = 4$ . There is only one critical point and it is unstable. When the population starts above 4 million it will tend towards 4 million. If it ever drops below 4 million, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business. There is no room for error. See Figure 1.9 on the following page.

Finally if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts. See Figure 1.10 on the next page.

<sup>‡</sup>The unstable points that have one of the arrows pointing towards the critical point are sometimes called *semistable*.

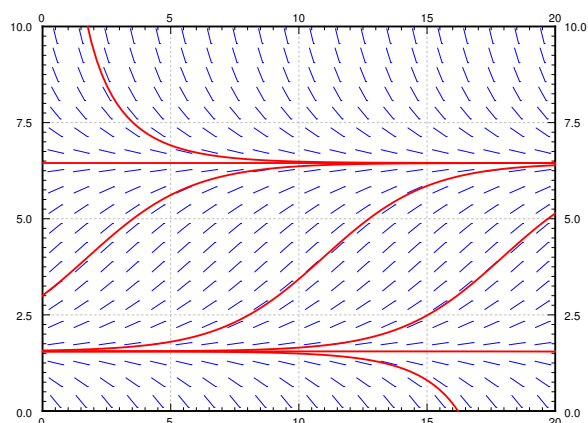


Figure 1.8: Slope field and some solutions of  $x' = 0.1x(8-x) - 1$ .

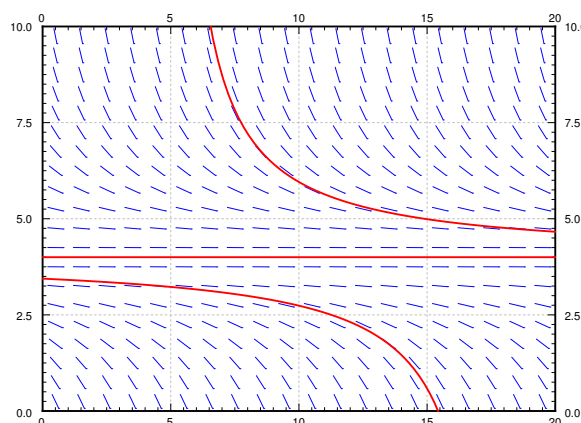


Figure 1.9: Slope field and some solutions of  $x' = 0.1x(8-x) - 1.6$ .

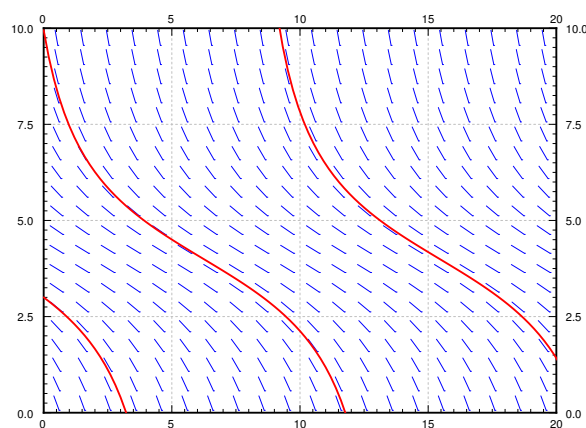


Figure 1.10: Slope field and some solutions of  $x' = 0.1x(8-x) - 2$ .

## 1.6.1 Exercises

**Exercise 1.6.3:** Let  $x' = x^2$ . a) Draw the phase diagram, find the critical points and mark them stable or unstable. b) Sketch typical solutions of the equation. c) Find  $\lim_{t \rightarrow \infty} x(t)$  for the solution with the initial condition  $x(0) = -1$ .

**Exercise 1.6.4:** Let  $x' = \sin x$ . a) Draw the phase diagram for  $-4\pi \leq x \leq 4\pi$ . On this interval mark the critical points stable or unstable. b) Sketch typical solutions of the equation. c) Find  $\lim_{t \rightarrow \infty} x(t)$  for the solution with the initial condition  $x(0) = 1$ .

**Exercise 1.6.5:** Suppose  $f(x)$  is positive for  $0 < x < 1$ , it is zero when  $x = 0$  and  $x = 1$ , and it is negative for all other  $x$ . a) Draw the phase diagram for  $x' = f(x)$ , find the critical points and mark



them stable or unstable. b) Sketch typical solutions of the equation. c) Find  $\lim_{t \rightarrow \infty} x(t)$  for the solution with the initial condition  $x(0) = 0.5$ .

**Exercise 1.6.6:** Start with the logistic equation  $\frac{dx}{dt} = kx(M - x)$ . Suppose that we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words we harvest  $hx$  per unit of time for some  $h > 0$  (Similar to earlier example with  $h$  replaced with  $hx$ ). a) Construct the differential equation. b) Show that if  $kM > h$ , then the equation is still logistic. c) What happens when  $kM < h$ ?

**Exercise 1.6.101:** Let  $x' = (x - 1)(x - 2)x^2$ . a) Sketch the phase diagram and find critical points. b) Classify the critical points. c) If  $x(0) = 0.5$  then find  $\lim_{t \rightarrow \infty} x(t)$ .

**Exercise 1.6.102:** Let  $x' = e^{-x}$ . a) Find and classify all critical points. b) Find  $\lim_{t \rightarrow \infty} x(t)$  given any initial condition.

**Exercise 1.6.103:** Assume that a population of fish in a lake satisfies  $\frac{dx}{dt} = kx(M - x)$ . Now suppose that fish are continually added at  $A$  fish per unit of time. a) Find the differential equation for  $x$ . b) What is the new limiting population.

## 1.7 Numerical methods: Euler's method

*Note: 1 lecture, §2.4 in [EP], §8.1 in [BD]*

At this point it may be good to first try the Lab II and/or Project II from the IODE website: <http://www.math.uiuc.edu/iode/>.

As we said before, unless  $f(x, y)$  is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

What if we want to find the value of the solution at some particular  $x$ ? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is *Euler's method*<sup>§</sup>. It works as follows: We take  $x_0$  and compute the slope  $k = f(x_0, y_0)$ . The slope is the change in  $y$  per unit change in  $x$ . We follow the line for an interval of length  $h$  on the  $x$  axis. Hence if  $y = y_0$  at  $x_0$ , then we will say that  $y_1$  (the approximate value of  $y$  at  $x_1 = x_0 + h$ ) will be  $y_1 = y_0 + hk$ . Rinse, repeat! That is, compute  $x_2$  and  $y_2$  using  $x_1$  and  $y_1$ . For an example of the first two steps of the method see Figure 1.11.

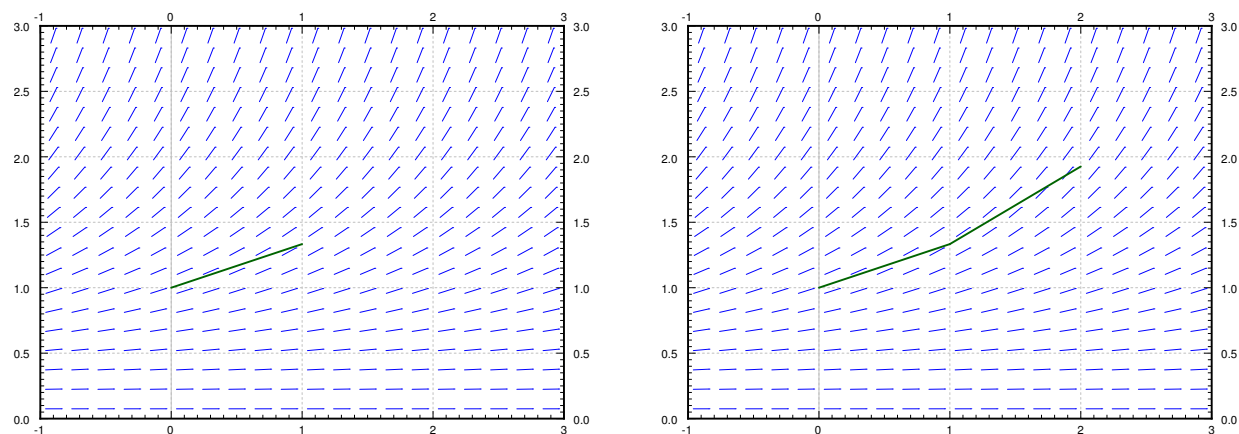


Figure 1.11: First two steps of Euler's method with  $h = 1$  for the equation  $y' = \frac{y^2}{3}$  with initial conditions  $y(0) = 1$ .

More abstractly, for any  $i = 1, 2, 3, \dots$ , we compute

$$x_{i+1} = x_i + h, \quad y_{i+1} = y_i + h f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 1.12 on the next page for the plot of the real solution and the approximation.

<sup>§</sup>Named after the Swiss mathematician Leonhard Paul Euler (1707 – 1783). Do note the correct pronunciation of the name sounds more like “oiler.”

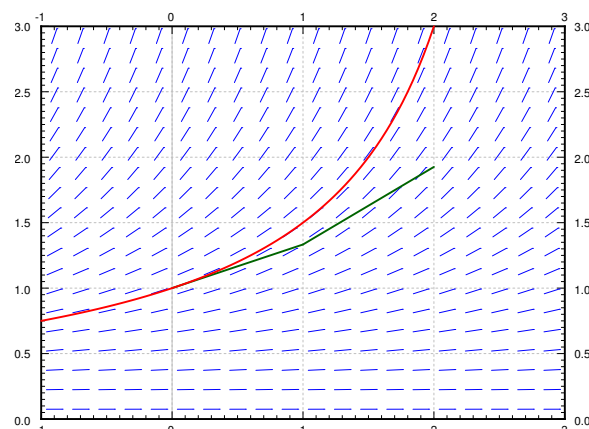


Figure 1.12: Two steps of Euler's method (step size 1) and the exact solution for the equation  $y' = \frac{y^2}{3}$  with initial conditions  $y(0) = 1$ .

Let us see what happens with the equation  $y' = y^2/3$ ,  $y(0) = 1$ . Let us try to approximate  $y(2)$  using Euler's method. In Figures 1.11 and 1.12 we have graphically approximated  $y(2)$  with step size 1. With step size 1 we have  $y(2) \approx 1.926$ . The real answer is 3. So we are approximately 1.074 off. Let us halve the step size. Computing  $y_4$  with  $h = 0.5$ , we find that  $y(2) \approx 2.209$ , so an error of about 0.791. Table 1.1 on the following page gives the values computed for various parameters.

**Exercise 1.7.1:** Solve this equation exactly and show that  $y(2) = 3$ .

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly ... what is the point of doing the approximation?

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a *first order method*. In the IODE Project II you are asked to implement a second order method. A second order method reduces the error to approximately one quarter every time we halve the interval.

Note that to get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at  $h = 1$ ) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than  $y^2/3$ ). Then the difference is 32 seconds versus about

$h$	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table 1.1: Euler's method approximation of  $y(2)$  where of  $y' = y^2/3$ ,  $y(0) = 1$ .

17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Note that we do not know the error! How do we know what is the right step size? Essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

**Exercise 1.7.2:** In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example  $y' = y^2/3$ ,  $y(0) = 1$ . Suppose that instead of the value  $y(2)$  we wish to find  $y(3)$ . The results of this effort are listed in Table 1.2 on the next page for successive halvings of  $h$ . What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at  $x = 3$ . In fact, the solution goes to infinity when you approach  $x = 3$ .

Another case when things can go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. In this case, the solution may exist at all points, but even a better approximation method than Euler would need an insanely small step size to compute the solution with reasonable precision. And computers might not be able to handle such a small step size anyway.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method (see exercises). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16.

$h$	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table 1.2: Attempts to use Euler's to approximate  $y(3)$  where of  $y' = y^2/3$ ,  $y(0) = 1$ .

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function  $f$  is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.
- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse.
- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called *stiff*. In the worst case the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

### 1.7.1 Exercises

**Exercise 1.7.3:** Consider  $\frac{dx}{dt} = (2t - x)^2$ ,  $x(0) = 2$ . Use Euler's method with step size  $h = 0.5$  to approximate  $x(1)$ .

**Exercise 1.7.4:** Consider  $\frac{dx}{dt} = t - x$ ,  $x(0) = 1$ . a) Use Euler's method with step sizes  $h = 1, 1/2, 1/4, 1/8$  to approximate  $x(1)$ . b) Solve the equation exactly. c) Describe what happens to the errors for each  $h$  you used. That is, find the factor by which the error changed each time you halved the interval.

**Exercise 1.7.5:** Approximate the value of  $e$  by looking at the initial value problem  $y' = y$  with  $y(0) = 1$  and approximating  $y(1)$  using Euler's method with a step size of 0.2.

**Exercise 1.7.6:** Example of numerical instability: Take  $y' = -5y$ ,  $y(0) = 1$ . We know that the solution should decay to zero as  $x$  grows. Using Euler's method, start with  $h = 1$  and compute  $y_1, y_2, y_3, y_4$  to try to approximate  $y(4)$ . What happened? Now halve the interval. Keep halving the interval and approximating  $y(4)$  until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

The simplest method used in practice is the *Runge-Kutta method*. Consider  $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$ , and a step size  $h$ . Everything is the same as in Euler's method, except the computation of  $y_{i+1}$  and  $x_{i+1}$ .

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f(x_i + h/2, y_i + k_1 h/2), & x_{i+1} &= x_i + h, \\ k_3 &= f(x_i + h/2, y_i + k_2 h/2), & y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} h, \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned}$$

**Exercise 1.7.7:** Consider  $\frac{dy}{dx} = yx^2$ ,  $y(0) = 1$ . a) Use Runge-Kutta (see above) with step sizes  $h = 1$  and  $h = 1/2$  to approximate  $y(1)$ . b) Use Euler's method with  $h = 1$  and  $h = 1/2$ . c) Solve exactly, find the exact value of  $y(1)$ , and compare.

**Exercise 1.7.101:** Let  $x' = \sin(xt)$ , and  $x(0) = 1$ . Approximate  $x(1)$  using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

**Exercise 1.7.102:** Let  $x' = 2t$ , and  $x(0) = 0$ . a) Approximate  $x(4)$  using Euler's method with step sizes 4, 2, and 1. b) Solve exactly, and compute the errors. c) Compute the factor by which the errors changed.

**Exercise 1.7.103:** Let  $x' = xe^{x+1}$ , and  $x(0) = 0$ . a) Approximate  $x(4)$  using Euler's method with step sizes 4, 2, and 1. b) Guess an exact solution based on part a) and compute the errors.

# Chapter 2

## Higher order linear ODEs

### 2.1 Second order linear ODEs

*Note: less than 1 lecture, first part of §3.1 in [EP], parts of §3.1 and §3.2 in [BD]*

Let us consider the general *second order linear differential equation*

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by  $A(x)$  to get

$$y'' + p(x)y' + q(x)y = f(x), \quad (2.1)$$

where  $p(x) = B(x)/A(x)$ ,  $q(x) = C(x)/A(x)$ , and  $f(x) = F(x)/A(x)$ . The word *linear* means that the equation contains no powers nor functions of  $y$ ,  $y'$ , and  $y''$ .

In the special case when  $f(x) = 0$  we have a so-called *homogeneous* equation

$$y'' + p(x)y' + q(x)y = 0. \quad (2.2)$$

We have already seen some second order linear homogeneous equations.

$y'' + k^2y = 0$	Two solutions are: $y_1 = \cos(kx), \quad y_2 = \sin(kx).$
$y'' - k^2y = 0$	Two solutions are: $y_1 = e^{kx}, \quad y_2 = e^{-kx}.$

If we know two solutions of a linear homogeneous equation, we know a lot more of them.

**Theorem 2.1.1** (Superposition). *Suppose  $y_1$  and  $y_2$  are two solutions of the homogeneous equation (2.2). Then*

$$y(x) = C_1y_1(x) + C_2y_2(x),$$

*also solves (2.2) for arbitrary constants  $C_1$  and  $C_2$ .*

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression  $C_1y_1 + C_2y_2$  a *linear combination* of  $y_1$  and  $y_2$ . Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof: Let  $y = C_1y_1 + C_2y_2$ . Then

$$\begin{aligned} y'' + py' + qy &= (C_1y_1 + C_2y_2)'' + p(C_1y_1 + C_2y_2)' + q(C_1y_1 + C_2y_2) \\ &= C_1y_1'' + C_2y_2'' + C_1py_1' + C_2py_2' + C_1qy_1 + C_2qy_2 \\ &= C_1(y_1'' + py_1' + qy_1) + C_2(y_2'' + py_2' + qy_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function is, but a function eats numbers and spits out numbers). Define the operator  $L$  by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes  $Ly = 0$ . The operator (and the equation)  $L$  being *linear* means that  $L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2$ . The proof above becomes

$$Ly = L(C_1y_1 + C_2y_2) = C_1Ly_1 + C_2Ly_2 = C_1 \cdot 0 + C_2 \cdot 0 = 0.$$

Two different solutions to the second equation  $y'' - k^2y = 0$  are  $y_1 = \cosh(kx)$  and  $y_2 = \sinh(kx)$ . Let us remind ourselves of the definition,  $\cosh x = \frac{e^x + e^{-x}}{2}$  and  $\sinh x = \frac{e^x - e^{-x}}{2}$ . Therefore, these are solutions by superposition as they are linear combinations of the two exponential solutions.

The functions  $\sinh$  and  $\cosh$  are sometimes more convenient to use than the exponential. Let us review some of their properties.

$$\begin{array}{ll} \cosh 0 = 1 & \sinh 0 = 0 \\ \frac{d}{dx} \cosh x = \sinh x & \frac{d}{dx} \sinh x = \cosh x \\ \cosh^2 x - \sinh^2 x = 1 & \end{array}$$

**Exercise 2.1.1:** Derive these properties using the definitions of  $\sinh$  and  $\cosh$  in terms of exponentials.

Linear equations have nice and simple answers to the existence and uniqueness question.

**Theorem 2.1.2** (Existence and uniqueness). Suppose  $p, q, f$  are continuous functions and  $a, b_0, b_1$  are constants. The equation

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution  $y(x)$  satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$



For example, the equation  $y'' + k^2y = 0$  with  $y(0) = b_0$  and  $y'(0) = b_1$  has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

The equation  $y'' - k^2y = 0$  with  $y(0) = b_0$  and  $y'(0) = b_1$  has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

Using  $\cosh$  and  $\sinh$  in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Question: Suppose we find two different solutions  $y_1$  and  $y_2$  to the homogeneous equation (2.2). Can every solution be written (using superposition) in the form  $y = C_1y_1 + C_2y_2$ ?

Answer is affirmative! Provided that  $y_1$  and  $y_2$  are different enough in the following sense. We will say  $y_1$  and  $y_2$  are *linearly independent* if one is not a constant multiple of the other.

**Theorem 2.1.3.** *Let  $p, q, f$  be continuous functions and take the homogeneous equation (2.2). Let  $y_1$  and  $y_2$  be two linearly independent solutions to (2.2). Then every other solution is of the form*

$$y = C_1y_1 + C_2y_2.$$

*That is,  $y = C_1y_1 + C_2y_2$  is the general solution.*

For example, we found the solutions  $y_1 = \sin x$  and  $y_2 = \cos x$  for the equation  $y'' + y = 0$ . It is not hard to see that sine and cosine are not constant multiples of each other. If  $\sin x = A \cos x$  for some constant  $A$ , we let  $x = 0$  and this would imply  $A = 0$ . But then  $\sin x = 0$  for all  $x$ , which is preposterous. So  $y_1$  and  $y_2$  are linearly independent. Hence

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to  $y'' + y = 0$ .

We will study the solution of nonhomogeneous equations in § 2.5. We will first focus on finding general solutions to homogeneous equations.

### 2.1.1 Exercises

**Exercise 2.1.2:** Show that  $y = e^x$  and  $y = e^{2x}$  are linearly independent.

**Exercise 2.1.3:** Take  $y'' + 5y = 10x + 5$ . Find (guess!) a solution.

**Exercise 2.1.4:** Prove the superposition principle for nonhomogeneous equations. Suppose that  $y_1$  is a solution to  $Ly_1 = f(x)$  and  $y_2$  is a solution to  $Ly_2 = g(x)$  (same linear operator  $L$ ). Show that  $y$  solves  $Ly = f(x) + g(x)$ .

**Exercise 2.1.5:** For the equation  $x^2y'' - xy' = 0$ , find two solutions, show that they are linearly independent and find the general solution. Hint: Try  $y = x^r$ .

Note that equations of the form  $ax^2y'' + bxy' + cy = 0$  are called *Euler's equations* or *Cauchy-Euler equations*. They are solved by trying  $y = x^r$  and solving for  $r$  (we can assume that  $x \geq 0$  for simplicity).

**Exercise 2.1.6:** Suppose that  $(b - a)^2 - 4ac > 0$ . a) Find a formula for the general solution of  $ax^2y'' + bxy' + cy = 0$ . Hint: Try  $y = x^r$  and find a formula for  $r$ . b) What happens when  $(b - a)^2 - 4ac = 0$  or  $(b - a)^2 - 4ac < 0$ ?

We will revisit the case when  $(b - a)^2 - 4ac < 0$  later.

**Exercise 2.1.7:** Same equation as in Exercise 2.1.6. Suppose  $(b - a)^2 - 4ac = 0$ . Find a formula for the general solution of  $ax^2y'' + bxy' + cy = 0$ . Hint: Try  $y = x^r \ln x$  for the second solution.

If you have one solution to a second order linear homogeneous equation you can find another one. This is the *reduction of order method*.

**Exercise 2.1.8** (reduction of order): Suppose  $y_1$  is a solution to  $y'' + p(x)y' + q(x)y = 0$ . Show that

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x) dx}}{(y_1(x))^2} dx$$

is also a solution.

Note: If you wish to come up with the formula for reduction of order yourself, start by trying  $y_2(x) = y_1(x)v(x)$ . Then plug  $y_2$  into the equation, use the fact that  $y_1$  is a solution, substitute  $w = v'$ , and you have a first order linear equation in  $w$ . Solve for  $w$  and then for  $v$ . When solving for  $w$ , make sure to include a constant of integration. Let us solve some famous equations using the method.

**Exercise 2.1.9** (Chebyshev's equation of order 1): Take  $(1 - x^2)y'' - xy' + y = 0$ . a) Show that  $y = x$  is a solution. b) Use reduction of order to find a second linearly independent solution. c) Write down the general solution.

**Exercise 2.1.10** (Hermite's equation of order 2): Take  $y'' - 2xy' + 4y = 0$ . a) Show that  $y = 1 - 2x^2$  is a solution. b) Use reduction of order to find a second linearly independent solution. c) Write down the general solution.

**Exercise 2.1.101:** Are  $\sin(x)$  and  $e^x$  linearly independent? Justify.

**Exercise 2.1.102:** Are  $e^x$  and  $e^{x+2}$  linearly independent? Justify.

**Exercise 2.1.103:** Guess a solution to  $y'' + y' + y = 5$ .

**Exercise 2.1.104:** Find the general solution to  $xy'' + y' = 0$ . Hint: notice that it is a first order ODE in  $y'$ .

## 2.2 Constant coefficient second order linear ODEs

*Note: more than 1 lecture, second part of §3.1 in [EP], §3.1 in [BD]*

Suppose we have the problem

$$y'' - 6y' + 8y = 0, \quad y(0) = -2, \quad y'(0) = 6.$$

This is a second order linear homogeneous equation with constant coefficients. *Constant coefficients* means that the functions in front of  $y''$ ,  $y'$ , and  $y$  are constants, not depending on  $x$ .

To guess a solution, think of a function that you know stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero.

Let us try a solution of the form  $y = e^{rx}$ . Then  $y' = re^{rx}$  and  $y'' = r^2e^{rx}$ . Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0, \\ r^2e^{rx} - 6re^{rx} + 8e^{rx} &= 0, \\ r^2 - 6r + 8 &= 0 \quad (\text{divide through by } e^{rx}), \\ (r - 2)(r - 4) &= 0. \end{aligned}$$

Hence, if  $r = 2$  or  $r = 4$ , then  $e^{rx}$  is a solution. So let  $y_1 = e^{2x}$  and  $y_2 = e^{4x}$ .

**Exercise 2.2.1:** Check that  $y_1$  and  $y_2$  are solutions.

The functions  $e^{2x}$  and  $e^{4x}$  are linearly independent. If they were not linearly independent we could write  $e^{4x} = Ce^{2x}$  for some constant  $C$ , implying that  $e^{2x} = C$  for all  $x$ , which is clearly not possible. Hence, we can write the general solution as

$$y = C_1e^{2x} + C_2e^{4x}.$$

We need to solve for  $C_1$  and  $C_2$ . To apply the initial conditions we first find  $y' = 2C_1e^{2x} + 4C_2e^{4x}$ . We plug in  $x = 0$  and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2, \\ 6 &= y'(0) = 2C_1 + 4C_2. \end{aligned}$$

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain  $3 = C_1 + 2C_2$ , and subtract the two equations to get  $5 = C_2$ . Then  $C_1 = -7$  as  $-2 = C_1 + 5$ . Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}.$$

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{2.3}$$

where  $a, b, c$  are constants. Try the solution  $y = e^{rx}$  to obtain

$$\begin{aligned} ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0, \\ ar^2 + br + c &= 0. \end{aligned}$$

The equation  $ar^2 + br + c = 0$  is called the *characteristic equation* of the ODE. Solve for the  $r$  by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore, we have  $e^{r_1x}$  and  $e^{r_2x}$  as solutions. There is still a difficulty if  $r_1 = r_2$ , but it is not hard to overcome.

**Theorem 2.2.1.** Suppose that  $r_1$  and  $r_2$  are the roots of the characteristic equation.

(i) If  $r_1$  and  $r_2$  are distinct and real (when  $b^2 - 4ac > 0$ ), then (2.3) has the general solution

$$y = C_1e^{r_1x} + C_2e^{r_2x}.$$

(ii) If  $r_1 = r_2$  (happens when  $b^2 - 4ac = 0$ ), then (2.3) has the general solution

$$y = (C_1 + C_2x)e^{r_1x}.$$

For another example of the first case, take the equation  $y'' - k^2y = 0$ . Here the characteristic equation is  $r^2 - k^2 = 0$  or  $(r - k)(r + k) = 0$ . Consequently,  $e^{-kx}$  and  $e^{kx}$  are the two linearly independent solutions.

**Example 2.2.1:** Find the general solution of

$$y'' - 8y' + 16y = 0.$$

The characteristic equation is  $r^2 - 8r + 16 = (r - 4)^2 = 0$ . The equation has a double root  $r_1 = r_2 = 4$ . The general solution is, therefore,

$$y = (C_1 + C_2x)e^{4x} = C_1e^{4x} + C_2xe^{4x}.$$

**Exercise 2.2.2:** Check that  $e^{4x}$  and  $xe^{4x}$  are linearly independent.

That  $e^{4x}$  solves the equation is clear. If  $xe^{4x}$  solves the equation, then we know we are done. Let us compute  $y' = e^{4x} + 4xe^{4x}$  and  $y'' = 8e^{4x} + 16xe^{4x}$ . Plug in

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0.$$

We should note that in practice, doubled root rarely happens. If coefficients are picked truly randomly we are very unlikely to get a doubled root.

Let us give a short proof for why the solution  $xe^{rx}$  works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that  $\frac{e^{r_2x} - e^{r_1x}}{r_2 - r_1}$  is a solution when the roots are distinct. When we take the limit as  $r_1$  goes to  $r_2$ , we are really taking the derivative of  $e^{rx}$  using  $r$  as the variable. Therefore, the limit is  $xe^{rx}$ , and hence this is a solution in the doubled root case.

### 2.2.1 Complex numbers and Euler's formula

It may happen that a polynomial has some complex roots. For example, the equation  $r^2 + 1 = 0$  has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers,  $(a, b)$ . We can think of a complex number as a point in the plane. We add complex numbers in the straightforward way,  $(a, b) + (c, d) = (a + c, b + d)$ . We define a multiplication by

$$(a, b) \times (c, d) \stackrel{\text{def}}{=} (ac - bd, ad + bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly  $(0, 1) \times (0, 1) = (-1, 0)$ .

Generally we just write  $(a, b)$  as  $a + ib$ , and we treat  $i$  as if it were an unknown. We can just do arithmetic with complex numbers just as we would do with polynomials. The property we just mentioned becomes  $i^2 = -1$ . So whenever we see  $i^2$ , we can replace it by  $-1$ . The numbers  $i$  and  $-i$  are roots of  $r^2 + 1 = 0$ .

Note that engineers often use the letter  $j$  instead of  $i$  for the square root of  $-1$ . We will use the mathematicians' convention and use  $i$ .

**Exercise 2.2.3:** Make sure you understand (that you can justify) the following identities:

- $i^2 = -1, i^3 = -i, i^4 = 1,$
- $\frac{1}{i} = -i,$
- $(3 - 7i)(-2 - 9i) = \dots = -69 - 13i,$
- $(3 - 2i)(3 + 2i) = 3^2 - (2i)^2 = 3^2 + 2^2 = 13,$
- $\frac{1}{3-2i} = \frac{1}{3-2i} \frac{3+2i}{3+2i} = \frac{3+2i}{13} = \frac{3}{13} + \frac{2}{13}i.$

We can also define the exponential  $e^{a+ib}$  of a complex number. We can do this by just writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, we note that many such properties still hold for the complex exponential. For example,  $e^{x+y} = e^x e^y$ . This means that  $e^{a+ib} = e^a e^{ib}$ . Hence if we can compute  $e^{ib}$ , we can compute  $e^{a+ib}$ . For  $e^{ib}$  we will use the so-called *Euler's formula*.

**Theorem 2.2.2** (Euler's formula).

$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta.$
--

**Exercise 2.2.4:** Using Euler's formula, check the identities:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

**Exercise 2.2.5:** Double angle identities: Start with  $e^{i(2\theta)} = (e^{i\theta})^2$ . Use Euler on each side and deduce:

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta \quad \text{and} \quad \sin(2\theta) = 2 \sin \theta \cos \theta.$$

For a complex number  $a + ib$  we call  $a$  the *real part* and  $b$  the *imaginary part* of the number. Often the following notation is used,

$$\operatorname{Re}(a + ib) = a \quad \text{and} \quad \operatorname{Im}(a + ib) = b.$$

## 2.2.2 Complex roots

Now suppose that the equation  $ay'' + by' + cy = 0$  has the characteristic equation  $ar^2 + br + c = 0$  that has complex roots. By quadratic formula the roots are  $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . These are complex if  $b^2 - 4ac < 0$ . In this case we can see that the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a}.$$

As you can see, we will always get a pair of roots of the form  $\alpha \pm i\beta$ . In this case we can still write the solution as

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$

However, the exponential is now complex valued. We would need to allow  $C_1$  and  $C_2$  to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's formula. Let

$$y_1 = e^{(\alpha + i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha - i\beta)x}.$$

Then note that

$$\begin{aligned} y_1 &= e^{\alpha x} \cos(\beta x) + i e^{\alpha x} \sin(\beta x), \\ y_2 &= e^{\alpha x} \cos(\beta x) - i e^{\alpha x} \sin(\beta x). \end{aligned}$$

Linear combinations of solutions are also solutions. Hence,

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{\alpha x} \cos(\beta x), \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{\alpha x} \sin(\beta x), \end{aligned}$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other). Therefore, we have the following theorem.

**Theorem 2.2.3.** Take the equation

$$ay'' + by' + cy = 0.$$

If the characteristic equation has the roots  $\alpha \pm i\beta$  (when  $b^2 - 4ac < 0$ ), then the general solution is

$$y = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x).$$

**Example 2.2.2:** Find the general solution of  $y'' + k^2 y = 0$ , for a constant  $k > 0$ .

The characteristic equation is  $r^2 + k^2 = 0$ . Therefore, the roots are  $r = \pm ik$  and by the theorem we have the general solution

$$y = C_1 \cos(kx) + C_2 \sin(kx).$$

**Example 2.2.3:** Find the solution of  $y'' - 6y' + 13y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 10$ .

The characteristic equation is  $r^2 - 6r + 13 = 0$ . By completing the square we get  $(r - 3)^2 + 2^2 = 0$  and hence the roots are  $r = 3 \pm 2i$ . By the theorem we have the general solution

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x).$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1.$$

Hence  $C_1 = 0$  and  $y = C_2 e^{3x} \sin(2x)$ . We differentiate

$$y' = 3C_2 e^{3x} \sin(2x) + 2C_2 e^{3x} \cos(2x).$$

We again plug in the initial condition and obtain  $10 = y'(0) = 2C_2$ , or  $C_2 = 5$ . Hence the solution we are seeking is

$$y = 5e^{3x} \sin(2x).$$

### 2.2.3 Exercises

**Exercise 2.2.6:** Find the general solution of  $2y'' + 2y' - 4y = 0$ .

**Exercise 2.2.7:** Find the general solution of  $y'' + 9y' - 10y = 0$ .

**Exercise 2.2.8:** Solve  $y'' - 8y' + 16y = 0$  for  $y(0) = 2$ ,  $y'(0) = 0$ .

**Exercise 2.2.9:** Solve  $y'' + 9y' = 0$  for  $y(0) = 1$ ,  $y'(0) = 1$ .

**Exercise 2.2.10:** Find the general solution of  $2y'' + 50y = 0$ .

**Exercise 2.2.11:** Find the general solution of  $y'' + 6y' + 13y = 0$ .

**Exercise 2.2.12:** Find the general solution of  $y'' = 0$  using the methods of this section.

**Exercise 2.2.13:** The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation  $2y' + 3y = 0$  using the methods of this section.

**Exercise 2.2.14:** Let us revisit Euler's equations of Exercise 2.1.6 on page 50. Suppose now that  $(b - a)^2 - 4ac < 0$ . Find a formula for the general solution of  $ax^2y'' + bxy' + cy = 0$ . Hint: Note that  $x^r = e^{r \ln x}$ .

**Exercise 2.2.101:** Find the general solution to  $y'' + 4y' + 2y = 0$ .

**Exercise 2.2.102:** Find the general solution to  $y'' - 6y' + 9y = 0$ .

**Exercise 2.2.103:** Find the solution to  $2y'' + y' + y = 0$ ,  $y(0) = 1$ ,  $y'(0) = -2$ .

**Exercise 2.2.104:** Find the solution to  $2y'' + y' - 3y = 0$ ,  $y(0) = a$ ,  $y'(0) = b$ .

**Exercise 2.2.105:** Find the solution to  $z''(t) = -2z'(t) - 2z(t)$ ,  $z(0) = 2$ ,  $z'(0) = -2$ .



## 2.3 Higher order linear ODEs

*Note: somewhat more than 1 lecture, §3.2 and §3.3 in [EP], §4.1 and §4.2 in [BD]*

After reading this lecture, it may be good to try Project III from the IODE website: <http://www.math.uiuc.edu/iode/>.

Equations that appear in applications tend to be second order. Higher order equations do appear from time to time, but it is a general assumption of modern physics that the world is “second order.”

The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by  $n$ . The important concept of linear independence is somewhat more complicated when more than two functions are involved.

For higher order constant coefficient ODEs, the methods are also somewhat harder to apply, but we will not dwell on these. We can always use the methods for systems of linear equations from chapter 3 to solve higher order constant coefficient equations.

So let us start with a general homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \quad (2.4)$$

**Theorem 2.3.1** (Superposition). *Suppose  $y_1, y_2, \dots, y_n$  are solutions of the homogeneous equation (2.4). Then*

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

*also solves (2.4) for arbitrary constants  $C_1, \dots, C_n$ .*

In other words, a *linear combination* of solutions to (2.4) is also a solution to (2.4). We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

**Theorem 2.3.2** (Existence and uniqueness). *Suppose  $p_0$  through  $p_{n-1}$ , and  $f$  are continuous functions and  $a, b_0, b_1, \dots, b_{n-1}$  are constants. The equation*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

*has exactly one solution  $y(x)$  satisfying the initial conditions*

$$y(a) = b_0, \quad y'(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1}.$$

### 2.3.1 Linear independence

When we had two functions  $y_1$  and  $y_2$  we said they were linearly independent if one was not the multiple of the other. Same idea holds for  $n$  functions. In this case it is easier to state as follows. The functions  $y_1, y_2, \dots, y_n$  are *linearly independent* if

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

has only the trivial solution  $c_1 = c_2 = \cdots = c_n = 0$ , where the equation must hold for all  $x$ . If we can solve equation with some constants where for example  $c_1 \neq 0$ , then we can solve for  $y_1$  as a linear combination of the others. If the functions are not linearly independent, they are *linearly dependent*.

**Example 2.3.1:** Show that  $e^x, e^{2x}, e^{3x}$  are linearly independent.

Let us give several ways to show this fact. Many textbooks (including [EP] and [F]) introduce Wronskians, but that is really not necessary here.

Let us write down

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We use rules of exponentials and write  $z = e^x$ . Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0.$$

The left hand side is a third degree polynomial in  $z$ . It can either be identically zero, or it can have at most 3 zeros. Therefore, it is identically zero,  $c_1 = c_2 = c_3 = 0$ , and the functions are linearly independent.

Let us try another way. As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

This equation has to hold for all  $x$ . What we could do is divide through by  $e^{3x}$  to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0.$$

As the equation is true for all  $x$ , let  $x \rightarrow \infty$ . After taking the limit we see that  $c_3 = 0$ . Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0.$$

Rinse, repeat!

How about yet another way. We again write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We can evaluate the equation and its derivatives at different values of  $x$  to obtain equations for  $c_1$ ,  $c_2$ , and  $c_3$ . Let us first divide by  $e^x$  for simplicity.

$$c_1 + c_2 e^x + c_3 e^{2x} = 0.$$

We set  $x = 0$  to get the equation  $c_1 + c_2 + c_3 = 0$ . Now differentiate both sides

$$c_2 e^x + 2c_3 e^{2x} = 0.$$

We set  $x = 0$  to get  $c_2 + 2c_3 = 0$ . We divide by  $e^x$  again and differentiate to get  $2c_3 e^x = 0$ . It is clear that  $c_3$  is zero. Then  $c_2$  must be zero as  $c_2 = -2c_3$ , and  $c_1$  must be zero because  $c_1 + c_2 + c_3 = 0$ .

There is no one best way to do it. All of these methods are perfectly valid.

**Example 2.3.2:** On the other hand, the functions  $e^x, e^{-x}$ , and  $\cosh x$  are linearly dependent. Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{or} \quad 2 \cosh x - e^x - e^{-x} = 0.$$

### 2.3.2 Constant coefficient higher order ODEs

When we have a higher order constant coefficient homogeneous linear equation. The song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is  $n^{\text{th}}$  order we need to find  $n$  linearly independent solutions. It is best seen by example.

**Example 2.3.3:** Find the general solution to

$$y''' - 3y'' - y' + 3y = 0. \quad (2.5)$$

Try:  $y = e^{rx}$ . We plug in and get

$$r^3 e^{rx} - 3r^2 e^{rx} - r e^{rx} + 3e^{rx} = 0.$$

We divide through by  $e^{rx}$ . Then

$$r^3 - 3r^2 - r + 3 = 0.$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always  $n$  roots for an  $n^{\text{th}}$  degree polynomial. They might be repeated and they might be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers  $r = -2, -1, 0, 1, 2, \dots$  and see if we get a hit (we can also try complex roots). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug  $r = -2$  into our polynomial and get  $-15$ ; we plug in  $r = 0$  and get  $3$ . That means there is a root between  $r = -2$  and  $r = 0$  because the sign changed. If we find one root, say  $r_1$ , then we know  $(r - r_1)$  is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with  $r = -1, 1$ , or  $0$ . These are easy to compute. Our polynomial happens to have two such roots,  $r_1 = -1$  and  $r_2 = 1$ . There should be 3 roots and the last root is reasonably easy to find. The constant term in a polynomial is the multiple of the negations of all the roots because  $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$ . In our case we see that

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3.$$

You should check that  $r_3 = 3$  really is a root. Hence we know that  $e^{-x}$ ,  $e^x$  and  $e^{3x}$  are solutions to (2.5). They are linearly independent as can easily be checked, and there are 3 of them, which happens to be exactly the number we need. Hence the general solution is

$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}.$$

Suppose we were given some initial conditions  $y(0) = 1$ ,  $y'(0) = 2$ , and  $y''(0) = 3$ . Then

$$\begin{aligned} 1 &= y(0) = C_1 + C_2 + C_3, \\ 2 &= y'(0) = -C_1 + C_2 + 3C_3, \\ 3 &= y''(0) = C_1 + C_2 + 9C_3. \end{aligned}$$

It is possible to find the solution by high school algebra, but it would be a pain. The only sensible way to solve a system of equations such as this is to use matrix algebra, see § 3.2. For now we note that the solution is  $C_1 = -1/4$ ,  $C_2 = 1$  and  $C_3 = 1/4$ . The specific solution to the ODE is

$$y = \frac{-1}{4} e^{-x} + e^x + \frac{1}{4} e^{3x}.$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root  $r$  repeated  $k$  times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, xe^{rx}, x^2 e^{rx}, \dots, x^{k-1} e^{rx}.$$

We take a linear combination of these solutions to find the general solution.

**Example 2.3.4:** Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0.$$

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0.$$

By inspection we note that  $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3$ . Hence the roots given with multiplicity are  $r = 0, 1, 1, 1$ . Thus the general solution is

$$y = \underbrace{(C_1 + C_2x + C_3x^2) e^x}_{\text{terms coming from } r=1} + \underbrace{C_4}_{\text{from } r=0}.$$

Similarly to the second order case we can handle complex roots. Complex roots always come in pairs  $r = \alpha \pm i\beta$ . Suppose we have two such complex roots, each repeated  $k$  times. The corresponding solution is

$$(C_0 + C_1x + \dots + C_{k-1}x^{k-1}) e^{\alpha x} \cos(\beta x) + (D_0 + D_1x + \dots + D_{k-1}x^{k-1}) e^{\alpha x} \sin(\beta x).$$

where  $C_0, \dots, C_{k-1}, D_0, \dots, D_{k-1}$  are arbitrary constants.

**Example 2.3.5:** Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0.$$

The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 8r^2 - 8r + 4 &= 0, \\ (r^2 - 2r + 2)^2 &= 0, \\ ((r-1)^2 + 1)^2 &= 0. \end{aligned}$$

Hence the roots are  $1 \pm i$ , both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2x) e^x \cos x + (C_3 + C_4x) e^x \sin x.$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

### 2.3.3 Exercises

**Exercise 2.3.1:** Find the general solution for  $y''' - y'' + y' - y = 0$ .

**Exercise 2.3.2:** Find the general solution for  $y^{(4)} - 5y''' + 6y'' = 0$ .

**Exercise 2.3.3:** Find the general solution for  $y''' + 2y'' + 2y' = 0$ .

**Exercise 2.3.4:** Suppose the characteristic equation for a differential equation is  $(r - 1)^2(r - 2)^2 = 0$ . a) Find such a differential equation. b) Find its general solution.

**Exercise 2.3.5:** Suppose that a fourth order equation has a solution  $y = 2e^{4x}x \cos x$ . a) Find such an equation. b) Find the initial conditions that the given solution satisfies.

**Exercise 2.3.6:** Find the general solution for the equation of Exercise 2.3.5.

**Exercise 2.3.7:** Let  $f(x) = e^x - \cos x$ ,  $g(x) = e^x + \cos x$ , and  $h(x) = \cos x$ . Are  $f(x)$ ,  $g(x)$ , and  $h(x)$  linearly independent? If so, show it, if not, find a linear combination that works.

**Exercise 2.3.8:** Let  $f(x) = 0$ ,  $g(x) = \cos x$ , and  $h(x) = \sin x$ . Are  $f(x)$ ,  $g(x)$ , and  $h(x)$  linearly independent? If so, show it, if not, find a linear combination that works.

**Exercise 2.3.9:** Are  $x$ ,  $x^2$ , and  $x^4$  linearly independent? If so, show it, if not, find a linear combination that works.

**Exercise 2.3.10:** Are  $e^x$ ,  $xe^x$ , and  $x^2e^x$  linearly independent? If so, show it, if not, find a linear combination that works.

**Exercise 2.3.101:** Find the general solution of  $y^{(5)} - y^{(4)} = 0$

**Exercise 2.3.102:** Suppose that the characteristic equation of a third order differential equation has roots  $3, \pm 2i$ . a) What is the characteristic equation? b) Find the corresponding differential equation. c) Find the general solution.

**Exercise 2.3.103:** Solve  $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 0$ ,  $y''(0) = 0$ .

**Exercise 2.3.104:** Are  $e^x$ ,  $e^{x+1}$ ,  $e^{2x}$ ,  $\sin(x)$  linearly independent? If so, show it, if not find a linear combination that works.

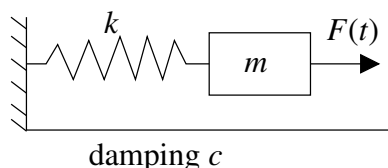
**Exercise 2.3.105:** Are  $\sin(x)$ ,  $x$ ,  $x \sin(x)$  linearly independent? If so, show it, if not find a linear combination that works.

## 2.4 Mechanical vibrations

*Note: 2 lectures, §3.4 in [EP], §3.7 in [BD]*

Let us look at some applications of linear second order constant coefficient equations.

### 2.4.1 Some examples



Our first example is a mass on a spring. Suppose we have a mass  $m > 0$  (in kilograms) connected by a spring with spring constant  $k > 0$  (in newtons per meter) to a fixed wall. There may be some external force  $F(t)$  (in newtons) acting on the mass. Finally, there is some friction measured by  $c \geq 0$  (in newton-seconds per meter) as the mass slides along the floor (or perhaps there is a damper connected).

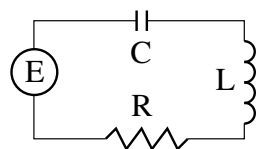
Let  $x$  be the displacement of the mass ( $x = 0$  is the rest position), with  $x$  growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is  $kx$  in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence  $mx'' = F(t) - cx' - kx$  or

$$mx'' + cx' + kx = F(t).$$

This is a linear second order constant coefficient ODE. We set up some terminology about this equation. We say the motion is

- (i) *forced*, if  $F \not\equiv 0$  (if  $F$  is not identically zero),
- (ii) *unforced* or *free*, if  $F \equiv 0$  (if  $F$  is identically zero),
- (iii) *damped*, if  $c > 0$ , and
- (iv) *undamped*, if  $c = 0$ .

This system appears in lots of applications even if it does not at first seem like it. Many real world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give 2 other examples.



Here is an example for electrical engineers. Suppose that we have the pictured RLC circuit. There is a resistor with a resistance of  $R$  ohms, an inductor with an inductance of  $L$  henries, and a capacitor with a capacitance of  $C$  farads. There is also an electric source (such as a battery) giving a voltage of  $E(t)$  volts at time  $t$  (measured in seconds). Let  $Q(t)$  be the charge

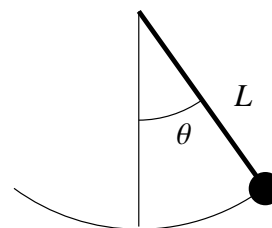
in coulombs on the capacitor and  $I(t)$  be the current in the circuit. The relation between the two is  $Q' = I$ . By elementary principles we have that  $LI' + RI + Q/C = E$ . If we differentiate we get

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

This is an nonhomogeneous second order constant coefficient linear equation. Further, as  $L, R$ , and  $C$  are all positive, this system behaves just like the mass and spring system. The position of the mass is replaced by the current. Mass is replaced by the inductance, damping is replaced by resistance and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function. Hence for constant voltage this is an unforced motion.

Our next example is going to behave like a mass and spring system only approximately. Suppose we have a mass  $m$  on a pendulum of length  $L$ . We wish to find an equation for the angle  $\theta(t)$ . Let  $g$  be the force of gravity. Elementary physics mandates that the equation is of the form

$$\theta'' + \frac{g}{L} \sin \theta = 0.$$



This equation can be derived using Newton's second law; force equals mass times acceleration. The acceleration is  $L\theta''$  and mass is  $m$ . So  $mL\theta''$  has to be equal to the tangential component of the force given by the gravity. This is  $mg \sin \theta$  in the opposite direction. The  $m$  curiously cancels from the equation.

Now we make our approximation. For small  $\theta$  we have that approximately  $\sin \theta \approx \theta$ . This can be seen by looking at the graph. In Figure 2.1 we can see that for approximately  $-0.5 < \theta < 0.5$  (in radians) the graphs of  $\sin \theta$  and  $\theta$  are almost the same.

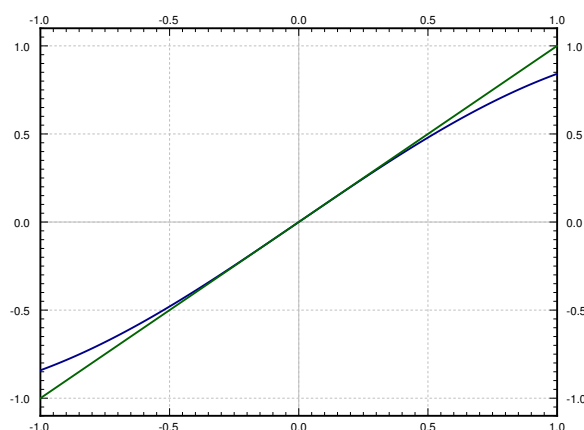


Figure 2.1: The graphs of  $\sin \theta$  and  $\theta$  (in radians).

Therefore, when the swings are small,  $\theta$  is always small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

Note that the errors that we get from the approximation build up. So after a very long time, the behavior of the real system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (for example if the pendulum is very long), the approximation is reasonably good.

In real world problems it is very often necessary to make these types of simplifications. Therefore, it is good to understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

### 2.4.2 Free undamped motion

In this section we will only consider free or unforced motion, as we cannot yet solve nonhomogeneous equations. Let us start with undamped motion where  $c = 0$ . We have the equation

$$mx'' + kx = 0.$$

If we divide by  $m$  and let  $\omega_0 = \sqrt{k/m}$ , then we can write the equation as

$$x'' + \omega_0^2 x = 0.$$

The general solution to this equation is

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

By a trigonometric identity, we have that for two different constants  $C$  and  $\gamma$ , we have

$$A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \gamma).$$

It is not hard to compute that  $C = \sqrt{A^2 + B^2}$  and  $\tan \gamma = B/A$ . Therefore, we let  $C$  and  $\gamma$  be our arbitrary constants and write  $x(t) = C \cos(\omega_0 t - \gamma)$ .

**Exercise 2.4.1:** Justify the above identity and verify the equations for  $C$  and  $\gamma$ . Hint: Start with  $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$  and multiply by  $C$ . Then think what should  $\alpha$  and  $\beta$  be.

While it is generally easier to use the first form with  $A$  and  $B$  to solve for the initial conditions, the second form is much more natural. The constants  $C$  and  $\gamma$  have very nice interpretation. We look at the form of the solution

$$x(t) = C \cos(\omega_0 t - \gamma).$$



We can see that the *amplitude* is  $C$ ,  $\omega_0$  is the (angular) frequency, and  $\gamma$  is the so-called *phase shift*. The phase shift just shifts the graph left or right. We call  $\omega_0$  the *natural (angular) frequency*. This entire setup is usually called *simple harmonic motion*.

Let us pause to explain the word angular before the word frequency. The units of  $\omega_0$  are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because we know one cycle is  $2\pi$  radians, the usual frequency is given by  $\frac{\omega_0}{2\pi}$ . It is simply a matter of where we put the constant  $2\pi$ , and that is a matter of taste.

The *period* of the motion is one over the frequency (in cycles per unit time) and hence  $\frac{2\pi}{\omega_0}$ . That is the amount of time it takes to complete one full oscillation.

**Example 2.4.1:** Suppose that  $m = 2$  kg and  $k = 8$  N/m. The whole mass and spring setup is sitting on a truck that was traveling at 1 m/s. The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at 1 m/s, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation and what is the amplitude. The units are the mks units (meters-kilograms-seconds).

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at 1 m/s. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \quad x(0) = 0.5, \quad x'(0) = 1.$$

We can directly compute  $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$ . Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is  $2/2\pi = 1/\pi \approx 0.318$ .

The general solution is

$$x(t) = A \cos(2t) + B \sin(2t).$$

Letting  $x(0) = 0.5$  means  $A = 0.5$ . Then  $x'(t) = -2(0.5) \sin(2t) + 2B \cos(2t)$ . Letting  $x'(0) = 1$  we get  $B = 0.5$ . Therefore, the amplitude is  $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$ . The solution is

$$x(t) = 0.5 \cos(2t) + 0.5 \sin(2t).$$

A plot of  $x(t)$  is shown in Figure 2.2 on the following page.

In general, for free undamped motion, a solution of the form

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

corresponds to the initial conditions  $x(0) = A$  and  $x'(0) = \omega_0 B$ . Therefore, it is easy to figure out  $A$  and  $B$  from the initial conditions. The amplitude and the phase shift can then be computed from  $A$  and  $B$ . In the example, we have already found the amplitude  $C$ . Let us compute the phase shift. We know that  $\tan \gamma = B/A = 1$ . We take the arctangent of 1 and get approximately 0.785. We still

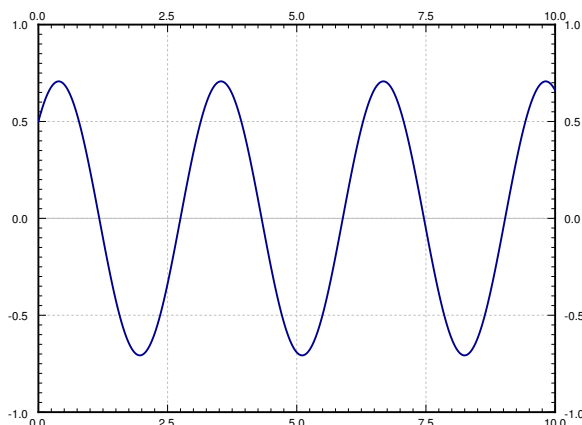


Figure 2.2: Simple undamped oscillation.

need to check if this  $\gamma$  is in the correct quadrant (and add  $\pi$  to  $\gamma$  if it is not). Since both  $A$  and  $B$  are positive, then  $\gamma$  should be in the first quadrant, and 0.785 radians really is in the first quadrant.

Note: Many calculators and computer software do not only have the `atan` function for arctangent, but also what is sometimes called `atan2`. This function takes two arguments,  $B$  and  $A$ , and returns a  $\gamma$  in the correct quadrant for you.

### 2.4.3 Free damped motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx'' + cx' + kx = 0,$$

as

$$x'' + 2px' + \omega_0^2 x = 0,$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}.$$

The characteristic equation is

$$r^2 + 2pr + \omega_0^2 = 0.$$

Using the quadratic formula we get that the roots are

$$r = -p \pm \sqrt{p^2 - \omega_0^2}.$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - \omega_0^2 = \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}.$$

The sign of  $p^2 - \omega_0^2$  is the same as the sign of  $c^2 - 4km$ . Thus we get real roots if and only if  $c^2 - 4km$  is nonnegative, or in other words if  $c^2 \geq 4km$ .

### Overdamping

When  $c^2 - 4km > 0$ , we say the system is *overdamped*. In this case, there are two distinct real roots  $r_1$  and  $r_2$ . Notice that both roots are negative. As  $\sqrt{p^2 - \omega_0^2}$  is always less than  $p$ , then  $-p \pm \sqrt{p^2 - \omega_0^2}$  is negative.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Since  $r_1, r_2$  are negative,  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions, see Figure 2.3.

Do note that no oscillation happens. In fact, the graph will cross the  $x$  axis at most once. To see why, we try to solve  $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ . Therefore,  $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$  and using laws of exponents we obtain

$$\frac{-C_1}{C_2} = e^{(r_2 - r_1)t}.$$

This equation has at most one solution  $t \geq 0$ . For some initial conditions the graph will never cross the  $x$  axis, as is evident from the sample graphs.

**Example 2.4.2:** Suppose the mass is released from rest. That is  $x(0) = x_0$  and  $x'(0) = 0$ . Then

$$x(t) = \frac{x_0}{r_1 - r_2} (r_1 e^{r_2 t} - r_2 e^{r_1 t}).$$

It is not hard to see that this satisfies the initial conditions.

### Critical damping

When  $c^2 - 4km = 0$ , we say the system is *critically damped*. In this case, there is one root of multiplicity 2 and this root is  $-p$ . Therefore, our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}.$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

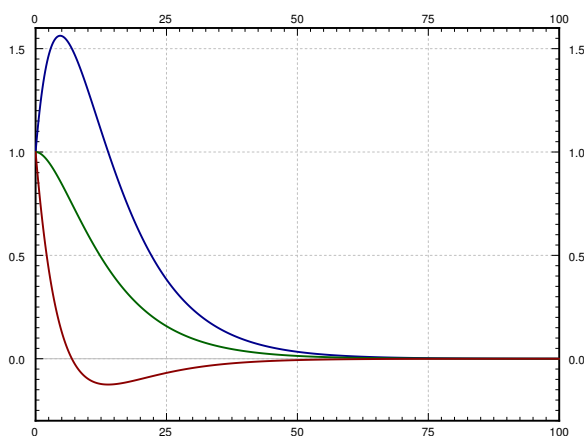


Figure 2.3: Overdamped motion for several different initial conditions.

### Underdamping

When  $c^2 - 4km < 0$ , we say the system is *underdamped*. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - \omega_0^2} \\ &= -p \pm \sqrt{-1} \sqrt{\omega_0^2 - p^2} \\ &= -p \pm i\omega_1, \end{aligned}$$

where  $\omega_1 = \sqrt{\omega_0^2 - p^2}$ . Our solution is

$$x(t) = e^{-pt}(A \cos(\omega_1 t) + B \sin(\omega_1 t)),$$

or

$$x(t) = Ce^{-pt} \cos(\omega_1 t - \gamma).$$

An example plot is given in Figure 2.4. Note that we still have that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In the figure we also show the *envelope curves*  $Ce^{-pt}$  and  $-Ce^{-pt}$ . The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example if you are bungee jumping, you are really interested in computing the envelope curve so that you do not hit the concrete with your head.

The phase shift  $\gamma$  just shifts the graph left or right but within the envelope curves (the envelope curves do not change if  $\gamma$  changes).

Finally note that the angular *pseudo-frequency* (we do not call it a frequency since the solution is not really a periodic function)  $\omega_1$  becomes smaller when the damping  $c$  (and hence  $p$ ) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making  $c$  larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if  $c^2$  approaches  $4km$ , we want  $\omega_1$  to approach 0.

On the other hand when  $c$  becomes smaller,  $\omega_1$  approaches  $\omega_0$  ( $\omega_1$  is always smaller than  $\omega_0$ ), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as  $c$  (and hence  $p$ ) goes to 0.

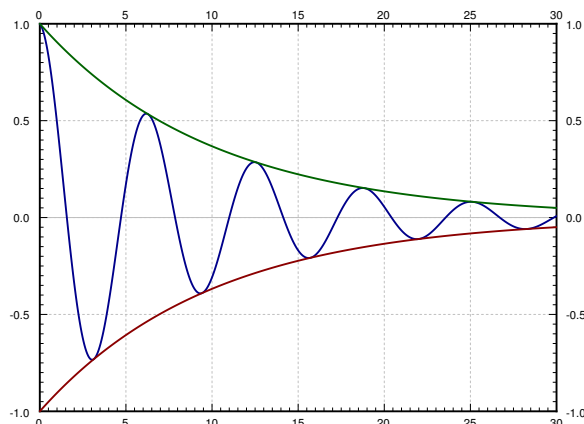


Figure 2.4: Underdamped motion with the envelope curves shown.

### 2.4.4 Exercises

**Exercise 2.4.2:** Consider a mass and spring system with a mass  $m = 2$ , spring constant  $k = 3$ , and damping constant  $c = 1$ . a) Set up and find the general solution of the system. b) Is the system underdamped, overdamped or critically damped? c) If the system is not critically damped, find a  $c$  that makes the system critically damped.

**Exercise 2.4.3:** Do Exercise 2.4.2 for  $m = 3$ ,  $k = 12$ , and  $c = 12$ .

**Exercise 2.4.4:** Using the mks units (meters-kilograms-seconds), suppose you have a spring with spring constant  $4 \text{ N/m}$ . You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion. a) You count and find that the frequency is  $0.8 \text{ Hz}$  (cycles per second). What is the mass? b) Find a formula for the mass  $m$  given the frequency  $\omega$  in  $\text{Hz}$ .

**Exercise 2.4.5:** Suppose we add possible friction to Exercise 2.4.4. Further, suppose you do not know the spring constant, but you have two reference weights  $1 \text{ kg}$  and  $2 \text{ kg}$  to calibrate your setup. You put each in motion on your spring and measure the frequency. For the  $1 \text{ kg}$  weight you measured  $1.1 \text{ Hz}$ , for the  $2 \text{ kg}$  weight you measured  $0.8 \text{ Hz}$ . a) Find  $k$  (spring constant) and  $c$  (damping constant). b) Find a formula for the mass in terms of the frequency in  $\text{Hz}$ . Note that there may be more than one possible mass for a given frequency. c) For an unknown object you measured  $0.2 \text{ Hz}$ , what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.

**Exercise 2.4.6:** Suppose you wish to measure the friction a mass of  $0.1 \text{ kg}$  experiences as it slides along a floor (you wish to find  $c$ ). You have a spring with spring constant  $k = 5 \text{ N/m}$ . You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency  $1 \text{ Hz}$ . What is the friction?

**Exercise 2.4.101:** A mass of  $2 \text{ kilograms}$  is on a spring with spring constant  $k$  newtons per meter with no damping. Suppose the system is at rest and at time  $t = 0$  the mass is kicked and starts travelling at  $2 \text{ meters per second}$ . How large does  $k$  have to be to so that the mass does not go further than  $3 \text{ meters}$  from the rest position.

**Exercise 2.4.102:** Suppose we have an RLC circuit with a resistor of  $100 \text{ miliohms}$  ( $0.1 \text{ ohms}$ ), inductor of inductance of  $50 \text{ millihenries}$  ( $0.05 \text{ henries}$ ), and a capacitor of  $5 \text{ farads}$ , with constant voltage. a) Set up the ODE equation for the current  $I$ . b) Find the general solution. c) Solve for  $I(0) = 10$  and  $I'(0) = 0$ .

**Exercise 2.4.103:** A  $5000 \text{ kg}$  railcar hits a bumper (a spring) at  $1 \text{ m/s}$ , and the spring compresses by  $0.1 \text{ m}$ . Assume no damping. a) Find  $k$ . b) Find out how far does the spring compress when a  $10000 \text{ kg}$  railcar hits the spring at the same speed. c) If the spring would break if it compresses further than  $0.3 \text{ m}$ , what is the maximum mass of a railcar that can hit it at  $1 \text{ m/s}$ ? d) What is the maximum mass of a railcar that can hit the spring without breaking at  $2 \text{ m/s}$ ?

## 2.5 Nonhomogeneous equations

*Note: 2 lectures, §3.5 in [EP], §3.5 and §3.6 in [BD]*

### 2.5.1 Solving nonhomogeneous equations

We have solved linear constant coefficient homogeneous equations. What about nonhomogeneous linear ODEs? For example, the equations for forced mechanical vibrations. That is, suppose we have an equation such as

$$y'' + 5y' + 6y = 2x + 1. \quad (2.6)$$

We will write  $Ly = 2x + 1$  when the exact form of the operator is not important. We solve (2.6) in the following manner. First, we find the general solution  $y_c$  to the *associated homogeneous equation*

$$y'' + 5y' + 6y = 0. \quad (2.7)$$

We call  $y_c$  the *complementary solution*. Next, we find a single *particular solution*  $y_p$  to (2.6) in some way. Then

$$y = y_c + y_p$$

is the general solution to (2.6). We have  $Ly_c = 0$  and  $Ly_p = 2x + 1$ . As  $L$  is a *linear operator* we verify that  $y$  is a solution,  $Ly = L(y_c + y_p) = Ly_c + Ly_p = 0 + (2x + 1)$ . Let us see why we obtain the *general* solution.

Let  $y_p$  and  $\tilde{y}_p$  be two different particular solutions to (2.6). Write the difference as  $w = y_p - \tilde{y}_p$ . Then plug  $w$  into the left hand side of the equation to get

$$w'' + 5w' + 6w = (y_p'' + 5y_p' + 6y_p) - (\tilde{y}_p'' + 5\tilde{y}_p' + 6\tilde{y}_p) = (2x + 1) - (2x + 1) = 0.$$

Using the operator notation the calculation becomes simpler. As  $L$  is a linear operator we write

$$Lw = L(y_p - \tilde{y}_p) = Ly_p - L\tilde{y}_p = (2x + 1) - (2x + 1) = 0.$$

So  $w = y_p - \tilde{y}_p$  is a solution to (2.7), that is  $Lw = 0$ . Any two solutions of (2.6) differ by a solution to the homogeneous equation (2.7). The solution  $y = y_c + y_p$  includes *all* solutions to (2.6), since  $y_c$  is the general solution to the associated homogeneous equation.

**Theorem 2.5.1.** *Let  $Ly = f(x)$  be a linear ODE (not necessarily constant coefficient). Let  $y_c$  be the general solution to the associated homogeneous equation  $Ly = 0$  and let  $y_p$  be any particular solution to  $Ly = f(x)$ . Then the general solution to  $Ly = f(x)$  is*

$$y = y_c + y_p.$$

The moral of the story is that we can find the particular solution in any old way. If we find a different particular solution (by a different method, or simply by guessing), then we still get the same general solution. The formula may look different, and the constants we will have to choose to satisfy the initial conditions may be different, but it is the same solution.

### 2.5.2 Undetermined coefficients

The trick is to somehow, in a smart way, guess one particular solution to (2.6). Note that  $2x + 1$  is a polynomial, and the left hand side of the equation will be a polynomial if we let  $y$  be a polynomial of the same degree. Let us try

$$y_p = Ax + B.$$

We plug in to obtain

$$y_p'' + 5y_p' + 6y_p = (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) = 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B).$$

So  $6Ax + (5A + 6B) = 2x + 1$ . Therefore,  $A = 1/3$  and  $B = -1/9$ . That means  $y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x-1}{9}$ . Solving the complementary problem (exercise!) we get

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}.$$

Hence the general solution to (2.6) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x-1}{9}.$$

Now suppose we are further given some initial conditions. For example,  $y(0) = 0$  and  $y'(0) = 1/3$ . First find  $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + 1/3$ . Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \quad \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}.$$

We solve to get  $C_1 = 1/3$  and  $C_2 = -2/9$ . The particular solution we want is

$$y(x) = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}.$$

**Exercise 2.5.1:** Check that  $y$  really solves the equation (2.6) and the given initial conditions.

Note: A common mistake is to solve for constants using the initial conditions with  $y_c$  and only add the particular solution  $y_p$  after that. That will *not* work. You need to first compute  $y = y_c + y_p$  and *only then* solve for the constants using the initial conditions.

A right hand side consisting of exponentials, sines, and cosines can be handled similarly. For example,

$$y'' + 2y' + 2y = \cos(2x).$$

Let us find some  $y_p$ . We start by guessing the solution includes some multiple of  $\cos(2x)$ . We may have to also add a multiple of  $\sin(2x)$  to our guess since derivatives of cosine are sines. We try

$$y_p = A \cos(2x) + B \sin(2x).$$

We plug  $y_p$  into the equation and we get

$$-4A \cos(2x) - 4B \sin(2x) - 4A \sin(2x) + 4B \cos(2x) + 2A \cos(2x) + 2B \sin(2x) = \cos(2x).$$

The left hand side must equal to right hand side. We group terms and we get that  $-4A + 4B + 2A = 1$  and  $-4B - 4A + 2B = 0$ . So  $-2A + 4B = 1$  and  $2A + B = 0$  and hence  $A = -1/10$  and  $B = 1/5$ . So

$$y_p = A \cos(2x) + B \sin(2x) = \frac{-\cos(2x) + 2 \sin(2x)}{10}.$$

Similarly, if the right hand side contains exponentials we try exponentials. For example, for

$$Ly = e^{3x},$$

we will try  $y = Ae^{3x}$  as our guess and try to solve for  $A$ .

When the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for  $y_p$  such that  $Ly_p$  is of the same form, and has all the terms needed to for the right hand side. For example,

$$Ly = (1 + 3x^2) e^{-x} \cos(\pi x).$$

For this equation, we will guess

$$y_p = (A + Bx + Cx^2) e^{-x} \cos(\pi x) + (D + Ex + Fx^2) e^{-x} \sin(\pi x).$$

We will plug in and then hopefully get equations that we can solve for  $A, B, C, D, E, F$ . As you can see this can make for a very long and tedious calculation very quickly. C'est la vie!

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation. That is, suppose we have

$$y'' - 9y = e^{3x}.$$

We would love to guess  $y = Ae^{3x}$ , but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}.$$

There is no way we can choose  $A$  to make the left hand side be  $e^{3x}$ . The trick in this case is to multiply our guess by  $x$  to get rid of duplication with the complementary solution. That is first we compute  $y_c$  (solution to  $Ly = 0$ )

$$y_c = C_1 e^{-3x} + C_2 e^{3x}$$

and we note that the  $e^{3x}$  term is a duplicate with our desired guess. We modify our guess to  $y = Axe^{3x}$  and notice there is no duplication anymore. Let us try. Note that  $y' = Ae^{3x} + 3Axe^{3x}$  and  $y'' = 6Ae^{3x} + 9Axe^{3x}$ . So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}.$$



So  $6Ae^{3x}$  is supposed to equal  $e^{3x}$ . Hence,  $6A = 1$  and so  $A = 1/6$ . Thus we can now write the general solution as

$$y = y_c + y_p = C_1e^{-3x} + C_2e^{3x} + \frac{1}{6}xe^{3x}.$$

It is possible that multiplying by  $x$  does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}.$$

The complementary solution is  $y_c = C_1e^{3x} + C_2xe^{3x}$ . Guessing  $y = Axe^{3x}$  would not get us anywhere. In this case we want to guess  $y_p = Ax^2e^{3x}$ . Basically, we want to multiply our guess by  $x$  until all duplication is gone. *But no more!* Multiplying too many times will not work.

Finally, what if the right hand side has several terms, such as

$$Ly = e^{2x} + \cos x.$$

In this case we find  $u$  that solves  $Lu = e^{2x}$  and  $v$  that solves  $Lv = \cos x$  (that is, do each term separately). Then note that if  $y = u + v$ , then  $Ly = e^{2x} + \cos x$ . This is because  $L$  is linear; we have  $Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x$ .

### 2.5.3 Variation of parameters

The method of undetermined coefficients will work for many basic problems that crop up. But it does not work all the time. It only works when the right hand side of the equation  $Ly = f(x)$  has only finitely many linearly independent derivatives, so that we can write a guess that consists of them all. Some equations are a bit tougher. Consider

$$y'' + y = \tan x.$$

Note that each new derivative of  $\tan x$  looks completely different and cannot be written as a linear combination of the previous derivatives. We get  $\sec^2 x$ ,  $2\sec^2 x \tan x$ , etc. . . .

This equation calls for a different method. We present the method of *variation of parameters*, which will handle any equation of the form  $Ly = f(x)$ , provided we can solve certain integrals. For simplicity, we will restrict ourselves to second order constant coefficient equations, but the method will work for higher order equations just as well (the computations will be more tedious). The method also works for equations with nonconstant coefficients, provided we can solve the associated homogeneous equation.

Perhaps it is best to explain this method by example. Let us try to solve the equation

$$Ly = y'' + y = \tan x.$$

First we find the complementary solution (solution to  $Ly_c = 0$ ). We get  $y_c = C_1y_1 + C_2y_2$ , where  $y_1 = \cos x$  and  $y_2 = \sin x$ . Now to try to find a solution to the nonhomogeneous equation we try

$$y_p = y = u_1y_1 + u_2y_2,$$

where  $u_1$  and  $u_2$  are *functions* and not constants. We are trying to satisfy  $Ly = \tan x$ . That gives us one condition on the functions  $u_1$  and  $u_2$ . Compute (note the product rule!)

$$y' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2').$$

We can still impose one more condition at our discretion to simplify computations (we have two unknown functions, so we should be allowed two conditions). We require that  $(u_1' y_1 + u_2' y_2) = 0$ . This makes computing the second derivative easier.

$$\begin{aligned} y' &= u_1 y_1' + u_2 y_2', \\ y'' &= (u_1' y_1' + u_2' y_2') + (u_1 y_1'' + u_2 y_2''). \end{aligned}$$

Since  $y_1$  and  $y_2$  are solutions to  $y'' + y = 0$ , we know that  $y_1'' = -y_1$  and  $y_2'' = -y_2$ . (Note: If the equation was instead  $y'' + p(x)y' + q(x)y = 0$  we would have  $y_i'' = -p(x)y_i' - q(x)y_i$ .) So

$$y'' = (u_1' y_1' + u_2' y_2') - (u_1 y_1 + u_2 y_2).$$

We have  $(u_1 y_1 + u_2 y_2) = y$  and so

$$y'' = (u_1' y_1' + u_2' y_2') - y,$$

and hence

$$y'' + y = Ly = u_1' y_1' + u_2' y_2'.$$

For  $y$  to satisfy  $Ly = f(x)$  we must have  $f(x) = u_1' y_1' + u_2' y_2'$ .

So what we need to solve are the two equations (conditions) we imposed on  $u_1$  and  $u_2$

$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}$
---

We can now solve for  $u_1'$  and  $u_2'$  in terms of  $f(x)$ ,  $y_1$  and  $y_2$ . We will always get these formulas for any  $Ly = f(x)$ , where  $Ly = y'' + p(x)y' + q(x)y$ . There is a general formula for the solution we can just plug into, but it is better to just repeat what we do below. In our case the two equations become

$$\begin{aligned} u_1' \cos(x) + u_2' \sin(x) &= 0, \\ -u_1' \sin(x) + u_2' \cos(x) &= \tan(x). \end{aligned}$$

Hence

$$\begin{aligned} u_1' \cos(x) \sin(x) + u_2' \sin^2(x) &= 0, \\ -u_1' \sin(x) \cos(x) + u_2' \cos^2(x) &= \tan(x) \cos(x) = \sin(x). \end{aligned}$$

And thus

$$\begin{aligned}u_2'(\sin^2(x) + \cos^2(x)) &= \sin(x), \\u_2' &= \sin(x), \\u_1' &= \frac{-\sin^2(x)}{\cos(x)} = -\tan(x) \sin(x).\end{aligned}$$

Now we need to integrate  $u_1'$  and  $u_2'$  to get  $u_1$  and  $u_2$ .

$$\begin{aligned}u_1 &= \int u_1' dx = \int -\tan(x) \sin(x) dx = \frac{1}{2} \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right| + \sin(x), \\u_2 &= \int u_2' dx = \int \sin(x) dx = -\cos(x).\end{aligned}$$

So our particular solution is

$$\begin{aligned}y_p = u_1 y_1 + u_2 y_2 &= \frac{1}{2} \cos(x) \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right| + \cos(x) \sin(x) - \cos(x) \sin(x) = \\&= \frac{1}{2} \cos(x) \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right|.\end{aligned}$$

The general solution to  $y'' + y = \tan x$  is, therefore,

$$y = C_1 \cos(x) + C_2 \sin(x) + \frac{1}{2} \cos(x) \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right|.$$

## 2.5.4 Exercises

**Exercise 2.5.2:** Find a particular solution of  $y'' - y' - 6y = e^{2x}$ .

**Exercise 2.5.3:** Find a particular solution of  $y'' - 4y' + 4y = e^{2x}$ .

**Exercise 2.5.4:** Solve the initial value problem  $y'' + 9y = \cos(3x) + \sin(3x)$  for  $y(0) = 2$ ,  $y'(0) = 1$ .

**Exercise 2.5.5:** Setup the form of the particular solution but do not solve for the coefficients for  $y^{(4)} - 2y''' + y'' = e^x$ .

**Exercise 2.5.6:** Setup the form of the particular solution but do not solve for the coefficients for  $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$ .

**Exercise 2.5.7:** a) Using variation of parameters find a particular solution of  $y'' - 2y' + y = e^x$ . b) Find a particular solution using undetermined coefficients. c) Are the two solutions you found the same? What is going on?

**Exercise 2.5.8:** Find a particular solution of  $y'' - 2y' + y = \sin(x^2)$ . It is OK to leave the answer as a definite integral.

**Exercise 2.5.101:** Find a particular solution to  $y'' - y' + y = 2 \sin(3x)$

**Exercise 2.5.102:** a) Find a particular solution to  $y'' + 2y = e^x + x^3$ . b) Find the general solution.

**Exercise 2.5.103:** Solve  $y'' + 2y' + y = x^2$ ,  $y(0) = 1$ ,  $y'(0) = 2$ .

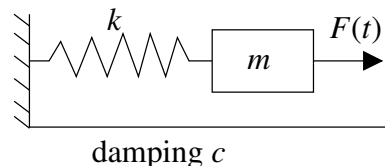
**Exercise 2.5.104:** Use variation of parameters to find a particular solution of  $y'' - y = \frac{1}{e^x + e^{-x}}$ .

## 2.6 Forced oscillations and resonance

Note: 2 lectures, §3.6 in [EP], §3.8 in [BD]

Let us return back to the mass on a spring example. We will now consider the case of forced oscillations. That is, we will consider the equation

$$mx'' + cx' + kx = F(t)$$



for some nonzero  $F(t)$ . The setup is again:  $m$  is mass,  $c$  is friction,  $k$  is the spring constant and  $F(t)$  is an external force acting on the mass.

What we are interested in is periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series in chapter 4 we will see that we cover all periodic functions by simply considering  $F(t) = F_0 \cos(\omega t)$  (or sine instead of cosine, the calculations will be essentially the same).

### 2.6.1 Undamped forced motion and resonance

First let us consider undamped ( $c = 0$ ) motion for simplicity. We have the equation

$$mx'' + kx = F_0 \cos(\omega t).$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),$$

where  $\omega_0 = \sqrt{k/m}$  is the *natural frequency* (angular). It is the frequency at which the system “wants to oscillate” without external interference.

Let us suppose that  $\omega_0 \neq \omega$ . We try the solution  $x_p = A \cos(\omega t)$  and solve for  $A$ . Note that we need not have sine in our trial solution as on the left hand side we will only get cosines anyway. If you include a sine it is fine; you will find that its coefficient will be zero (I could not find a rhyme).

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

or written another way

$$x = C \cos(\omega_0 t - \gamma) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Hence it is a superposition of two cosine waves at different frequencies.

**Example 2.6.1:** Take

$$0.5x'' + 8x = 10 \cos(\pi t), \quad x(0) = 0, \quad x'(0) = 0.$$

Let us compute. First we read off the parameters:  $\omega = \pi$ ,  $\omega_0 = \sqrt{8/0.5} = 4$ ,  $F_0 = 10$ ,  $m = 0.5$ . The general solution is

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16 - \pi^2} \cos(\pi t).$$

Solve for  $C_1$  and  $C_2$  using the initial conditions. It is easy to see that  $C_1 = \frac{-20}{16 - \pi^2}$  and  $C_2 = 0$ . Hence

$$x = \frac{20}{16 - \pi^2} (\cos(\pi t) - \cos(4t)).$$

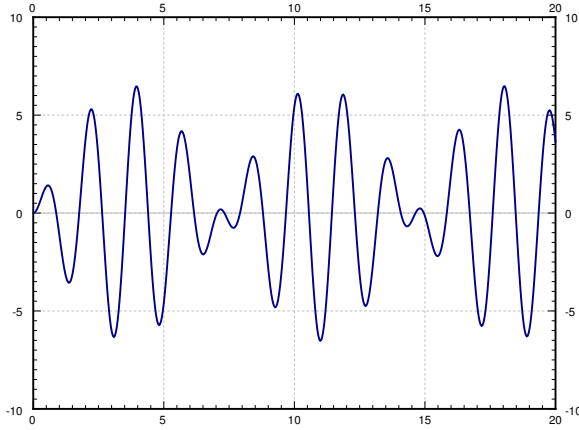


Figure 2.5: Graph of  $\frac{20}{16 - \pi^2} (\cos(\pi t) - \cos(4t))$ .

that  $\cos(\omega t)$  solves the associated homogeneous equation. Therefore, we need to try  $x_p = At \cos(\omega t) + Bt \sin(\omega t)$ . This time we do need the sine term since the second derivative of  $t \cos(\omega t)$  does contain sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m} \cos(\omega t).$$

Plugging into the left hand side we get

$$2B\omega \cos(\omega t) - 2A\omega \sin(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

Notice the “beating” behavior in Figure 2.5. First use the trigonometric identity

$$2 \sin\left(\frac{A - B}{2}\right) \sin\left(\frac{A + B}{2}\right) = \cos B - \cos A$$

to get that

$$x = \frac{20}{16 - \pi^2} \left( 2 \sin\left(\frac{4 - \pi}{2} t\right) \sin\left(\frac{4 + \pi}{2} t\right) \right).$$

Notice that  $x$  is a high frequency wave modulated by a low frequency wave.

Now suppose that  $\omega_0 = \omega$ . Obviously, we cannot try the solution  $A \cos(\omega t)$  and then use the method of undetermined coefficients. We notice

Hence  $A = 0$  and  $B = \frac{F_0}{2m\omega}$ . Our particular solution is  $\frac{F_0}{2m\omega} t \sin(\omega t)$  and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t).$$

The important term is the last one (the particular solution we found). We can see that this term grows without bound as  $t \rightarrow \infty$ . In fact it oscillates between  $\frac{F_0 t}{2m\omega}$  and  $-\frac{F_0 t}{2m\omega}$ . The first two terms only oscillate between  $\pm \sqrt{C_1^2 + C_2^2}$ , which becomes smaller and smaller in proportion to the oscillations of the last term as  $t$  gets larger. In Figure 2.6 we see the graph with  $C_1 = C_2 = 0$ ,  $F_0 = 2$ ,  $m = 1$ ,  $\omega = \pi$ .

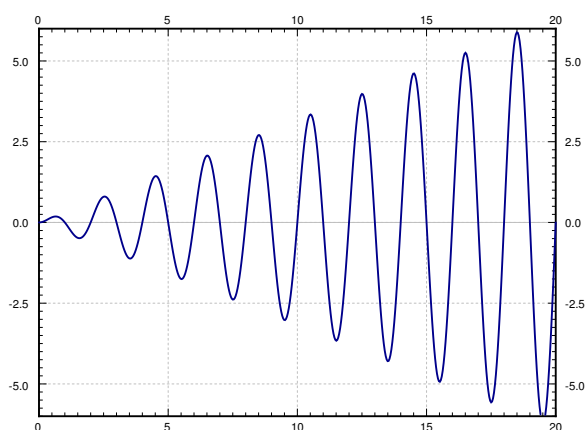


Figure 2.6: Graph of  $\frac{1}{\pi}t \sin(\pi t)$ .

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called *resonance* or sometimes *pure resonance*. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the correct “frequency”? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance

frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play there\*.

## 2.6.2 Damped forced motion and practical resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \quad (2.8)$$

for some  $c > 0$ . We have solved the homogeneous problem before. We let

$$p = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

---

\*K. Billah and R. Scanlan, *Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks*, American Journal of Physics, 59(2), 1991, 118–124, <http://www.ketchum.org/billah/Billah-Scanlan.pdf>

We replace equation (2.8) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t).$$

We find the roots of the characteristic equation of the associated homogeneous problem are  $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$ . The form of the general solution of the associated homogeneous equation depends on the sign of  $p^2 - \omega_0^2$ , or equivalently on the sign of  $c^2 - 4km$ , as we have seen before. That is

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } c^2 > 4km, \\ C_1 e^{-pt} + C_2 t e^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt}(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)) & \text{if } c^2 < 4km, \end{cases}$$

where  $\omega_1 = \sqrt{\omega_0^2 - p^2}$ . In any case, we can see that  $x_c(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Furthermore, there can be no conflicts when trying to solve for the undetermined coefficients by trying  $x_p = A \cos(\omega t) + B \sin(\omega t)$ . Let us plug in and solve for  $A$  and  $B$ . We get (the tedious details are left to reader)

$$((\omega_0^2 - \omega^2)B - 2\omega pA) \sin(\omega t) + ((\omega_0^2 - \omega^2)A + 2\omega pB) \cos(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

We get that

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2},$$

$$B = \frac{2\omega pF_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}.$$

We also compute  $C = \sqrt{A^2 + B^2}$  to be

$$C = \frac{F_0}{m \sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}.$$

Thus our particular solution is

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega pF_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t).$$

Or in the alternative notation we have amplitude  $C$  and phase shift  $\gamma$  where (if  $\omega \neq \omega_0$ )

$$\tan \gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}.$$



Hence we have

$$x_p = \frac{F_0}{m \sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}} \cos(\omega t - \gamma).$$

If  $\omega = \omega_0$  we see that  $A = 0$ ,  $B = C = \frac{F_0}{2m\omega p}$ , and  $\gamma = \pi/2$ .

The exact formula is not as important as the idea. You should not memorize the above formula, you should remember the ideas involved. For different forcing function  $F$ , you will get a different formula for  $x_p$ . So there is no point in memorizing this specific formula. You can always recompute it later or look it up if you really need it.

For reasons we will explain in a moment, we will call  $x_c$  the *transient solution* and denote it by  $x_{tr}$ . We will call the  $x_p$  we found above the *steady periodic solution* and denote it by  $x_{sp}$ . The general solution to our problem is

$$x = x_c + x_p = x_{tr} + x_{sp}.$$

We note that  $x_c = x_{tr}$  goes to zero as  $t \rightarrow \infty$ , as all the terms involve an exponential with a negative exponent. Hence for large  $t$ , the effect of  $x_{tr}$  is negligible and we will essentially only see  $x_{sp}$ . Hence the name transient. Notice that  $x_{sp}$  involves no arbitrary constants, and the initial conditions will only affect  $x_{tr}$ . This means that the effect of the initial conditions will be negligible after some period of time. Because of this behavior, we might as well focus on the steady periodic solution and ignore the transient solution. See Figure 2.7 for a graph of different initial conditions.

Notice that the speed at which  $x_{tr}$  goes to zero depends on  $p$  (and hence  $c$ ). The bigger  $p$  is (the bigger  $c$  is), the “faster”  $x_{tr}$  becomes negligible. So the smaller the damping, the longer the “transient region.” This agrees with the observation that when  $c = 0$ , the initial conditions affect the behavior for all time (i.e. an infinite “transient region”).

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. What we will look at however is the maximum value of the amplitude of the steady periodic solution. Let  $C$  be the amplitude of  $x_{sp}$ . If we plot  $C$  as a function of  $\omega$  (with all other parameters fixed) we can find its maximum. We call the  $\omega$  that achieves this maximum the *practical resonance frequency*. We call

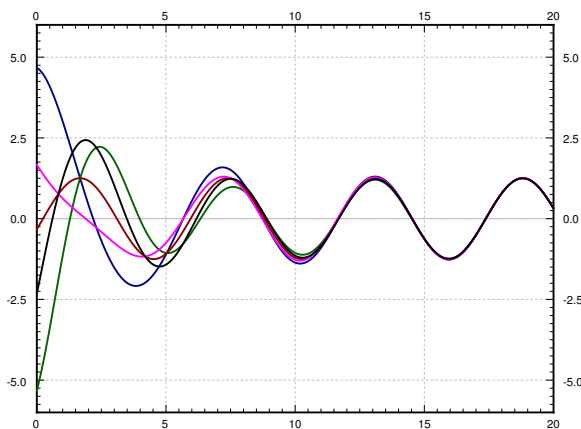


Figure 2.7: Solutions with different initial conditions for parameters  $k = 1$ ,  $m = 1$ ,  $F_0 = 1$ ,  $c = 0.7$ , and  $\omega = 1.1$ .

the maximal amplitude  $C(\omega)$  the *practical resonance amplitude*. Thus when damping is present we talk of *practical resonance* rather than pure resonance. A sample plot for three different values of  $c$  is given in Figure 2.8. As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

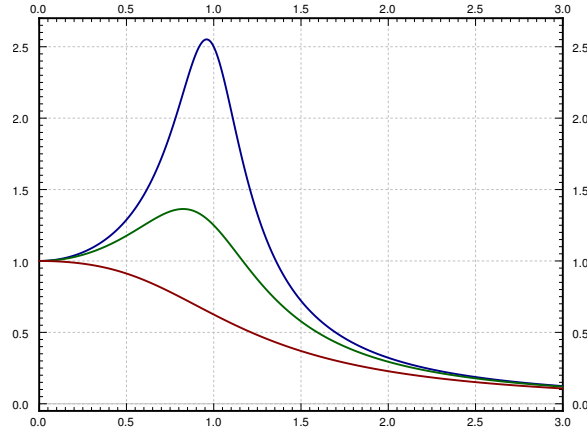


Figure 2.8: Graph of  $C(\omega)$  showing practical resonance with parameters  $k = 1$ ,  $m = 1$ ,  $F_0 = 1$ . The top line is with  $c = 0.4$ , the middle line with  $c = 0.8$ , and the bottom line with  $c = 1.6$ .

To find the maximum we need to find the derivative  $C'(\omega)$ . Computation shows

$$C'(\omega) = \frac{-4\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m((2\omega p)^2 + (\omega_0^2 - \omega^2)^2)^{3/2}}.$$

This is zero either when  $\omega = 0$  or when  $2p^2 + \omega^2 - \omega_0^2 = 0$ . In other words,  $C'(\omega) = 0$  when

$$\omega = \sqrt{\omega_0^2 - 2p^2} \quad \text{or} \quad \omega = 0.$$

It can be shown that if  $\omega_0^2 - 2p^2$  is positive, then  $\sqrt{\omega_0^2 - 2p^2}$  is the practical resonance frequency (that is the point where  $C(\omega)$  is maximal, note that in this case  $C'(\omega) > 0$  for small  $\omega$ ). If  $\omega = 0$  is the maximum, then essentially there is no practical resonance since we assume that  $\omega > 0$  in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than  $\omega_0$ . As the damping  $c$  (and hence  $p$ ) becomes smaller, the practical resonance frequency goes to  $\omega_0$ . So when damping is very small,  $\omega_0$  is a good estimate of the resonance frequency. This behavior agrees with the observation that when  $c = 0$ , then  $\omega_0$  is the resonance frequency.

The behavior will be more complicated if the forcing function is not an exact cosine wave, but for example a square wave. It will be good to come back to this section once we have learned about the Fourier series.

### 2.6.3 Exercises

**Exercise 2.6.1:** Derive a formula for  $x_{sp}$  if the equation is  $mx'' + cx' + kx = F_0 \sin(\omega t)$ . Assume  $c > 0$ .

**Exercise 2.6.2:** Derive a formula for  $x_{sp}$  if the equation is  $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$ . Assume  $c > 0$ .

**Exercise 2.6.3:** Take  $mx'' + cx' + kx = F_0 \cos(\omega t)$ . Fix  $m > 0$  and  $k > 0$ . Now think of the function  $C(\omega)$ . For what values of  $c$  (solve in terms of  $m$ ,  $k$ , and  $F_0$ ) will there be no practical resonance (that is, for what values of  $c$  is there no maximum of  $C(\omega)$  for  $\omega > 0$ ).

**Exercise 2.6.4:** Take  $mx'' + cx' + kx = F_0 \cos(\omega t)$ . Fix  $c > 0$  and  $k > 0$ . Now think of the function  $C(\omega)$ . For what values of  $m$  (solve in terms of  $c$ ,  $k$ , and  $F_0$ ) will there be no practical resonance (that is, for what values of  $m$  is there no maximum of  $C(\omega)$  for  $\omega > 0$ ).

**Exercise 2.6.5:** Suppose a water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as a mass and the support acts as the spring, where the induced vibrations are horizontal. Suppose that the container with water has a mass of  $m = 10,000$  kg. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving).

Suppose that an earthquake induces an external force  $F(t) = mA\omega^2 \cos(\omega t)$ .

a) What is the natural frequency of the water tower.

b) If  $\omega$  is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.

c) Suppose  $A = 1$  and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations. Suppose that if the water tower moves more than 1.5 meter, the tower collapses. Will the tower collapse?

**Exercise 2.6.101:** A mass of 4 kg on a spring with  $k = 4$  and a damping constant  $c = 1$ . Suppose that  $F_0 = 2$ . Using forcing of  $F_0 \cos(\omega t)$ . Find the  $\omega$  that causes practical resonance and find the amplitude.

**Exercise 2.6.102:** Derive a formula for  $x_{sp}$  for  $mx'' + cx' + kx = F_0 \cos(\omega t) + A$  where  $A$  is some constant. Assume  $c > 0$ .

**Exercise 2.6.103:** Suppose there is no damping in a mass and spring system with  $m = 5$ ,  $k = 20$ , and  $F_0 = 5$ . Suppose that  $\omega$  is chosen to be precisely the resonance frequency. a) Find  $\omega$ . b) Find the amplitude of the oscillations at time  $t = 100$ .



# Chapter 3

## Systems of ODEs

### 3.1 Introduction to systems of ODEs

*Note: 1 lecture, §4.1 in [EP], §7.1 in [BD]*

Often we do not have just one dependent variable and one equation. And as we will see, we may end up with systems of several equations and several dependent variables even if we start with a single equation.

If we have several dependent variables, suppose  $y_1, y_2, \dots, y_n$ , then we can have a differential equation involving all of them and their derivatives. For example,  $y_1'' = f(y_1', y_2', y_1, y_2, x)$ . Usually, when we have two dependent variables we would have two equations such as

$$\begin{aligned}y_1'' &= f_1(y_1', y_2', y_1, y_2, x), \\y_2'' &= f_2(y_1', y_2', y_1, y_2, x),\end{aligned}$$

for some functions  $f_1$  and  $f_2$ . We call the above a *system of differential equations*. More precisely, the above is a second order system of ODEs.

**Example 3.1.1:** Sometimes a system is easy to solve by solving for one variable and then for the second variable. Take the first order system

$$\begin{aligned}y_1' &= y_1, \\y_2' &= y_1 - y_2,\end{aligned}$$

with initial conditions of the form  $y_1(0) = 1, y_2(0) = 2$ .

We note that  $y_1 = C_1 e^x$  is the general solution of the first equation. We can then plug this  $y_1$  into the second equation and get the equation  $y_2' = C_1 e^x - y_2$ , which is a linear first order equation that is easily solved for  $y_2$ . By the method of integrating factor we get

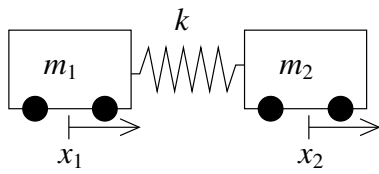
$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2,$$

or  $y_2 = \frac{C_1}{2}e^x + C_2e^{-x}$ . The general solution to the system is, therefore,

$$\begin{aligned} y_1 &= C_1e^x, \\ y_2 &= \frac{C_1}{2}e^x + C_2e^{-x}. \end{aligned}$$

We can now solve for  $C_1$  and  $C_2$  given the initial conditions. We substitute  $x = 0$  and find that  $C_1 = 1$  and  $C_2 = 3/2$ . Thus the solution is  $y_1 = e^x$ , and  $y_2 = (1/2)e^x + (3/2)e^{-x}$ .

Generally, we will not be so lucky to be able to solve for each variable separately as in the example above, and we will have to solve for all variables at once.



As an example application, let us think of mass and spring systems again. Suppose we have one spring with constant  $k$ , but two masses  $m_1$  and  $m_2$ . We can think of the masses as carts, and we will suppose that they ride along a straight track with no friction. Let  $x_1$  be the displacement of the first cart and  $x_2$  be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then  $x_1$  measures how far the first cart is from its zero position, and  $x_2$  measures how far the second cart is from its zero position. The force exerted by the spring on the first cart is  $k(x_2 - x_1)$ , since  $x_2 - x_1$  is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign. Newton's second law states that force equals mass times acceleration. So the system of equations governing the setup is

$$\begin{aligned} m_1 x_1'' &= k(x_2 - x_1), \\ m_2 x_2'' &= -k(x_2 - x_1). \end{aligned}$$

In this system we cannot solve for the  $x_1$  or  $x_2$  variable separately. That we must solve for both  $x_1$  and  $x_2$  at once is intuitively clear, since where the first cart goes depends exactly on where the second cart goes and vice-versa.

Before we talk about how to handle systems, let us note that in some sense we need only consider first order systems. Let us take an  $n^{\text{th}}$  order differential equation

$$y^{(n)} = F(y^{(n-1)}, \dots, y', y, x).$$

We define new variables  $u_1, \dots, u_n$  and write the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ &\vdots \\ u_{n-1}' &= u_n, \\ u_n' &= F(u_n, u_{n-1}, \dots, u_2, u_1, x). \end{aligned}$$

We solve this system for  $u_1, u_2, \dots, u_n$ . Once we have solved for the  $u$ 's, we can discard  $u_2$  through  $u_n$  and let  $y = u_1$ . We note that this  $y$  solves the original equation.

A similar process can be followed for a system of higher order differential equations. For example, a system of  $k$  differential equations in  $k$  unknowns, all of order  $n$ , can be transformed into a first order system of  $n \times k$  equations and  $n \times k$  unknowns.

**Example 3.1.2:** Sometimes we can use this idea in reverse as well. Let us take the system

$$x' = 2y - x, \quad y' = x,$$

where the independent variable is  $t$ . We wish to solve for the initial conditions  $x(0) = 1, y(0) = 0$ .

If we differentiate the second equation we get  $y'' = x'$ . We know what  $x'$  is in terms of  $x$  and  $y$ , and we know that  $x = y'$ .

$$y'' = x' = 2y - x = 2y - y'.$$

So we now have an equation  $y'' + y' - 2y = 0$ . We know how to solve this equation and we find that  $y = C_1 e^{-2t} + C_2 e^t$ . Once we have  $y$  we can plug in to get  $x$ .

$$x = y' = -2C_1 e^{-2t} + C_2 e^t.$$

We solve for the initial conditions  $1 = x(0) = -2C_1 + C_2$  and  $0 = y(0) = C_1 + C_2$ . Hence,  $C_1 = -C_2$  and  $1 = 3C_2$ . So  $C_1 = -1/3$  and  $C_2 = 1/3$ . Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \quad y = \frac{-e^{-2t} + e^t}{3}.$$

**Exercise 3.1.1:** Plug in and check that this really is the solution.

It is useful to go back and forth between systems and higher order equations for other reasons. For example, the ODE approximation methods are generally only given as solutions for first order systems. It is not very hard to adapt the code for the Euler method for a first order equation to first order systems. We essentially just treat the dependent variable not as a number but as a vector. In many mathematical computer languages there is almost no distinction in syntax.

In fact, this is what IODE was doing when you had it solve a second order equation numerically in the IODE Project III if you have done that project.

The above example was what we will call a *linear first order system*, as none of the dependent variables appear in any functions or with any higher powers than one. It is also *autonomous* as the equations do not depend on the independent variable  $t$ .

For autonomous systems we can easily draw the so-called *direction field* or *vector field*. That is, a plot similar to a slope field, but instead of giving a slope at each point, we give a direction (and a magnitude). The previous example  $x' = 2y - x, y' = x$  says that at the point  $(x, y)$  the direction in which we should travel to satisfy the equations should be the direction of the vector  $(2y - x, x)$  with the speed equal to the magnitude of this vector. So we draw the vector  $(2y - x, x)$  based at the point

$(x, y)$  and we do this for many points on the  $xy$ -plane. We may want to scale down the size of our vectors to fit many of them on the same direction field. See Figure 3.1.

We can now draw a path of the solution in the plane. That is, suppose the solution is given by  $x = f(t)$ ,  $y = g(t)$ , then we can pick an interval of  $t$  (say  $0 \leq t \leq 2$  for our example) and plot all the points  $(f(t), g(t))$  for  $t$  in the selected range. The resulting picture is usually called the *phase portrait* (or phase plane portrait). The particular curve obtained we call the *trajectory* or *solution curve*. An example plot is given in Figure 3.2. In this figure the line starts at  $(1, 0)$  and travels along the vector field for a distance of 2 units of  $t$ . Since we solved this system precisely we can compute  $x(2)$  and  $y(2)$ . We get that  $x(2) \approx 2.475$  and  $y(2) \approx 2.457$ . This point corresponds to the top right end of the plotted solution curve in the figure.

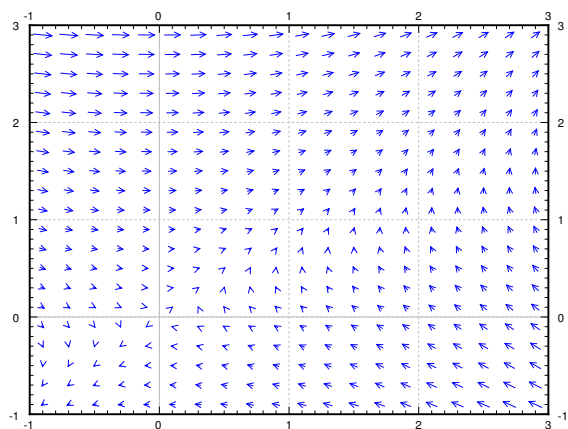


Figure 3.1: The direction field for  $x' = 2y - x$ ,  $y' = x$ .

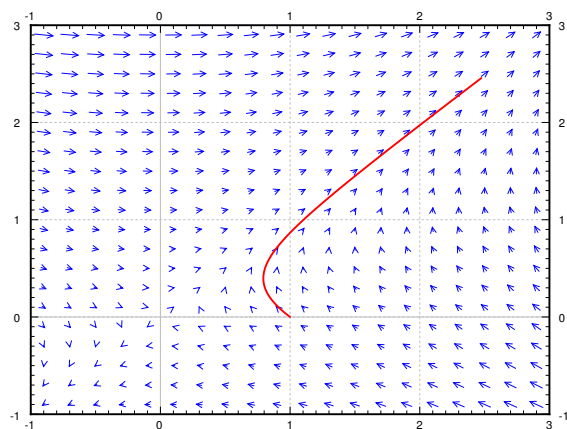


Figure 3.2: The direction field for  $x' = 2y - x$ ,  $y' = x$  with the trajectory of the solution starting at  $(1, 0)$  for  $0 \leq t \leq 2$ .

Notice the similarity to the diagrams we drew for autonomous systems in one dimension. But now note how much more complicated things become if we allow just one more dimension.

Also note that we can draw phase portraits and trajectories in the  $xy$ -plane even if the system is not autonomous. In this case however we cannot draw the direction field, since the field changes as  $t$  changes. For each  $t$  we would get a different direction field.

### 3.1.1 Exercises

**Exercise 3.1.2:** Find the general solution of  $x'_1 = x_2 - x_1 + t$ ,  $x'_2 = x_2$ .

**Exercise 3.1.3:** Find the general solution of  $x'_1 = 3x_1 - x_2 + e^t$ ,  $x'_2 = x_1$ .

**Exercise 3.1.4:** Write  $ay'' + by' + cy = f(x)$  as a first order system of ODEs.



**Exercise 3.1.5:** Write  $x'' + y^2 y' - x^3 = \sin(t)$ ,  $y'' + (x' + y')^2 - x = 0$  as a first order system of ODEs.

**Exercise 3.1.101:** Find the general solution to  $y'_1 = 3y_1$ ,  $y'_2 = y_1 + y_2$ ,  $y'_3 = y_1 + y_3$ .

**Exercise 3.1.102:** Solve  $y' = 2x$ ,  $x' = x + y$ ,  $x(0) = 1$ ,  $y(0) = 3$ .

**Exercise 3.1.103:** Write  $x''' = x + t$  as a first order system.

**Exercise 3.1.104:** Write  $y''_1 + y_1 + y_2 = t$ ,  $y''_2 + y_1 - y_2 = t^2$  as a first order system.

## 3.2 Matrices and linear systems

*Note: 1 and a half lectures, first part of §5.1 in [EP], §7.2 and §7.3 in [BD]*

### 3.2.1 Matrices and vectors

Before we can start talking about linear systems of ODEs, we will need to talk about matrices, so let us review these briefly. A *matrix* is an  $m \times n$  array of numbers ( $m$  rows and  $n$  columns). For example, we denote a  $3 \times 5$  matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}.$$

By a *vector* we will usually mean a *column vector*, that is an  $m \times 1$  matrix. If we mean a *row vector* we will explicitly say so (a row vector is a  $1 \times n$  matrix). We will usually denote matrices by upper case letters and vectors by lower case letters with an arrow such as  $\vec{x}$  or  $\vec{b}$ . By  $\vec{0}$  we will mean the vector of all zeros.

It is easy to define some operations on matrices. Note that we will want  $1 \times 1$  matrices to really act like numbers, so our operations will have to be compatible with this viewpoint.

First, we can multiply by a *scalar* (a number). This means just multiplying each entry by the same number. For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Matrix addition is also easy. We add matrices element by element. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 7 & 10 \end{bmatrix}.$$

If the sizes do not match, then addition is not defined.

If we denote by  $0$  the matrix of with all zero entries, by  $c, d$  some scalars, and by  $A, B, C$  some matrices, we have the following familiar rules.

$$\begin{aligned} A + 0 &= A = 0 + A, \\ A + B &= B + A, \\ (A + B) + C &= A + (B + C), \\ c(A + B) &= cA + cB, \\ (c + d)A &= cA + dA. \end{aligned}$$

Another useful operation for matrices is the so-called *transpose*. This operation just swaps rows and columns of a matrix. The transpose of  $A$  is denoted by  $A^T$ . Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

### 3.2.2 Matrix multiplication

Let us now define matrix multiplication. First we define the so-called *dot product* (or *inner product*) of two vectors. Usually this will be a row vector multiplied with a column vector of the same size. For the dot product we multiply each pair of entries from the first and the second vector and we sum these products. The result is a single number. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

And similarly for larger (or smaller) vectors.

Armed with the dot product we can define the *product of matrices*. First let us denote by  $\text{row}_i(A)$  the  $i^{\text{th}}$  row of  $A$  and by  $\text{column}_j(A)$  the  $j^{\text{th}}$  column of  $A$ . For an  $m \times n$  matrix  $A$  and an  $n \times p$  matrix  $B$  we can define the product  $AB$ . We let  $AB$  be an  $m \times p$  matrix whose  $ij^{\text{th}}$  entry is

$$\text{row}_i(A) \cdot \text{column}_j(B).$$

Do note how the sizes match up. Example:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} &= \\ = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot (-1) + 2 \cdot 1 + 3 \cdot 0 \\ 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 & 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0 & 4 \cdot (-1) + 5 \cdot 1 + 6 \cdot 0 \end{bmatrix} &= \begin{bmatrix} 6 & 2 & 1 \\ 15 & 5 & 1 \end{bmatrix} \end{aligned}$$

For multiplication we will want an analogue of a 1. This is the so-called *identity matrix*. The identity matrix is a square matrix with 1s on the main diagonal and zeros everywhere else. It is usually denoted by  $I$ . For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, the  $I_3$  would be the  $3 \times 3$  identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have the following rules for matrix multiplication. Suppose that  $A, B, C$  are matrices of the correct sizes so that the following make sense. Let  $\alpha$  denote a scalar (number).

$$\begin{aligned} A(BC) &= (AB)C, \\ A(B + C) &= AB + AC, \\ (B + C)A &= BA + CA, \\ \alpha(AB) &= (\alpha A)B = A(\alpha B), \\ IA &= A = AI. \end{aligned}$$

A few warnings are in order.

- (i)  $AB \neq BA$  in general (it may be true by fluke sometimes). That is, matrices do not commute. For example take  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ .
- (ii)  $AB = AC$  does not necessarily imply  $B = C$ , even if  $A$  is not 0.
- (iii)  $AB = 0$  does not necessarily mean that  $A = 0$  or  $B = 0$ . For example take  $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ .

For the last two items to hold we would need to “divide” by a matrix. This is where the *matrix inverse* comes in. Suppose that  $A$  and  $B$  are  $n \times n$  matrices such that

$$AB = I = BA.$$

Then we call  $B$  the inverse of  $A$  and we denote  $B$  by  $A^{-1}$ . If the inverse of  $A$  exists, then we call  $A$  *invertible*. If  $A$  is not invertible we sometimes say  $A$  is *singular*.

If  $A$  is invertible, then  $AB = AC$  does imply that  $B = C$  (in particular the inverse of  $A$  is unique). We just multiply both sides by  $A^{-1}$  to get  $A^{-1}AB = A^{-1}AC$  or  $IB = IC$  or  $B = C$ . It is also not hard to see that  $(A^{-1})^{-1} = A$ .

### 3.2.3 The determinant

We can now talk about *determinants* of square matrices. We define the determinant of a  $1 \times 1$  matrix as the value of its only entry. For a  $2 \times 2$  matrix we define

$$\det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \stackrel{\text{def}}{=} ad - bc.$$

Before trying to compute the determinant for larger matrices, let us first note the meaning of the determinant. Consider an  $n \times n$  matrix as a mapping of the  $n$  dimensional euclidean space  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . In particular, a  $2 \times 2$  matrix  $A$  is a mapping of the plane to itself, where  $\vec{x}$  gets sent to  $A\vec{x}$ . Then the determinant of  $A$  is the factor by which the area of objects gets changed. If we take the unit square

(square of side 1) in the plane, then  $A$  takes the square to a parallelogram of area  $|\det(A)|$ . The sign of  $\det(A)$  denotes changing of orientation (if the axes got flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then  $\det(A) = 1 + 1 = 2$ . Let us see where the square with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$  gets sent. Clearly  $(0, 0)$  gets sent to  $(0, 0)$ .

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

So the image of the square is another square. The image square has a side of length  $\sqrt{2}$  and is therefore of area 2.

If you think back to high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices  $(0, 0)$ ,  $(a, c)$ ,  $(b, d)$  and  $(a + b, c + d)$ . And it is precisely

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right|.$$

The vertical lines above mean absolute value. The matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  carries the unit square to the given parallelogram.

Now we can define the determinant for larger matrices. We define  $A_{ij}$  as the matrix  $A$  with the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column deleted. To compute the determinant of a matrix, pick one row, say the  $i^{\text{th}}$  row and compute.

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

For the first row we get

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \dots \begin{cases} +a_{1n} \det(A_{1n}) & \text{if } n \text{ is odd,} \\ -a_{1n} \det(A_{1n}) & \text{if } n \text{ even.} \end{cases}$$

We alternately add and subtract the determinants of the submatrices  $A_{ij}$  for a fixed  $i$  and all  $j$ . For a  $3 \times 3$  matrix, picking the first row, we would get  $\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$ . For example,

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned}$$

The numbers  $(-1)^{i+j} \det(A_{ij})$  are called *cofactors* of the matrix and this way of computing the determinant is called the *cofactor expansion*. It is also possible to compute the determinant by expanding along columns (picking a column instead of a row above).

Note that a common notation for the determinant is a pair of vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

I personally find this notation confusing as vertical lines usually mean a positive quantity, while determinants can be negative. I will not use this notation in this book.

One of the most important properties of determinants (in the context of this course) is the following theorem.

**Theorem 3.2.1.** *An  $n \times n$  matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .*

In fact, there is a formula for the inverse of a  $2 \times 2$  matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Notice the determinant of the matrix in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

### 3.2.4 Solving linear systems

One application of matrices we will need is to solve systems of linear equations. This may be best shown by example. Suppose that we have the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Without changing the solution, we could swap equations in this system, we could multiply any of the equations by a nonzero number, and we could add a multiple of one equation to another equation. It turns out these operations always suffice to find a solution.

It is easier to write the system as a matrix equation. Note that the system can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}.$$

To solve the system we put the coefficient matrix (the matrix on the left hand side of the equation) together with the vector on the right and side and get the so-called *augmented matrix*

$$\left[ \begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

We apply the following three elementary operations.

- (i) Swap two rows.
- (ii) Multiply a row by a nonzero number.
- (iii) Add a multiple of one row to another row.

We will keep doing these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution, for example if we come up with an equation such as  $0 = 1$ .

Let us work through the example. First multiply the first row by  $1/2$  to obtain

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

Now subtract the first row from the second and third row.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array} \right]$$

Multiply the last row by  $1/3$  and the second row by  $1/2$ .

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array} \right]$$

Swap rows 2 and 3.

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Subtract the last row from the first, then subtract the second row from the first.

$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

If we think about what equations this augmented matrix represents, we see that  $x_1 = -4$ ,  $x_2 = 3$ , and  $x_3 = 2$ . We try this solution in the original system and, voilà, it works!

**Exercise 3.2.1:** Check that the solution above really solves the given equations.

If we write this equation in matrix notation as

$$A\vec{x} = \vec{b},$$

where  $A$  is the matrix  $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}$  and  $\vec{b}$  is the vector  $\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$ . The solution can be also computed via the inverse,

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

One last note to make about linear systems of equations is that it is possible that the solution is not unique (or that no solution exists). It is easy to tell if a solution does not exist. If during the row reduction you come up with a row where all the entries except the last one are zero (the last entry in a row corresponds to the right hand side of the equation) the system is *inconsistent* and has no solution. For example if for a system of 3 equations and 3 unknowns you find a row such as  $[0 \ 0 \ 0 \mid 1]$  in the augmented matrix, you know the system is inconsistent.

You generally try to use row operations until the following conditions are satisfied. The first nonzero entry in each row is called the *leading entry*.

- (i) There is only one leading entry in each column.
- (ii) All the entries above and below a leading entry are zero.
- (iii) All leading entries are 1.

Such a matrix is said to be in *reduced row echelon form*. The variables corresponding to columns with no leading entries are said to be *free variables*. Free variables mean that we can pick those variables to be anything we want and then solve for the rest of the unknowns.

**Example 3.2.1:** The following augmented matrix is in reduced row echelon form.

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Suppose the variables are  $x_1$ ,  $x_2$ , and  $x_3$ . Then  $x_2$  is the free variable,  $x_1 = 3 - 2x_2$ , and  $x_3 = 1$ .

On the other hand if during the row reduction process you come up with the matrix

$$\left[ \begin{array}{ccc|c} 1 & 2 & 13 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array} \right],$$

there is no need to go further. The last row corresponds to the equation  $0x_1 + 0x_2 + 0x_3 = 3$ , which is preposterous. Hence, no solution exists.



### 3.2.5 Computing the inverse

If the coefficient matrix is square and there exists a unique solution  $\vec{x}$  to  $A\vec{x} = \vec{b}$  for any  $\vec{b}$ , then  $A$  is invertible. In fact by multiplying both sides by  $A^{-1}$  you can see that  $\vec{x} = A^{-1}\vec{b}$ . So it is useful to compute the inverse if you want to solve the equation for many different right hand sides  $\vec{b}$ .

The  $2 \times 2$  inverse can be given by a formula, but it is also not hard to compute inverses of larger matrices. While we will not have too much occasion to compute inverses for larger matrices than  $2 \times 2$  by hand, let us touch on how to do it. Finding the inverse of  $A$  is actually just solving a bunch of linear equations. If we can solve  $A\vec{x}_k = \vec{e}_k$  where  $\vec{e}_k$  is the vector with all zeros except a 1 at the  $k^{\text{th}}$  position, then the inverse is the matrix with the columns  $\vec{x}_k$  for  $k = 1, \dots, n$  (exercise: why?). Therefore, to find the inverse we can write a larger  $n \times 2n$  augmented matrix  $[A \mid I]$ , where  $I$  is the identity. We then perform row reduction. The reduced row echelon form of  $[A \mid I]$  will be of the form  $[I \mid A^{-1}]$  if and only if  $A$  is invertible. We can then just read off the inverse  $A^{-1}$ .

### 3.2.6 Exercises

**Exercise 3.2.2:** Solve  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$  by using matrix inverse.

**Exercise 3.2.3:** Compute determinant of  $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$ .

**Exercise 3.2.4:** Compute determinant of  $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 \\ 8 & 0 & 10 & 1 \end{bmatrix}$ . Hint: Expand along the proper row or column to make the calculations simpler.

**Exercise 3.2.5:** Compute inverse of  $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ .

**Exercise 3.2.6:** For which  $h$  is  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{bmatrix}$  not invertible? Is there only one such  $h$ ? Are there several? Infinitely many?

**Exercise 3.2.7:** For which  $h$  is  $\begin{bmatrix} h & 1 & 1 \\ 0 & h & 0 \\ 1 & 1 & h \end{bmatrix}$  not invertible? Find all such  $h$ .

**Exercise 3.2.8:** Solve  $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ .

**Exercise 3.2.9:** Solve  $\begin{bmatrix} 5 & 3 & 7 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ .

**Exercise 3.2.10:** Solve  $\begin{bmatrix} 3 & 2 & 3 & 0 \\ 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 3 & 4 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$ .

**Exercise 3.2.11:** Find 3 nonzero  $2 \times 2$  matrices  $A$ ,  $B$ , and  $C$  such that  $AB = AC$  but  $B \neq C$ .

**Exercise 3.2.101:** Compute determinant of  $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{bmatrix}$

**Exercise 3.2.102:** Find  $t$  such that  $\begin{bmatrix} 1 & t \\ -1 & 2 \end{bmatrix}$  is not invertible.

**Exercise 3.2.103:** Solve  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$ .

**Exercise 3.2.104:** Suppose  $a, b, c$  are nonzero numbers. Let  $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ ,  $N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ . a) Compute  $M^{-1}$ . b) Compute  $N^{-1}$ .

### 3.3 Linear systems of ODEs

*Note: less than 1 lecture, second part of §5.1 in [EP], §7.4 in [BD]*

First let us talk about matrix or vector valued functions. Such a function is just a matrix whose entries depend on some variable. Let us say the independent variable is  $t$ . Then we write a *vector valued function*  $\vec{x}(t)$  as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Similarly a *matrix valued function*  $A(t)$  is

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

We can talk about the derivative  $A'(t)$  or  $\frac{dA}{dt}$ . This is just the matrix valued function whose  $ij^{\text{th}}$  entry is  $a'_{ij}(t)$ .

Rules of differentiation of matrix valued functions are similar to rules for normal functions. Let  $A(t)$  and  $B(t)$  be matrix valued functions. Let  $c$  a scalar and let  $C$  be a constant matrix. Then

$$\begin{aligned} (A(t) + B(t))' &= A'(t) + B'(t), \\ (A(t)B(t))' &= A'(t)B(t) + A(t)B'(t), \\ (cA(t))' &= cA'(t), \\ (CA(t))' &= CA'(t), \\ (A(t)C)' &= A'(t)C. \end{aligned}$$

Note the order of the multiplication in the last two expressions.

A *first order linear system of ODEs* is a system that can be written as the vector equation

$$\vec{x}'(t) = P(t)\vec{x}(t) + \vec{f}(t),$$

where  $P(t)$  is a matrix valued function, and  $\vec{x}(t)$  and  $\vec{f}(t)$  are vector valued functions. We will often suppress the dependence on  $t$  and only write  $\vec{x}' = P\vec{x} + \vec{f}$ . A solution of the system is a vector valued function  $\vec{x}$  satisfying the vector equation.

For example, the equations

$$\begin{aligned} x_1' &= 2tx_1 + e^t x_2 + t^2, \\ x_2' &= \frac{x_1}{t} - x_2 + e^t, \end{aligned}$$

can be written as

$$\vec{x}' = \begin{bmatrix} 2t & e^t \\ 1/t & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}.$$

We will mostly concentrate on equations that are not just linear, but are in fact *constant coefficient* equations. That is, the matrix  $P$  will be constant; it will not depend on  $t$ .

When  $\vec{f} = \vec{0}$  (the zero vector), then we say the system is *homogeneous*. For homogeneous linear systems we have the principle of superposition, just like for single homogeneous equations.

**Theorem 3.3.1** (Superposition). *Let  $\vec{x}' = P\vec{x}$  be a linear homogeneous system of ODEs. Suppose that  $\vec{x}_1, \dots, \vec{x}_n$  are  $n$  solutions of the equation, then*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n, \quad (3.1)$$

*is also a solution. Furthermore, if this is a system of  $n$  equations ( $P$  is  $n \times n$ ), and  $\vec{x}_1, \dots, \vec{x}_n$  are linearly independent, then every solution can be written as (3.1).*

Linear independence for vector valued functions is the same idea as for normal functions. The vector valued functions  $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$  are linearly independent when

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = \vec{0}$$

has only the solution  $c_1 = c_2 = \dots = c_n = 0$ , where the equation must hold for all  $t$ .

**Example 3.3.1:**  $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}$ ,  $\vec{x}_2 = \begin{bmatrix} 0 \\ 1+t \end{bmatrix}$ ,  $\vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$  are linearly dependent because  $\vec{x}_1 + \vec{x}_3 = \vec{x}_2$ , and this holds for all  $t$ . So  $c_1 = 1$ ,  $c_2 = -1$ , and  $c_3 = 1$  above will work.

On the other hand if we change the example just slightly  $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}$ ,  $\vec{x}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix}$ ,  $\vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$ , then the functions are linearly independent. First write  $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$  and note that it has to hold for all  $t$ . We get that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{bmatrix} c_1t^2 - c_3t^3 \\ c_1t + c_2t + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words  $c_1t^2 - c_3t^3 = 0$  and  $c_1t + c_2t + c_3 = 0$ . If we set  $t = 0$ , then the second equation becomes  $c_3 = 0$ . But then the first equation becomes  $c_1t^2 = 0$  for all  $t$  and so  $c_1 = 0$ . Thus the second equation is just  $c_2t = 0$ , which means  $c_2 = 0$ . So  $c_1 = c_2 = c_3 = 0$  is the only solution and  $\vec{x}_1$ ,  $\vec{x}_2$ , and  $\vec{x}_3$  are linearly independent.

The linear combination  $c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$  could always be written as

$$X(t)\vec{c},$$

where  $X(t)$  is the matrix with columns  $\vec{x}_1, \dots, \vec{x}_n$ , and  $\vec{c}$  is the column vector with entries  $c_1, \dots, c_n$ . The matrix valued function  $X(t)$  is called the *fundamental matrix*, or the *fundamental matrix solution*.

To solve nonhomogeneous first order linear systems, we use the same technique as we applied to solve single linear nonhomogeneous equations.

**Theorem 3.3.2.** Let  $\vec{x}' = P\vec{x} + \vec{f}$  be a linear system of ODEs. Suppose  $\vec{x}_p$  is one particular solution. Then every solution can be written as

$$\vec{x} = \vec{x}_c + \vec{x}_p,$$

where  $\vec{x}_c$  is a solution to the associated homogeneous equation ( $\vec{x}' = P\vec{x}$ ).

So the procedure will be the same as for single equations. We find a particular solution to the nonhomogeneous equation, then we find the general solution to the associated homogeneous equation, and finally we add the two together.

Alright, suppose you have found the general solution  $\vec{x}' = P\vec{x} + \vec{f}$ . Now you are given an initial condition of the form  $\vec{x}(t_0) = \vec{b}$  for some constant vector  $\vec{b}$ . Suppose that  $X(t)$  is the fundamental matrix solution of the associated homogeneous equation (i.e. columns of  $X(t)$  are solutions). The general solution can be written as

$$\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t).$$

We are seeking a vector  $\vec{c}$  such that

$$\vec{b} = \vec{x}(t_0) = X(t_0)\vec{c} + \vec{x}_p(t_0).$$

In other words, we are solving for  $\vec{c}$  the nonhomogeneous system of linear equations

$$X(t_0)\vec{c} = \vec{b} - \vec{x}_p(t_0).$$

**Example 3.3.2:** In § 3.1 we solved the system

$$\begin{aligned} x_1' &= x_1, \\ x_2' &= x_1 - x_2, \end{aligned}$$

with initial conditions  $x_1(0) = 1$ ,  $x_2(0) = 2$ .

This is a homogeneous system, so  $\vec{f}(t) = \vec{0}$ . We write the system and the initial conditions as

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We found the general solution was  $x_1 = c_1 e^t$  and  $x_2 = \frac{c_1}{2} e^t + c_2 e^{-t}$ . Letting  $c_1 = 1$  and  $c_2 = 0$ , we obtain the solution  $\begin{bmatrix} e^t \\ (1/2)e^t \end{bmatrix}$ . Letting  $c_1 = 0$  and  $c_2 = 1$ , we obtain  $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$ . These two solutions are linearly independent, as can be seen by setting  $t = 0$ , and noting that the resulting constant vectors are linearly independent. In matrix notation, the fundamental matrix solution is, therefore,

$$X(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix}.$$

Hence to solve the initial problem we solve the equation

$$X(0)\vec{c} = \vec{b},$$

or in other words,

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

After a single elementary row operation we find that  $\vec{c} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$ . Hence our solution is

$$\vec{x}(t) = X(t)\vec{c} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}.$$

This agrees with our previous solution.

### 3.3.1 Exercises

**Exercise 3.3.1:** Write the system  $x'_1 = 2x_1 - 3tx_2 + \sin t$ ,  $x'_2 = e^t x_1 + 3x_2 + \cos t$  in the form  $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$ .

**Exercise 3.3.2:** a) Verify that the system  $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x}$  has the two solutions  $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$ . b) Write down the general solution. c) Write down the general solution in the form  $x_1 = ?$ ,  $x_2 = ?$  (i.e. write down a formula for each element of the solution).

**Exercise 3.3.3:** Verify that  $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$  are linearly independent. Hint: Just plug in  $t = 0$ .

**Exercise 3.3.4:** Verify that  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t$  and  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t$  and  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$  are linearly independent. Hint: You must be a bit more tricky than in the previous exercise.

**Exercise 3.3.5:** Verify that  $\begin{bmatrix} t \\ t^2 \end{bmatrix}$  and  $\begin{bmatrix} t^3 \\ t^4 \end{bmatrix}$  are linearly independent.

**Exercise 3.3.101:** Are  $\begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$  and  $\begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$  linearly independent? Justify.

**Exercise 3.3.102:** Are  $\begin{bmatrix} \cosh(t) \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} e^t \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$  linearly independent? Justify.

**Exercise 3.3.103:** Write  $x' = 3x - y + e^t$ ,  $y' = tx$  in matrix notation.

**Exercise 3.3.104:** a) Write  $x'_1 = 2tx_2$ ,  $x'_2 = 2tx_2$  in matrix notation. b) Solve and write the solution in matrix notation.

## 3.4 Eigenvalue method

*Note: 2 lectures, §5.2 in [EP], part of §7.3, §7.5, and §7.6 in [BD]*

In this section we will learn how to solve linear homogeneous constant coefficient systems of ODEs by the eigenvalue method. Suppose we have a linear constant coefficient homogeneous system

$$\vec{x}' = P\vec{x},$$

where  $P$  is a constant square matrix. Suppose we try to adapt the method for the single constant coefficient equation by trying the function  $e^{\lambda t}$ . However,  $\vec{x}$  is a vector. So we try  $\vec{x} = \vec{v}e^{\lambda t}$ , where  $\vec{v}$  is an arbitrary constant vector. We plug this  $\vec{x}$  into the equation to get

$$\lambda \vec{v}e^{\lambda t} = P\vec{v}e^{\lambda t}.$$

We divide by  $e^{\lambda t}$  and notice that we are looking for a scalar  $\lambda$  and a vector  $\vec{v}$  that satisfy the equation

$$\lambda \vec{v} = P\vec{v}.$$

To solve this equation we need a little bit more linear algebra, which we now review.

### 3.4.1 Eigenvalues and eigenvectors of a matrix

Let  $A$  be a constant square matrix. Suppose there is a scalar  $\lambda$  and a nonzero vector  $\vec{v}$  such that

$$A\vec{v} = \lambda\vec{v}.$$

We then call  $\lambda$  an *eigenvalue* of  $A$  and  $\vec{v}$  is said to be a corresponding *eigenvector*.

**Example 3.4.1:** The matrix  $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$  has an eigenvalue of  $\lambda = 2$  with a corresponding eigenvector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  because

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let us see how to compute the eigenvalues for any matrix. We rewrite the equation for an eigenvalue as

$$(A - \lambda I)\vec{v} = \vec{0}.$$

We notice that this equation has a nonzero solution  $\vec{v}$  only if  $A - \lambda I$  is not invertible. Were it invertible, we could write  $(A - \lambda I)^{-1}(A - \lambda I)\vec{v} = (A - \lambda I)^{-1}\vec{0}$ , which implies  $\vec{v} = \vec{0}$ . Therefore,  $A$  has the eigenvalue  $\lambda$  if and only if  $\lambda$  solves the equation

$$\det(A - \lambda I) = 0.$$

Consequently, we will be able to find an eigenvalue of  $A$  without finding a corresponding eigenvector. An eigenvector will have to be found later, once  $\lambda$  is known.

**Example 3.4.2:** Find all eigenvalues of  $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .

We write

$$\det \left( \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \det \left( \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \right) =$$

$$= (2-\lambda)((2-\lambda)^2 - 1) = -(\lambda-1)(\lambda-2)(\lambda-3).$$

and so the eigenvalues are  $\lambda = 1$ ,  $\lambda = 2$ , and  $\lambda = 3$ .

Note that for an  $n \times n$  matrix, the polynomial we get by computing  $\det(A - \lambda I)$  will be of degree  $n$ , and hence we will in general have  $n$  eigenvalues. Some may be repeated, some may be complex.

To find an eigenvector corresponding to an eigenvalue  $\lambda$ , we write

$$(A - \lambda I)\vec{v} = \vec{0},$$

and solve for a nontrivial (nonzero) vector  $\vec{v}$ . If  $\lambda$  is an eigenvalue, this will always be possible.

**Example 3.4.3:** Find an eigenvector of  $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$  corresponding to the eigenvalue  $\lambda = 3$ .

We write

$$(A - \lambda I)\vec{v} = \left( \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}.$$

It is easy to solve this system of linear equations. We write down the augmented matrix

$$\left[ \begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right],$$

and perform row operations (exercise: which ones?) until we get

$$\left[ \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

The equations the entries of  $\vec{v}$  have to satisfy are, therefore,  $v_1 - v_2 = 0$ ,  $v_3 = 0$ , and  $v_2$  is a free variable. We can pick  $v_2$  to be arbitrary (but nonzero) and let  $v_1 = v_2$  and of course  $v_3 = 0$ . For example,  $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ . Let us verify that we really have an eigenvector corresponding to  $\lambda = 3$ :

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Yay! It worked.



**Exercise 3.4.1** (easy): Are eigenvectors unique? Can you find a different eigenvector for  $\lambda = 3$  in the example above? How are the two eigenvectors related?

**Exercise 3.4.2:** Note that when the matrix is  $2 \times 2$  you do not need to write down the augmented matrix and do row operations when computing eigenvectors (if you have computed the eigenvalues correctly). Can you see why? Try it for the matrix  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ .

### 3.4.2 The eigenvalue method with distinct real eigenvalues

OK. We have the system of equations

$$\vec{x}' = P\vec{x}.$$

We find the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of the matrix  $P$ , and corresponding eigenvectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ . Now we notice that the functions  $\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}, \dots, \vec{v}_n e^{\lambda_n t}$  are solutions of the system of equations and hence  $\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}$  is a solution.

**Theorem 3.4.1.** Take  $\vec{x}' = P\vec{x}$ . If  $P$  is an  $n \times n$  constant matrix that has  $n$  distinct real eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , then there exist  $n$  linearly independent corresponding eigenvectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ , and the general solution to  $\vec{x}' = P\vec{x}$  can be written as

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}.$$

The corresponding fundamental matrix solution is  $X(t) = [\vec{v}_1 e^{\lambda_1 t} \quad \vec{v}_2 e^{\lambda_2 t} \quad \dots \quad \vec{v}_n e^{\lambda_n t}]$ . That is,  $X(t)$  is the matrix whose  $j^{\text{th}}$  column is  $\vec{v}_j e^{\lambda_j t}$ .

**Example 3.4.4:** Suppose we take the system

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}.$$

Find the general solution.

We have found the eigenvalues 1, 2, 3 earlier. We have found the eigenvector  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  for the eigenvalue 3. Similarly we find the eigenvector  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  for the eigenvalue 1, and  $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$  for the eigenvalue 2 (exercise: check). Hence our general solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} = \begin{bmatrix} c_1 e^t + c_3 e^{3t} \\ -c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \end{bmatrix}.$$

In terms of a fundamental matrix solution

$$\vec{x} = X(t) \vec{c} = \begin{bmatrix} e^t & 0 & e^{3t} \\ -e^t & e^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

**Exercise 3.4.3:** Check that this really solves the system.

Note: If we write a homogeneous linear constant coefficient  $n^{\text{th}}$  order equation as a first order system (as we did in § 3.1), then the eigenvalue equation

$$\det(P - \lambda I) = 0$$

is essentially the same as the characteristic equation we got in § 2.2 and § 2.3.

### 3.4.3 Complex eigenvalues

A matrix might very well have complex eigenvalues even if all the entries are real. For example, suppose that we have the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

Let us compute the eigenvalues of the matrix  $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ .

$$\det(P - \lambda I) = \det \left( \begin{bmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{bmatrix} \right) = (1 - \lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0.$$

Thus  $\lambda = 1 \pm i$ . The corresponding eigenvectors will also be complex. First take  $\lambda = 1 - i$ ,

$$(P - (1 - i)I)\vec{v} = \vec{0},$$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \vec{v} = \vec{0}.$$

The equations  $iv_1 + v_2 = 0$  and  $-v_1 + iv_2 = 0$  are multiples of each other. So we only need to consider one of them. After picking  $v_2 = 1$ , for example, we have an eigenvector  $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ . In similar fashion we find that  $\begin{bmatrix} -i \\ 1 \end{bmatrix}$  is an eigenvector corresponding to the eigenvalue  $1 + i$ .

We could write the solution as

$$\vec{x} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} c_1 i e^{(1-i)t} - c_2 i e^{(1+i)t} \\ c_1 e^{(1-i)t} + c_2 e^{(1+i)t} \end{bmatrix}.$$

We would then need to look for complex values  $c_1$  and  $c_2$  to solve any initial conditions. It is perhaps not completely clear that we get a real solution. We could use Euler's formula and do the whole song and dance we did before, but we will not. We will do something a bit smarter first.

We claim that we did not have to look for a second eigenvector (nor for the second eigenvalue). All complex eigenvalues come in pairs (because the matrix  $P$  is real).

First a small side note. The real part of a complex number  $z$  can be computed as  $\frac{z + \bar{z}}{2}$ , where the bar above  $z$  means  $\overline{a + ib} = a - ib$ . This operation is called the *complex conjugate*. Note that if  $a$  is

a real number, then  $\bar{a} = a$ . Similarly we can bar whole vectors or matrices. If a matrix  $P$  is real, then  $\bar{P} = P$ . We note that  $\overline{P\vec{x}} = \bar{P}\bar{\vec{x}} = P\bar{\vec{x}}$ . Therefore,

$$\overline{(P - \lambda I)\vec{v}} = (P - \bar{\lambda}I)\bar{\vec{v}}.$$

So if  $\vec{v}$  is an eigenvector corresponding to the eigenvalue  $\lambda = a + ib$ , then  $\bar{\vec{v}}$  is an eigenvector corresponding to the eigenvalue  $\bar{\lambda} = a - ib$ .

Suppose that  $a + ib$  is a complex eigenvalue of  $P$ , and  $\vec{v}$  is a corresponding eigenvector. Then

$$\vec{x}_1 = \vec{v}e^{(a+ib)t}$$

is a solution (complex valued) of  $\vec{x}' = P\vec{x}$ . Then note that  $\overline{e^{a+ib}} = e^{a-ib}$ , and so

$$\vec{x}_2 = \bar{\vec{x}}_1 = \bar{\vec{v}}e^{(a-ib)t}$$

is also a solution. The function

$$\vec{x}_3 = \operatorname{Re} \vec{x}_1 = \operatorname{Re} \vec{v}e^{(a+ib)t} = \frac{\vec{x}_1 + \bar{\vec{x}}_1}{2} = \frac{\vec{x}_1 + \vec{x}_2}{2}$$

is also a solution. And  $\vec{x}_3$  is real-valued! Similarly as  $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$  is the imaginary part, we find that

$$\vec{x}_4 = \operatorname{Im} \vec{x}_1 = \frac{\vec{x}_1 - \bar{\vec{x}}_1}{2i}.$$

is also a real-valued solution. It turns out that  $\vec{x}_3$  and  $\vec{x}_4$  are linearly independent. We will use Euler's formula to separate out the real and imaginary part.

Returning to our problem,

$$\vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} = \begin{bmatrix} i \\ 1 \end{bmatrix} (e^t \cos t - ie^t \sin t) = \begin{bmatrix} ie^t \cos t + e^t \sin t \\ e^t \cos t - ie^t \sin t \end{bmatrix}.$$

Then

$$\begin{aligned} \operatorname{Re} \vec{x}_1 &= \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix}, \\ \operatorname{Im} \vec{x}_1 &= \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix}, \end{aligned}$$

are the two real-valued linearly independent solutions we seek.

**Exercise 3.4.4:** Check that these really are solutions.

The general solution is

$$\vec{x} = c_1 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + c_2 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix} = \begin{bmatrix} c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t - c_2 e^t \sin t \end{bmatrix}.$$

This solution is real-valued for real  $c_1$  and  $c_2$ . Now we can solve for any initial conditions that we may have.

Let us summarize as a theorem.

**Theorem 3.4.2.** *Let  $P$  be a real-valued constant matrix. If  $P$  has a complex eigenvalue  $a + ib$  and a corresponding eigenvector  $\vec{v}$ , then  $P$  also has a complex eigenvalue  $a - ib$  with a corresponding eigenvector  $\bar{\vec{v}}$ . Furthermore,  $\vec{x}' = P\vec{x}$  has two linearly independent real-valued solutions*

$$\vec{x}_1 = \operatorname{Re} \vec{v} e^{(a+ib)t}, \quad \text{and} \quad \vec{x}_2 = \operatorname{Im} \vec{v} e^{(a+ib)t}.$$

So for each pair of complex eigenvalues we get two real-valued linearly independent solutions. We then go on to the next eigenvalue, which is either a real eigenvalue or another complex eigenvalue pair. If we had  $n$  distinct eigenvalues (real or complex), then we will end up with  $n$  linearly independent solutions.

We can now find a real-valued general solution to any homogeneous system where the matrix has distinct eigenvalues. When we have repeated eigenvalues, matters get a bit more complicated and we will look at that situation in § 3.7.

### 3.4.4 Exercises

**Exercise 3.4.5** (easy): Let  $A$  be a  $3 \times 3$  matrix with an eigenvalue of 3 and a corresponding eigenvector  $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ . Find  $A\vec{v}$ .

**Exercise 3.4.6:** a) Find the general solution of  $x'_1 = 2x_1$ ,  $x'_2 = 3x_2$  using the eigenvalue method (first write the system in the form  $\vec{x}' = A\vec{x}$ ). b) Solve the system by solving each equation separately and verify you get the same general solution.

**Exercise 3.4.7:** Find the general solution of  $x'_1 = 3x_1 + x_2$ ,  $x'_2 = 2x_1 + 4x_2$  using the eigenvalue method.

**Exercise 3.4.8:** Find the general solution of  $x'_1 = x_1 - 2x_2$ ,  $x'_2 = 2x_1 + x_2$  using the eigenvalue method. Do not use complex exponentials in your solution.

**Exercise 3.4.9:** a) Compute eigenvalues and eigenvectors of  $A = \begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$ . b) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.4.10:** Compute eigenvalues and eigenvectors of  $\begin{bmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{bmatrix}$ .

**Exercise 3.4.11:** Let  $a, b, c, d, e, f$  be numbers. Find the eigenvalues of  $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ .

**Exercise 3.4.101:** a) Compute eigenvalues and eigenvectors of  $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$ . b) Solve the system  $\vec{x}' = A\vec{x}$ .

**Exercise 3.4.102:** a) Compute eigenvalues and eigenvectors of  $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ . b) Solve the system  $\vec{x}' = A\vec{x}$ .

**Exercise 3.4.103:** Solve  $x'_1 = x_2$ ,  $x'_2 = x_1$  using the eigenvalue method.

**Exercise 3.4.104:** Solve  $x'_1 = x_2$ ,  $x'_2 = -x_1$  using the eigenvalue method.

### 3.5 Two dimensional systems and their vector fields

*Note: 1 lecture, should really be in [EP] §5.2, but is in [EP] §6.2, parts of §7.5 and §7.6 in [BD]*

Let us take a moment to talk about constant coefficient linear homogeneous systems in the plane. Much intuition can be obtained by studying this simple case. Suppose we have a  $2 \times 2$  matrix  $P$  and the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix}. \quad (3.2)$$

The system is autonomous (compare this section to § 1.6) and so we will be able to draw a vector field. We will be able to visually tell what the vector field looks like and how the solutions behave, once we find the eigenvalues and eigenvectors of the matrix  $P$ .

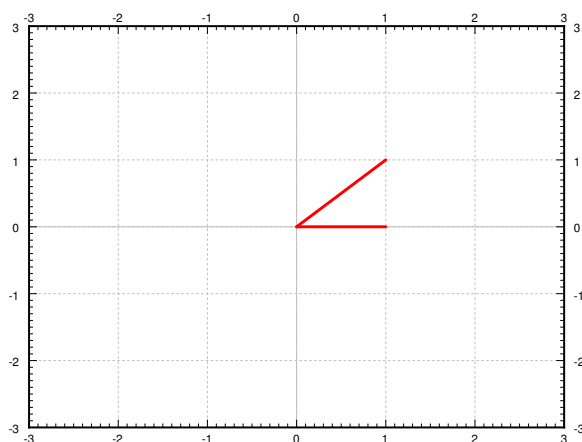


Figure 3.3: Eigenvectors of  $P$ .

*Case 1.* Suppose that the eigenvalues of  $P$  are real and positive. We find two corresponding eigenvectors and plot them in the plane. For example, take the matrix  $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ . The eigenvalues are 1 and 2 and corresponding eigenvectors are  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . See Figure 3.3.

Now suppose that  $x$  and  $y$  are on the line determined by an eigenvector  $\vec{v}$  for an eigenvalue  $\lambda$ . That is,  $\begin{bmatrix} x \\ y \end{bmatrix} = a\vec{v}$  for some scalar  $a$ . Then

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} = P(a\vec{v}) = a(P\vec{v}) = a\lambda\vec{v}.$$

The derivative is a multiple of  $\vec{v}$  and hence points along the line determined by  $\vec{v}$ . As  $\lambda > 0$ , the derivative points in the direction of  $\vec{v}$  when  $a$  is positive and in the opposite direction when  $a$  is

negative. Let us draw the lines determined by the eigenvectors, and let us draw arrows on the lines to indicate the directions. See Figure 3.4 on the facing page.

We fill in the rest of the arrows and we also draw a few solutions. See Figure 3.5 on the next page. Notice that the picture looks like a source with arrows coming out from the origin. Hence we call this type of picture a *source* or sometimes an *unstable node*.

*Case 2.* Suppose both eigenvalues were negative. For example, take the negation of the matrix in case 1,  $\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$ . The eigenvalues are  $-1$  and  $-2$  and corresponding eigenvectors are the same,  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . The calculation and the picture are almost the same. The only difference is that the eigenvalues are negative and hence all arrows are reversed. We get the picture in Figure 3.6 on the facing page. We call this kind of picture a *sink* or sometimes a *stable node*.

*Case 3.* Suppose one eigenvalue is positive and one is negative. For example the matrix  $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$ . The eigenvalues are 1 and  $-2$  and corresponding eigenvectors are  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ . We reverse the

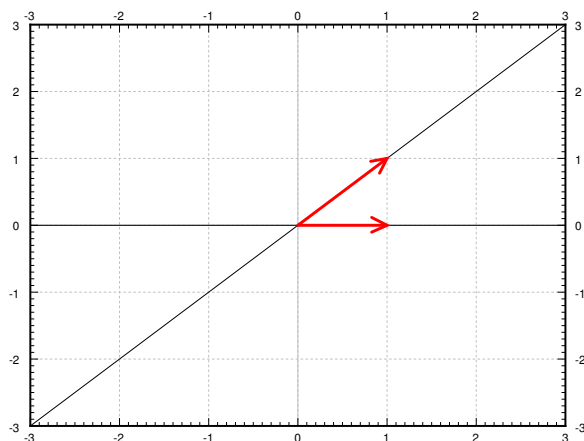
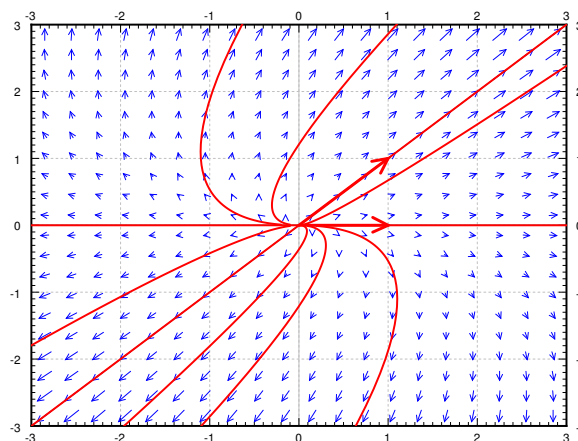
Figure 3.4: Eigenvectors of  $P$  with directions.

Figure 3.5: Example source vector field with eigenvectors and solutions.

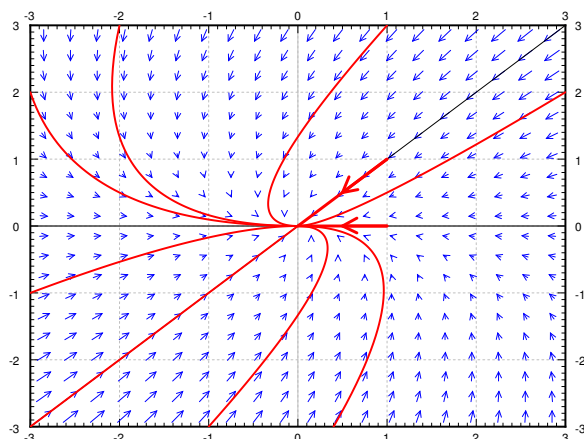


Figure 3.6: Example sink vector field with eigenvectors and solutions.

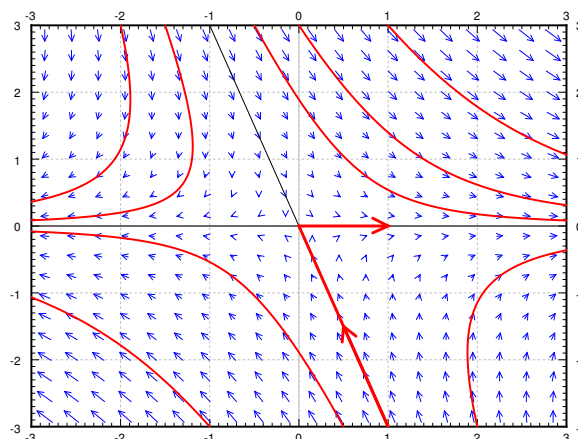


Figure 3.7: Example saddle vector field with eigenvectors and solutions.

arrows on one line (corresponding to the negative eigenvalue) and we obtain the picture in Figure 3.7. We call this picture a *saddle point*.

For the next three cases we will assume the eigenvalues are complex. In this case the eigenvectors are also complex and we cannot just plot them on the plane.

*Case 4.* Suppose the eigenvalues are purely imaginary. That is, suppose the eigenvalues are  $\pm ib$ . For example, let  $P = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ . The eigenvalues turn out to be  $\pm 2i$  and eigenvectors are  $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$ . We take the eigenvalue  $2i$  and its eigenvector  $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$  and note that the real and imaginary parts of  $\vec{v}e^{i2t}$

are

$$\begin{aligned}\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} &= \begin{bmatrix} \cos(2t) \\ -2 \sin(2t) \end{bmatrix}, \\ \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} &= \begin{bmatrix} \sin(2t) \\ 2 \cos(2t) \end{bmatrix}.\end{aligned}$$

We can take any linear combination of them, and which one we take depends on the initial conditions. For example, the real part is a parametric equation for an ellipse. Same with the imaginary part and in fact any linear combination of them. It is not difficult to see that this is what happens in general when the eigenvalues are purely imaginary. So when the eigenvalues are purely imaginary, we get *ellipses* for the solutions. This type of picture is sometimes called a *center*. See Figure 3.8.

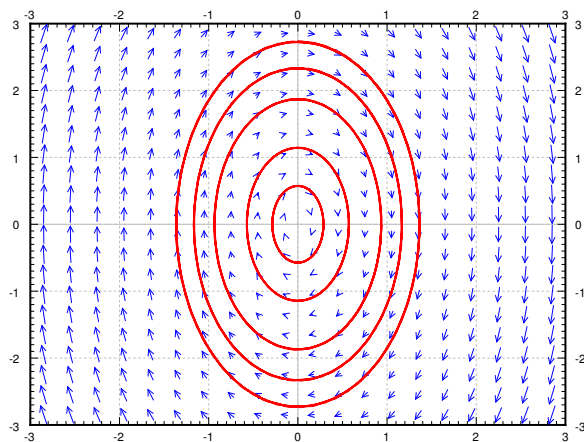


Figure 3.8: Example center vector field.

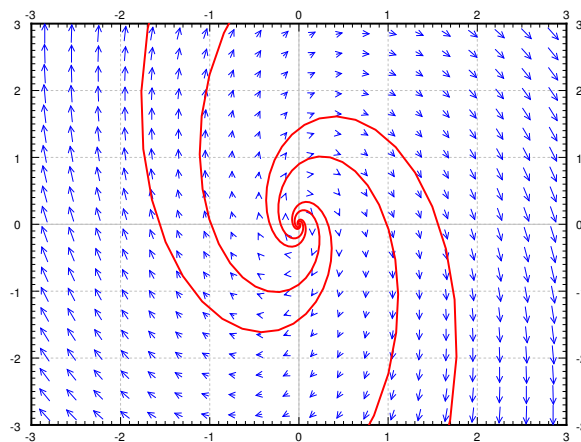


Figure 3.9: Example spiral source vector field.

*Case 5.* Now suppose the complex eigenvalues have a positive real part. That is, suppose the eigenvalues are  $a \pm ib$  for some  $a > 0$ . For example, let  $P = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ . The eigenvalues turn out to be  $1 \pm 2i$  and eigenvectors are  $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$ . We take  $1 + 2i$  and its eigenvector  $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$  and find the real and imaginary of  $\vec{v}e^{(1+2i)t}$  are

$$\begin{aligned}\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} &= e^t \begin{bmatrix} \cos(2t) \\ -2 \sin(2t) \end{bmatrix}, \\ \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} &= e^t \begin{bmatrix} \sin(2t) \\ 2 \cos(2t) \end{bmatrix}.\end{aligned}$$

Now note the  $e^t$  in front of the solutions. This means that the solutions grow in magnitude while spinning around the origin. Hence we get a *spiral source*. See Figure 3.9.

*Case 6.* Finally suppose the complex eigenvalues have a negative real part. That is, suppose the eigenvalues are  $-a \pm ib$  for some  $a > 0$ . For example, let  $P = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$ . The eigenvalues turn out to



be  $-1 \pm 2i$  and eigenvectors are  $\begin{bmatrix} 1 \\ -2i \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$ . We take  $-1 - 2i$  and its eigenvector  $\begin{bmatrix} 1 \\ 2i \end{bmatrix}$  and find the real and imaginary of  $\vec{v}e^{(-1-2i)t}$  are

$$\begin{aligned} \operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} &= e^{-t} \begin{bmatrix} \cos(2t) \\ 2 \sin(2t) \end{bmatrix}, \\ \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} &= e^{-t} \begin{bmatrix} -\sin(2t) \\ 2 \cos(2t) \end{bmatrix}. \end{aligned}$$

Now note the  $e^{-t}$  in front of the solutions. This means that the solutions shrink in magnitude while spinning around the origin. Hence we get a *spiral sink*. See Figure 3.10.

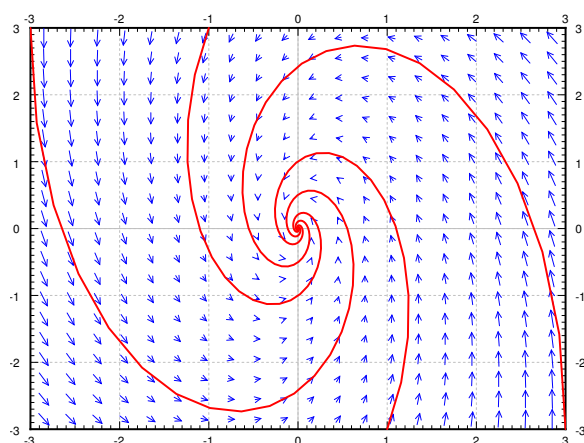


Figure 3.10: Example spiral sink vector field.

We summarize the behavior of linear homogeneous two dimensional systems in Table 3.1.

Eigenvalues	Behavior
real and both positive	source / unstable node
real and both negative	sink / stable node
real and opposite signs	saddle
purely imaginary	center point / ellipses
complex with positive real part	spiral source
complex with negative real part	spiral sink

Table 3.1: Summary of behavior of linear homogeneous two dimensional systems.

### 3.5.1 Exercises

**Exercise 3.5.1:** Take the equation  $mx'' + cx' + kx = 0$ , with  $m > 0$ ,  $c \geq 0$ ,  $k > 0$  for the mass-spring system. a) Convert this to a system of first order equations. b) Classify for what  $m, c, k$  do you get which behavior. c) Can you explain from physical intuition why you do not get all the different kinds of behavior here?

**Exercise 3.5.2:** Can you find what happens in the case when  $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ? In this case the eigenvalue is repeated and there is only one eigenvector. What picture does this look like?

**Exercise 3.5.3:** Can you find what happens in the case when  $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ? Does this look like any of the pictures we have drawn?

**Exercise 3.5.101:** Describe the behavior of the following systems without solving:

a)  $x' = x + y$ ,  $y' = x - y$ .

b)  $x'_1 = x_1 + x_2$ ,  $x'_2 = 2x_2$ .

c)  $x'_1 = -2x_2$ ,  $x'_2 = 2x_1$ .

d)  $x' = x + 3y$ ,  $y' = -2x - 4y$ .

e)  $x' = x - 4y$ ,  $y' = -4x + y$ .

**Exercise 3.5.102:** Suppose that  $\vec{x}' = A\vec{x}$  where  $A$  is a 2 by 2 matrix with eigenvalues  $2 \pm i$ . Describe the behavior.

**Exercise 3.5.103:** Take  $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ . Draw the vector field and describe the behavior. Is it one of the behaviours that we have seen before?

## 3.6 Second order systems and applications

*Note: more than 2 lectures, §5.3 in [EP], not in [BD]*

### 3.6.1 Undamped mass-spring systems

While we did say that we will usually only look at first order systems, it is sometimes more convenient to study the system in the way it arises naturally. For example, suppose we have 3 masses connected by springs between two walls. We could pick any higher number, and the math would be essentially the same, but for simplicity we pick 3 right now. Let us also assume no friction, that is, the system is undamped. The masses are  $m_1$ ,  $m_2$ , and  $m_3$  and the spring constants are  $k_1$ ,  $k_2$ ,  $k_3$ , and  $k_4$ . Let  $x_1$  be the displacement from rest position of the first mass, and  $x_2$  and  $x_3$  the displacement of the second and third mass. We will make, as usual, positive values go right (as  $x_1$  grows the first mass is moving right). See Figure 3.11.

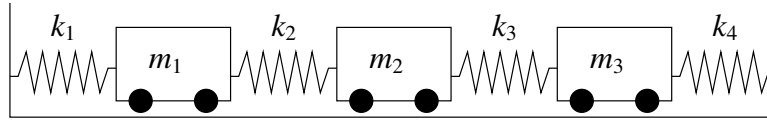


Figure 3.11: System of masses and springs.

This simple system turns up in unexpected places. For example, our world really consists of many small particles of matter interacting together. When we try the above system with many more masses, we obtain a good approximation to how an elastic material will behave. By somehow taking a limit of the number of masses going to infinity, we obtain the continuous one dimensional wave equation (that we study in § 4.7). But we digress.

Let us set up the equations for the three mass system. By Hooke's law we have that the force acting on the mass equals the spring compression times the spring constant. By Newton's second law we have that force is mass times acceleration. So if we sum the forces acting on each mass and put the right sign in front of each term, depending on the direction in which it is acting, we end up with the desired system of equations.

$$\begin{aligned}
 m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) &= -(k_1 + k_2)x_1 + k_2 x_2, \\
 m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3, \\
 m_3 x_3'' &= -k_3(x_3 - x_2) - k_4 x_3 &= k_3 x_2 - (k_3 + k_4)x_3.
 \end{aligned}$$

We define the matrices

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} -(k_1 + k_2) & k_2 & 0 \\ k_2 & -(k_2 + k_3) & k_3 \\ 0 & k_3 & -(k_3 + k_4) \end{bmatrix}.$$

We write the equation simply as

$$M\vec{x}'' = K\vec{x}.$$

At this point we could introduce 3 new variables and write out a system of 6 equations. We claim this simple setup is easier to handle as a second order system. We will call  $\vec{x}$  the *displacement vector*,  $M$  the *mass matrix*, and  $K$  the *stiffness matrix*.

**Exercise 3.6.1:** Repeat this setup for 4 masses (find the matrices  $M$  and  $K$ ). Do it for 5 masses. Can you find a prescription to do it for  $n$  masses?

As with a single equation we will want to “divide by  $M$ .” This means computing the inverse of  $M$ . The masses are all nonzero and  $M$  is a diagonal matrix, so computing the inverse is easy.

$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}.$$

This fact follows readily by how we multiply diagonal matrices. You should verify that  $MM^{-1} = M^{-1}M = I$  as an exercise.

Let  $A = M^{-1}K$ . We look at the system  $\vec{x}'' = M^{-1}K\vec{x}$ , or

$$\vec{x}'' = A\vec{x}.$$

Many real world systems can be modeled by this equation. For simplicity, we will only talk about the given masses-and-springs problem. We try a solution of the form

$$\vec{x} = \vec{v}e^{\alpha t}.$$

We compute that for this guess,  $\vec{x}'' = \alpha^2\vec{v}e^{\alpha t}$ . We plug our guess into the equation and get

$$\alpha^2\vec{v}e^{\alpha t} = A\vec{v}e^{\alpha t}.$$

We can divide by  $e^{\alpha t}$  to get that  $\alpha^2\vec{v} = A\vec{v}$ . Hence if  $\alpha^2$  is an eigenvalue of  $A$  and  $\vec{v}$  is a corresponding eigenvector, we have found a solution.

In our example, and in other common applications, it turns out that  $A$  has only real negative eigenvalues (and possibly a zero eigenvalue). So we will study only this case. When an eigenvalue  $\lambda$  is negative, it means that  $\alpha^2 = \lambda$  is negative. Hence there is some real number  $\omega$  such that  $-\omega^2 = \lambda$ . Then  $\alpha = \pm i\omega$ . The solution we guessed was

$$\vec{x} = \vec{v}(\cos(\omega t) + i \sin(\omega t)).$$

By taking real and imaginary parts (note that  $\vec{v}$  is real), we find that  $\vec{v}\cos(\omega t)$  and  $\vec{v}\sin(\omega t)$  are linearly independent solutions.

If an eigenvalue is zero, it turns out that both  $\vec{v}$  and  $\vec{v}t$  are solutions, where  $\vec{v}$  is a corresponding eigenvector.

**Exercise 3.6.2:** Show that if  $A$  has a zero eigenvalue and  $\vec{v}$  is a corresponding eigenvector, then  $\vec{x} = \vec{v}(a + bt)$  is a solution of  $\vec{x}'' = A\vec{x}$  for arbitrary constants  $a$  and  $b$ .

**Theorem 3.6.1.** Let  $A$  be an  $n \times n$  matrix with  $n$  distinct real negative eigenvalues we denote by  $-\omega_1^2 > -\omega_2^2 > \cdots > -\omega_n^2$ , and corresponding eigenvectors by  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ . If  $A$  is invertible (that is, if  $\omega_1 > 0$ ), then

$$\vec{x}(t) = \sum_{i=1}^n \vec{v}_i (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)),$$

is the general solution of

$$\vec{x}'' = A\vec{x},$$

for some arbitrary constants  $a_i$  and  $b_i$ . If  $A$  has a zero eigenvalue, that is  $\omega_1 = 0$ , and all other eigenvalues are distinct and negative, then the general solution can be written as

$$\vec{x}(t) = \vec{v}_1(a_1 + b_1 t) + \sum_{i=2}^n \vec{v}_i (a_i \cos(\omega_i t) + b_i \sin(\omega_i t)).$$

Note that we can use this solution and the setup from the introduction of this section even when some of the masses and springs are missing. For example, when there are say 2 masses and only 2 springs, simply take only the equations for the two masses and set all the spring constants for the springs that are missing to zero.

### 3.6.2 Examples

**Example 3.6.1:** Suppose we have the system in Figure 3.12, with  $m_1 = 2$ ,  $m_2 = 1$ ,  $k_1 = 4$ , and  $k_2 = 2$ .

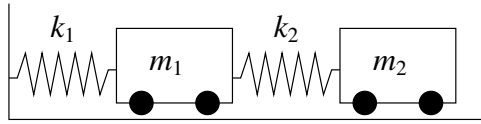


Figure 3.12: System of masses and springs.

The equations we write down are

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -(4+2) & 2 \\ 2 & -2 \end{bmatrix} \vec{x},$$

or

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We find the eigenvalues of  $A$  to be  $\lambda = -1, -4$  (exercise). Now we find corresponding eigenvectors to be  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  respectively (exercise).

We check the theorem and note that  $\omega_1 = 1$  and  $\omega_2 = 2$ . Hence the general solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)).$$

The two terms in the solution represent the two so-called *natural* or *normal modes of oscillation*. And the two (angular) frequencies are the *natural frequencies*. The two modes are plotted in Figure 3.13.

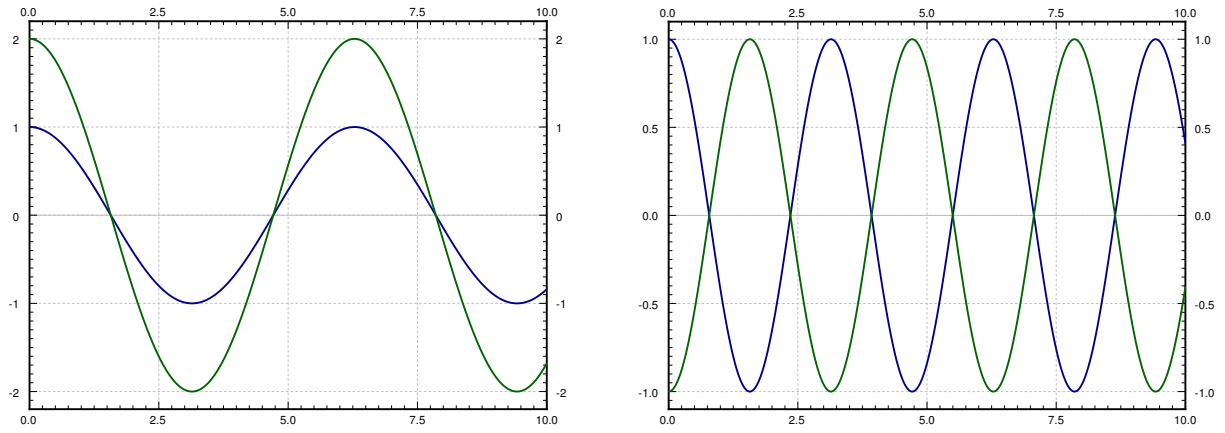


Figure 3.13: The two modes of the mass-spring system. In the left plot the masses are moving in unison and the right plot are masses moving in the opposite direction.

Let us write the solution as

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 \cos(t - \alpha_1) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} c_2 \cos(2t - \alpha_2).$$

The first term,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 \cos(t - \alpha_1) = \begin{bmatrix} c_1 \cos(t - \alpha_1) \\ 2c_1 \cos(t - \alpha_1) \end{bmatrix},$$

corresponds to the mode where the masses move synchronously in the same direction.

The second term,

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} c_2 \cos(2t - \alpha_2) = \begin{bmatrix} c_2 \cos(2t - \alpha_2) \\ -c_2 \cos(2t - \alpha_2) \end{bmatrix},$$

corresponds to the mode where the masses move synchronously but in opposite directions.

The general solution is a combination of the two modes. That is, the initial conditions determine the amplitude and phase shift of each mode.

**Example 3.6.2:** We have two toy rail cars. Car 1 of mass 2 kg is traveling at 3 m/s towards the second rail car of mass 1 kg. There is a bumper on the second rail car that engages at the moment the cars hit (it connects to two cars) and does not let go. The bumper acts like a spring of spring constant  $k = 2 \text{ N/m}$ . The second car is 10 meters from a wall. See Figure 3.14.

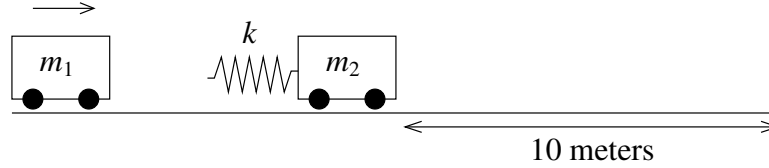


Figure 3.14: The crash of two rail cars.

We want to ask several questions. At what time after the cars link does impact with the wall happen? What is the speed of car 2 when it hits the wall?

OK, let us first set the system up. Let  $t = 0$  be the time when the two cars link up. Let  $x_1$  be the displacement of the first car from the position at  $t = 0$ , and let  $x_2$  be the displacement of the second car from its original location. Then the time when  $x_2(t) = 10$  is exactly the time when impact with the wall occurs. For this  $t$ ,  $x'_2(t)$  is the speed at impact. This system acts just like the system of the previous example but without  $k_1$ . Hence the equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

or

$$\vec{x}'' = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We compute the eigenvalues of  $A$ . It is not hard to see that the eigenvalues are 0 and  $-3$  (exercise). Furthermore, eigenvectors are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$  respectively (exercise). We note that  $\omega_2 = \sqrt{3}$  and we use the second part of the theorem to find our general solution to be

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (a_1 + b_1 t) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} (a_2 \cos(\sqrt{3} t) + b_2 \sin(\sqrt{3} t)) \\ &= \begin{bmatrix} a_1 + b_1 t + a_2 \cos(\sqrt{3} t) + b_2 \sin(\sqrt{3} t) \\ a_1 + b_1 t - 2a_2 \cos(\sqrt{3} t) - 2b_2 \sin(\sqrt{3} t) \end{bmatrix} \end{aligned}$$

We now apply the initial conditions. First the cars start at position 0 so  $x_1(0) = 0$  and  $x_2(0) = 0$ . The first car is traveling at 3 m/s, so  $x'_1(0) = 3$  and the second car starts at rest, so  $x'_2(0) = 0$ . The first conditions says

$$\vec{0} = \vec{x}(0) = \begin{bmatrix} a_1 + a_2 \\ a_1 - 2a_2 \end{bmatrix}.$$

It is not hard to see that this implies that  $a_1 = a_2 = 0$ . We plug  $a_1$  and  $a_2$  and differentiate to get

$$\vec{x}'(t) = \begin{bmatrix} b_1 + \sqrt{3} b_2 \cos(\sqrt{3} t) \\ b_1 - 2\sqrt{3} b_2 \cos(\sqrt{3} t) \end{bmatrix}.$$

So

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} b_1 + \sqrt{3} b_2 \\ b_1 - 2\sqrt{3} b_2 \end{bmatrix}.$$

It is not hard to solve these two equations to find  $b_1 = 2$  and  $b_2 = \frac{1}{\sqrt{3}}$ . Hence the position of our cars is (until the impact with the wall)

$$\vec{x} = \begin{bmatrix} 2t + \frac{1}{\sqrt{3}} \sin(\sqrt{3} t) \\ 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3} t) \end{bmatrix}.$$

Note how the presence of the zero eigenvalue resulted in a term containing  $t$ . This means that the carts will be traveling in the positive direction as time grows, which is what we expect.

What we are really interested in is the second expression, the one for  $x_2$ . We have  $x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3} t)$ . See Figure 3.15 for the plot of  $x_2$  versus time.

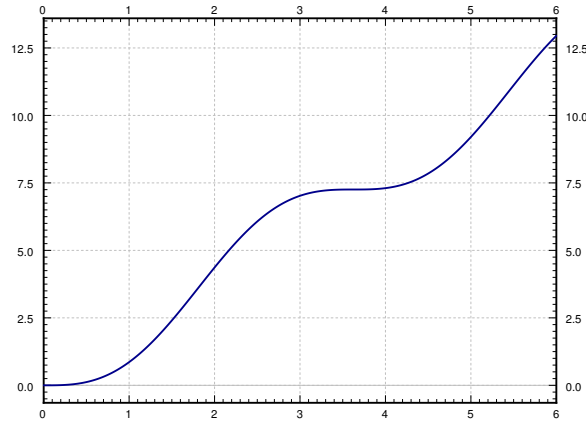


Figure 3.15: Position of the second car in time (ignoring the wall).

Just from the graph we can see that time of impact will be a little more than 5 seconds from time zero. For this we have to solve the equation  $10 = x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3} t)$ . Using a computer (or even a graphing calculator) we find that  $t_{\text{impact}} \approx 5.22$  seconds.

As for the speed we note that  $x_2' = 2 - 2 \cos(\sqrt{3} t)$ . At time of impact (5.22 seconds from  $t = 0$ ) we get that  $x_2'(t_{\text{impact}}) \approx 3.85$ .

The maximum speed is the maximum of  $2 - 2 \cos(\sqrt{3} t)$ , which is 4. We are traveling at almost the maximum speed when we hit the wall.



Now suppose that Bob is a tiny person sitting on car 2. Bob has a Martini in his hand and would like to not spill it. Let us suppose Bob would not spill his Martini when the first car links up with car 2, but if car 2 hits the wall at any speed greater than zero, Bob will spill his drink. Suppose Bob can move car 2 a few meters towards or away from the wall (he cannot go all the way to the wall, nor can he get out of the way of the first car). Is there a “safe” distance for him to be in? A distance such that the impact with the wall is at zero speed?

The answer is yes. Looking at Figure 3.15 on the facing page, we note the “plateau” between  $t = 3$  and  $t = 4$ . There is a point where the speed is zero. To find it we need to solve  $x_2'(t) = 0$ . This is when  $\cos(\sqrt{3}t) = 1$  or in other words when  $t = \frac{2\pi}{\sqrt{3}}, \frac{4\pi}{\sqrt{3}}, \dots$  and so on. We plug in the first value to obtain  $x_2\left(\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}} \approx 7.26$ . So a “safe” distance is about 7 and a quarter meters from the wall.

Alternatively Bob could move away from the wall towards the incoming car 2 where another safe distance is  $\frac{8\pi}{\sqrt{3}} \approx 14.51$  and so on, using all the different  $t$  such that  $x_2'(t) = 0$ . Of course  $t = 0$  is always a solution here, corresponding to  $x_2 = 0$ , but that means standing right at the wall.

### 3.6.3 Forced oscillations

Finally we move to forced oscillations. Suppose that now our system is

$$\vec{x}'' = A\vec{x} + \vec{F} \cos(\omega t). \quad (3.3)$$

That is, we are adding periodic forcing to the system in the direction of the vector  $\vec{F}$ .

Just like before this system just requires us to find one particular solution  $\vec{x}_p$ , add it to the general solution of the associated homogeneous system  $\vec{x}_c$  and we will have the general solution to (3.3). Let us suppose that  $\omega$  is not one of the natural frequencies of  $\vec{x}'' = A\vec{x}$ , then we can guess

$$\vec{x}_p = \vec{c} \cos(\omega t),$$

where  $\vec{c}$  is an unknown constant vector. Note that we do not need to use sine since there are only second derivatives. We solve for  $\vec{c}$  to find  $\vec{x}_p$ . This is really just the method of *undetermined coefficients* for systems. Let us differentiate  $\vec{x}_p$  twice to get

$$\vec{x}_p'' = -\omega^2 \vec{c} \cos(\omega t).$$

Now plug into the equation

$$-\omega^2 \vec{c} \cos(\omega t) = A\vec{c} \cos(\omega t) + \vec{F} \cos(\omega t)$$

We can cancel out the cosine and rearrange the equation to obtain

$$(A + \omega^2 I)\vec{c} = -\vec{F}.$$

So

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}).$$

Of course this is possible only if  $(A + \omega^2 I) = (A - (-\omega^2)I)$  is invertible. That matrix is invertible if and only if  $-\omega^2$  is not an eigenvalue of  $A$ . That is true if and only if  $\omega$  is not a natural frequency of the system.

**Example 3.6.3:** Let us take the example in Figure 3.12 on page 117 with the same parameters as before:  $m_1 = 2$ ,  $m_2 = 1$ ,  $k_1 = 4$ , and  $k_2 = 2$ . Now suppose that there is a force  $2 \cos(3t)$  acting on the second cart.

The equation is

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t).$$

We have solved the associated homogeneous equation before and found the complementary solution to be

$$\vec{x}_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)).$$

We note that the natural frequencies were 1 and 2. Hence 3 is not a natural frequency, we can try  $\vec{c} \cos(3t)$ . We can invert  $(A + 3^2 I)$

$$\left( \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} + 3^2 I \right)^{-1} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix}.$$

Hence,

$$\vec{c} = (A + \omega^2 I)^{-1} (-\vec{F}) = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix}.$$

Combining with what we know the general solution of the associated homogeneous problem to be we get that the general solution to  $\vec{x}'' = A\vec{x} + \vec{F} \cos(\omega t)$  is

$$\vec{x} = \vec{x}_c + \vec{x}_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (a_1 \cos(t) + b_1 \sin(t)) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_2 \cos(2t) + b_2 \sin(2t)) + \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix} \cos(3t).$$

The constants  $a_1$ ,  $a_2$ ,  $b_1$ , and  $b_2$  must then be solved for given any initial conditions.

If  $\omega$  is a natural frequency of the system *resonance* occurs because we will have to try a particular solution of the form

$$\vec{x}_p = \vec{c} t \sin(\omega t) + \vec{d} \cos(\omega t).$$

That is assuming that all eigenvalues of the coefficient matrix are distinct. Note that the amplitude of this solution grows without bound as  $t$  grows.

### 3.6.4 Exercises

**Exercise 3.6.3:** Find a particular solution to

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t).$$

**Exercise 3.6.4** (challenging): Let us take the example in Figure 3.12 on page 117 with the same parameters as before:  $m_1 = 2$ ,  $k_1 = 4$ , and  $k_2 = 2$ , except for  $m_2$ , which is unknown. Suppose that there is a force  $\cos(5t)$  acting on the first mass. Find an  $m_2$  such that there exists a particular solution where the first mass does not move.

*Note:* This idea is called dynamic damping. In practice there will be a small amount of damping and so any transient solution will disappear and after long enough time, the first mass will always come to a stop.

**Exercise 3.6.5:** Let us take the Example 3.6.2 on page 119, but that at time of impact, cart 2 is moving to the left at the speed of 3 m/s. a) Find the behavior of the system after linkup. b) Will the second car hit the wall, or will it be moving away from the wall as time goes on. c) at what speed would the first car have to be traveling for the system to essentially stay in place after linkup.

**Exercise 3.6.6:** Let us take the example in Figure 3.12 on page 117 with parameters  $m_1 = m_2 = 1$ ,  $k_1 = k_2 = 1$ . Does there exist a set of initial conditions for which the first cart moves but the second cart does not? If so, find those conditions. If not, argue why not.

**Exercise 3.6.101:** Find the general solution to  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 6 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos(2t) \\ 0 \\ 0 \end{bmatrix}$ .

**Exercise 3.6.102:** Suppose there are three carts of equal mass  $m$  and connected by two springs of constant  $k$  (and no connections to walls). Set up the system and find its general solution.

**Exercise 3.6.103:** Suppose a cart of mass 2 kg is attached by a spring of constant  $k = 1$  to a cart of mass 3 kg, which is attached to the wall by a spring also of constant  $k = 1$ . Suppose that the initial position of the first cart is 1 meter in the positive direction from the rest position, and the second mass starts at the rest position. The masses are not moving and are let go. Find the position of the second mass as a function of time.

### 3.7 Multiple eigenvalues

*Note: 1 or 1.5 lectures, §5.4 in [EP], §7.8 in [BD]*

It may very well happen that a matrix has some “repeated” eigenvalues. That is, the characteristic equation  $\det(A - \lambda I) = 0$  may have repeated roots. As we have said before, this is actually unlikely to happen for a random matrix. If we take a small perturbation of  $A$  (we change the entries of  $A$  slightly), then we will get a matrix with distinct eigenvalues. As any system we will want to solve in practice is an approximation to reality anyway, it is not indispensable to know how to solve these corner cases. It may happen on occasion that it is easier or desirable to solve such a system directly.

#### 3.7.1 Geometric multiplicity

Take the diagonal matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

$A$  has an eigenvalue 3 of multiplicity 2. We call the multiplicity of the eigenvalue in the characteristic equation the *algebraic multiplicity*. In this case, there also exist 2 linearly independent eigenvectors,  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  corresponding to the eigenvalue 3. This means that the so-called *geometric multiplicity* of this eigenvalue is also 2.

In all the theorems where we required a matrix to have  $n$  distinct eigenvalues, we only really needed to have  $n$  linearly independent eigenvectors. For example,  $\vec{x}' = A\vec{x}$  has the general solution

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}.$$

Let us restate the theorem about real eigenvalues. In the following theorem we will repeat eigenvalues according to (algebraic) multiplicity. So for the above matrix  $A$ , we would say that it has eigenvalues 3 and 3.

**Theorem 3.7.1.** *Take  $\vec{x}' = P\vec{x}$ . Suppose the matrix  $P$  is  $n \times n$ , has  $n$  real eigenvalues (not necessarily distinct),  $\lambda_1, \dots, \lambda_n$ , and there are  $n$  linearly independent corresponding eigenvectors  $\vec{v}_1, \dots, \vec{v}_n$ . Then the general solution to the ODE can be written as*

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}.$$

The *geometric multiplicity* of an eigenvalue of algebraic multiplicity  $n$  is equal to the number of corresponding linearly independent eigenvectors. The geometric multiplicity is always less than or equal to the algebraic multiplicity. We have handled the case when these two multiplicities are equal. If the geometric multiplicity is equal to the algebraic multiplicity, then we say the eigenvalue is *complete*.

In other words, the hypothesis of the theorem could be stated as saying that if all the eigenvalues of  $P$  are complete, then there are  $n$  linearly independent eigenvectors and thus we have the given general solution.

Note that if the geometric multiplicity of an eigenvalue is 2 or greater, then the set of linearly independent eigenvectors is not unique up to multiples as it was before. For example, for the diagonal matrix  $A$  above we could also pick eigenvectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , or in fact any pair of two linearly independent vectors. The number of linearly independent eigenvectors corresponding to  $\lambda$  is the number of free variables we obtain when solving  $A\vec{v} = \lambda\vec{v}$ . We then pick values for those free variables to obtain the eigenvectors. If you pick different values, you may get different eigenvectors.

### 3.7.2 Defective eigenvalues

If an  $n \times n$  matrix has less than  $n$  linearly independent eigenvectors, it is said to be *deficient*. Then there is at least one eigenvalue with an algebraic multiplicity that is higher than its geometric multiplicity. We call this eigenvalue *defective* and the difference between the two multiplicities we call the *defect*.

**Example 3.7.1:** The matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

has an eigenvalue 3 of algebraic multiplicity 2. Let us try to compute eigenvectors.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}.$$

We must have that  $v_2 = 0$ . Hence any eigenvector is of the form  $\begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ . Any two such vectors are linearly dependent, and hence the geometric multiplicity of the eigenvalue is 1. Therefore, the defect is 1, and we can no longer apply the eigenvalue method directly to a system of ODEs with such a coefficient matrix.

The key observation we will use here is that if  $\lambda$  is an eigenvalue of  $A$  of algebraic multiplicity  $m$ , then we will be able to find  $m$  linearly independent vectors solving the equation  $(A - \lambda I)^m \vec{v} = \vec{0}$ . We will call these *generalized eigenvectors*.

Let us continue with the example  $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  and the equation  $\vec{x}' = A\vec{x}$ . We have an eigenvalue  $\lambda = 3$  of (algebraic) multiplicity 2 and defect 1. We have found one eigenvector  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . We have the solution

$$\vec{x}_1 = \vec{v}_1 e^{3t}.$$

In this case, let us try (in the spirit of repeated roots of the characteristic equation for a single equation) another solution of the form

$$\vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t) e^{3t}.$$

We differentiate to get

$$\vec{x}_2' = \vec{v}_1 e^{3t} + 3(\vec{v}_2 + \vec{v}_1 t) e^{3t} = (3\vec{v}_2 + \vec{v}_1) e^{3t} + 3\vec{v}_1 t e^{3t}.$$

As we are assuming that  $\vec{x}_2$  is a solution,  $\vec{x}_2'$  must equal  $A\vec{x}_2$ , and

$$A\vec{x}_2 = A(\vec{v}_2 + \vec{v}_1 t) e^{3t} = A\vec{v}_2 e^{3t} + A\vec{v}_1 t e^{3t}.$$

By looking at the coefficients of  $e^{3t}$  and  $t e^{3t}$  we see  $3\vec{v}_2 + \vec{v}_1 = A\vec{v}_2$  and  $3\vec{v}_1 = A\vec{v}_1$ . This means that

$$(A - 3I)\vec{v}_2 = \vec{v}_1, \quad \text{and} \quad (A - 3I)\vec{v}_1 = \vec{0}.$$

Therefore,  $\vec{x}_2$  is a solution if these two equations are satisfied. We know the second of these two equations is satisfied as  $\vec{v}_1$  is an eigenvector. If we plug the first equation into the second we obtain

$$(A - 3I)(A - 3I)\vec{v}_2 = \vec{0}, \quad \text{or} \quad (A - 3I)^2 \vec{v}_2 = \vec{0}.$$

If we can, therefore, find a  $\vec{v}_2$  that solves  $(A - 3I)^2 \vec{v}_2 = \vec{0}$  and such that  $(A - 3I)\vec{v}_2 = \vec{v}_1$ , then we are done. This is just a bunch of linear equations to solve and we are by now very good at that.

We notice that in this simple case  $(A - 3I)^2$  is just the zero matrix (exercise). Hence, any vector  $\vec{v}_2$  solves  $(A - 3I)^2 \vec{v}_2 = \vec{0}$ . We just have to make sure that  $(A - 3I)\vec{v}_2 = \vec{v}_1$ . Write

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By inspection we see that letting  $a = 0$  ( $a$  could be anything in fact) and  $b = 1$  does the job. Hence we can take  $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Our general solution to  $\vec{x}' = A\vec{x}$  is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right) e^{3t} = \begin{bmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_2 e^{3t} \end{bmatrix}.$$

Let us check that we really do have the solution. First  $x_1' = c_1 3e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} = 3x_1 + x_2$ . Good. Now  $x_2' = 3c_2 e^{3t} = 3x_2$ . Good.

Note that the system  $\vec{x}' = A\vec{x}$  has a simpler solution since  $A$  is a so-called *upper triangular matrix*, that is every entry below the diagonal is zero. In particular, the equation for  $x_2$  does not depend on  $x_1$ . Mind you, not every defective matrix is triangular.

**Exercise 3.7.1:** Solve  $\vec{x}' = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \vec{x}$  by first solving for  $x_2$  and then for  $x_1$  independently. Now check that you got the same solution as we did above.

Let us describe the general algorithm. First for  $\lambda$  of multiplicity 2, defect 1. First find an eigenvector  $\vec{v}_1$  of  $\lambda$ . Now find a vector  $\vec{v}_2$  such that

$$\begin{aligned} (A - \lambda I)^2 \vec{v}_2 &= \vec{0}, \\ (A - \lambda I) \vec{v}_2 &= \vec{v}_1. \end{aligned}$$

This gives us two linearly independent solutions

$$\begin{aligned}\vec{x}_1 &= \vec{v}_1 e^{\lambda t}, \\ \vec{x}_2 &= (\vec{v}_2 + \vec{v}_1 t) e^{\lambda t}.\end{aligned}$$

This machinery can also be generalized to larger matrices and higher defects. We will not go over this method in detail, but let us just sketch the ideas. Suppose that  $A$  has a multiplicity  $m$  eigenvalue  $\lambda$ . We find vectors such that

$$(A - \lambda I)^k \vec{v} = \vec{0}, \quad \text{but} \quad (A - \lambda I)^{k-1} \vec{v} \neq \vec{0}.$$

Such vectors are called *generalized eigenvectors*. For every eigenvector  $\vec{v}_1$  we find a chain of generalized eigenvectors  $\vec{v}_2$  through  $\vec{v}_k$  such that:

$$\begin{aligned}(A - \lambda I)\vec{v}_1 &= \vec{0}, \\ (A - \lambda I)\vec{v}_2 &= \vec{v}_1, \\ &\vdots \\ (A - \lambda I)\vec{v}_k &= \vec{v}_{k-1}.\end{aligned}$$

We form the linearly independent solutions

$$\begin{aligned}\vec{x}_1 &= \vec{v}_1 e^{\lambda t}, \\ \vec{x}_2 &= (\vec{v}_2 + \vec{v}_1 t) e^{\lambda t}, \\ &\vdots \\ \vec{x}_k &= \left( \vec{v}_k + \vec{v}_{k-1} t + \vec{v}_{k-2} \frac{t^2}{2} + \cdots + \vec{v}_2 \frac{t^{k-2}}{(k-2)!} + \vec{v}_1 \frac{t^{k-1}}{(k-1)!} \right) e^{\lambda t}.\end{aligned}$$

Recall that  $k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$  is the factorial. We proceed to find chains until we form  $m$  linearly independent solutions ( $m$  is the multiplicity). You may need to find several chains for every eigenvalue.

### 3.7.3 Exercises

**Exercise 3.7.2:** Let  $A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ . Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.3:** Let  $A = \begin{bmatrix} 5 & -4 & 4 \\ 0 & 3 & 0 \\ -2 & 4 & -1 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.4:** Let  $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$  in two different ways and verify you get the same answer.

**Exercise 3.7.5:** Let  $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & -2 \\ -4 & 4 & 7 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.6:** Let  $A = \begin{bmatrix} 0 & 4 & -2 \\ -1 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.7:** Let  $A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.8:** Suppose that  $A$  is a  $2 \times 2$  matrix with a repeated eigenvalue  $\lambda$ . Suppose that there are two linearly independent eigenvectors. Show that  $A = \lambda I$ .

**Exercise 3.7.101:** Let  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.102:** Let  $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.103:** Let  $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 9 \\ 0 & -1 & 5 \end{bmatrix}$ . a) What are the eigenvalues? b) What is/are the defect(s) of the eigenvalue(s)? c) Find the general solution of  $\vec{x}' = A\vec{x}$ .

**Exercise 3.7.104:** Let  $A = \begin{bmatrix} a & a \\ b & c \end{bmatrix}$ , where  $a$ ,  $b$ , and  $c$  are unknowns. Suppose that 5 is a doubled eigenvalue of defect 1, and suppose that  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is the eigenvector. Find  $A$  and show that there is only one solution.



## 3.8 Matrix exponentials

*Note: 2 lectures, §5.5 in [EP], §7.7 in [BD]*

### 3.8.1 Definition

In this section we present a different way of finding the fundamental matrix solution of a system. Suppose that we have the constant coefficient equation

$$\vec{x}' = P\vec{x},$$

as usual. Now suppose that this was one equation ( $P$  is a number or a  $1 \times 1$  matrix). Then the solution to this would be

$$\vec{x} = e^{Pt}.$$

It turns out the same computation works for matrices when we define  $e^{Pt}$  properly. First let us write down the Taylor series for  $e^{at}$  for some number  $a$ .

$$e^{at} = 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \frac{(at)^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}.$$

Recall  $k! = 1 \cdot 2 \cdot 3 \cdots k$  is the factorial, and  $0! = 1$ . We differentiate this series term by term

$$\frac{d}{dt}(e^{at}) = a + a^2t + \frac{a^3t^2}{2} + \frac{a^4t^3}{6} + \cdots = a \left( 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots \right) = ae^{at}.$$

Maybe we can write try the same trick here. Suppose that for an  $n \times n$  matrix  $A$  we define the *matrix exponential* as

$$e^A \stackrel{\text{def}}{=} I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots$$

Let us not worry about convergence. The series really does always converge. We usually write  $Pt$  as  $tP$  by convention when  $P$  is a matrix. With this small change and by the exact same calculation as above we have that

$$\frac{d}{dt}(e^{tP}) = Pe^{tP}.$$

Now  $P$  and hence  $e^{tP}$  is an  $n \times n$  matrix. What we are looking for is a vector. We note that in the  $1 \times 1$  case we would at this point multiply by an arbitrary constant to get the general solution. In the matrix case we multiply by a column vector  $\vec{c}$ .

**Theorem 3.8.1.** *Let  $P$  be an  $n \times n$  matrix. Then the general solution to  $\vec{x}' = P\vec{x}$  is*

$$\vec{x} = e^{tP}\vec{c},$$

where  $\vec{c}$  is an arbitrary constant vector. In fact  $\vec{x}(0) = \vec{c}$ .

Let us check.

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}(e^{tP}\vec{c}) = Pe^{tP}\vec{c} = P\vec{x}.$$

Hence  $e^{tP}$  is the fundamental matrix solution of the homogeneous system. If we find a way to compute the matrix exponential, we will have another method of solving constant coefficient homogeneous systems. It also makes it easy to solve for initial conditions. To solve  $\vec{x}' = A\vec{x}$ ,  $\vec{x}(0) = \vec{b}$ , we take the solution

$$\vec{x} = e^{tA}\vec{b}.$$

This equation follows because  $e^{0A} = I$ , so  $\vec{x}(0) = e^{0A}\vec{b} = \vec{b}$ .

We mention a drawback of matrix exponentials. In general  $e^{A+B} \neq e^A e^B$ . The trouble is that matrices do not commute, that is, in general  $AB \neq BA$ . If you try to prove  $e^{A+B} \neq e^A e^B$  using the Taylor series, you will see why the lack of commutativity becomes a problem. However, it is still true that if  $AB = BA$ , that is, if  $A$  and  $B$  commute, then  $e^{A+B} = e^A e^B$ . We will find this fact useful. Let us restate this as a theorem to make a point.

**Theorem 3.8.2.** *If  $AB = BA$ , then  $e^{A+B} = e^A e^B$ . Otherwise  $e^{A+B} \neq e^A e^B$  in general.*

### 3.8.2 Simple cases

In some instances it may work to just plug into the series definition. Suppose the matrix is diagonal. For example,  $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ . Then

$$D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix},$$

and

$$e^D = I + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \cdots = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}.$$

So by this rationale we have that

$$e^I = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \quad \text{and} \quad e^{aI} = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix}.$$

This makes exponentials of certain other matrices easy to compute. Notice for example that the matrix  $A = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$  can be written as  $3I + B$  where  $B = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$ . Notice that  $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . So  $B^k = 0$  for all  $k \geq 2$ . Therefore,  $e^B = I + B$ . Suppose we actually want to compute  $e^{tA}$ . The matrices  $3tI$  and  $tB$  commute (exercise: check this) and  $e^{tB} = I + tB$ , since  $(tB)^2 = t^2 B^2 = 0$ . We write

$$\begin{aligned} e^{tA} &= e^{3tI+tB} = e^{3tI} e^{tB} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} (I + tB) = \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} = \begin{bmatrix} (1+2t)e^{3t} & 4te^{3t} \\ -te^{3t} & (1-2t)e^{3t} \end{bmatrix}. \end{aligned}$$

So we have found the fundamental matrix solution for the system  $\vec{x}' = A\vec{x}$ . Note that this matrix has a repeated eigenvalue with a defect; there is only one eigenvector for the eigenvalue 3. So we have found a perhaps easier way to handle this case. In fact, if a matrix  $A$  is  $2 \times 2$  and has an eigenvalue  $\lambda$  of multiplicity 2, then either  $A$  is diagonal, or  $A = \lambda I + B$  where  $B^2 = 0$ . This is a good exercise.

**Exercise 3.8.1:** Suppose that  $A$  is  $2 \times 2$  and  $\lambda$  is the only eigenvalue. Then show that  $(A - \lambda I)^2 = 0$ . Then we can write  $A = \lambda I + B$ , where  $B^2 = 0$ . Hint: First write down what does it mean for the eigenvalue to be of multiplicity 2. You will get an equation for the entries. Now compute the square of  $B$ .

Matrices  $B$  such that  $B^k = 0$  for some  $k$  are called *nilpotent*. Computation of the matrix exponential for nilpotent matrices is easy by just writing down the first  $k$  terms of the Taylor series.

### 3.8.3 General matrices

In general, the exponential is not as easy to compute as above. We cannot usually write any matrix as a sum of commuting matrices where the exponential is simple for each one. But fear not, it is still not too difficult provided we can find enough eigenvectors. First we need the following interesting result about matrix exponentials. For any two square matrices  $A$  and  $B$ , we have

$$e^{BAB^{-1}} = Be^AB^{-1}.$$

This can be seen by writing down the Taylor series. First note that

$$(BAB^{-1})^2 = BAB^{-1}BAB^{-1} = BAIAB^{-1} = BA^2B^{-1}.$$

And hence by the same reasoning  $(BAB^{-1})^k = BA^k B^{-1}$ . Now write down the Taylor series for  $e^{BAB^{-1}}$ .

$$\begin{aligned} e^{BAB^{-1}} &= I + BAB^{-1} + \frac{1}{2}(BAB^{-1})^2 + \frac{1}{6}(BAB^{-1})^3 + \cdots \\ &= BB^{-1} + BAB^{-1} + \frac{1}{2}BA^2B^{-1} + \frac{1}{6}BA^3B^{-1} + \cdots \\ &= B(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots)B^{-1} \\ &= Be^AB^{-1}. \end{aligned}$$

Sometimes we can write a matrix  $A$  as  $EDE^{-1}$ , where  $D$  is diagonal. This procedure is called *diagonalization*. If we can do that, you can see that the computation of the exponential becomes easy. Adding  $t$  into the mix we see that

$$e^{tA} = Ee^{tD}E^{-1}.$$

Now to do this we will need  $n$  linearly independent eigenvectors of  $A$ . Otherwise this method does not work and we need to be trickier, but we will not get into such details in this course. We

let  $E$  be the matrix with the eigenvectors as columns. Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues and let  $\vec{v}_1, \dots, \vec{v}_n$  be the eigenvectors, then  $E = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$ . Let  $D$  be the diagonal matrix with the eigenvalues on the main diagonal. That is

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

We compute

$$\begin{aligned} AE &= A[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n] \\ &= [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n] \\ &= [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \ \cdots \ \lambda_n\vec{v}_n] \\ &= [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]D \\ &= ED. \end{aligned}$$

The columns of  $E$  are linearly independent as these are linearly independent eigenvectors of  $A$ . Hence  $E$  is invertible. Since  $AE = ED$ , we right multiply by  $E^{-1}$  and we get

$$A = EDE^{-1}.$$

This means that  $e^A = Ee^DE^{-1}$ . Multiplying the matrix by  $t$  we obtain

$$e^{tA} = Ee^{tD}E^{-1} = E \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} E^{-1}. \quad (3.4)$$

The formula (3.4), therefore, gives the formula for computing the fundamental matrix solution  $e^{tA}$  for the system  $\vec{x}' = A\vec{x}$ , in the case where we have  $n$  linearly independent eigenvectors.

Notice that this computation still works when the eigenvalues and eigenvectors are complex, though then you will have to compute with complex numbers. Note that it is clear from the definition that if  $A$  is real, then  $e^{tA}$  is real. So you will only need complex numbers in the computation and you may need to apply Euler's formula to simplify the result. If simplified properly the final matrix will not have any complex numbers in it.

**Example 3.8.1:** Compute the fundamental matrix solution using the matrix exponentials for the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then compute the particular solution for the initial conditions  $x(0) = 4$  and  $y(0) = 2$ .

Let  $A$  be the coefficient matrix  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ . We first compute (exercise) that the eigenvalues are 3 and  $-1$  and corresponding eigenvectors are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . Hence we write

$$\begin{aligned} e^{tA} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -e^{3t} - e^{-t} & -e^{3t} + e^{-t} \\ -e^{3t} + e^{-t} & -e^{3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}. \end{aligned}$$

The initial conditions are  $x(0) = 4$  and  $y(0) = 2$ . Hence, by the property that  $e^{0A} = I$  we find that the particular solution we are looking for is  $e^{tA}\vec{b}$  where  $\vec{b}$  is  $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ . Then the particular solution we are looking for is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} + 2e^{-t} + e^{3t} - e^{-t} \\ 2e^{3t} - 2e^{-t} + e^{3t} + e^{-t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} - e^{-t} \end{bmatrix}.$$

### 3.8.4 Fundamental matrix solutions

We note that if you can compute the fundamental matrix solution in a different way, you can use this to find the matrix exponential  $e^{tA}$ . The fundamental matrix solution of a system of ODEs is not unique. The exponential is the fundamental matrix solution with the property that for  $t = 0$  we get the identity matrix. So we must find the right fundamental matrix solution. Let  $X$  be any fundamental matrix solution to  $\vec{x}' = A\vec{x}$ . Then we claim

$$e^{tA} = X(t) [X(0)]^{-1}.$$

Clearly, if we plug  $t = 0$  into  $X(t) [X(0)]^{-1}$  we get the identity. We can multiply a fundamental matrix solution on the right by any constant invertible matrix and we still get a fundamental matrix solution. All we are doing is changing what the arbitrary constants are in the general solution  $\vec{x}(t) = X(t)\vec{c}$ .

### 3.8.5 Approximations

If you think about it, the computation of any fundamental matrix solution  $X$  using the eigenvalue method is just as difficult as the computation of  $e^{tA}$ . So perhaps we did not gain much by this new tool. However, the Taylor series expansion actually gives us a very easy way to approximate solutions, which the eigenvalue method did not.

The simplest thing we can do is to just compute the series up to a certain number of terms. There are better ways to approximate the exponential\*. In many cases however, few terms of the Taylor series give a reasonable approximation for the exponential and may suffice for the application. For example, let us compute the first 4 terms of the series for the matrix  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ .

$$\begin{aligned} e^{tA} &\approx I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 = I + t \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + t^2 \begin{bmatrix} \frac{5}{2} & 2 \\ 2 & \frac{5}{2} \end{bmatrix} + t^3 \begin{bmatrix} \frac{13}{6} & \frac{7}{3} \\ \frac{7}{3} & \frac{13}{6} \end{bmatrix} = \\ &= \begin{bmatrix} 1 + t + \frac{5}{2}t^2 + \frac{13}{6}t^3 & 2t + 2t^2 + \frac{7}{3}t^3 \\ 2t + 2t^2 + \frac{7}{3}t^3 & 1 + t + \frac{5}{2}t^2 + \frac{13}{6}t^3 \end{bmatrix}. \end{aligned}$$

Just like the scalar version of the Taylor series approximation, the approximation will be better for small  $t$  and worse for larger  $t$ . For larger  $t$ , we will generally have to compute more terms. Let us see how we stack up against the real solution with  $t = 0.1$ . The approximate solution is approximately (rounded to 8 decimal places)

$$e^{0.1A} \approx I + 0.1A + \frac{0.1^2}{2}A^2 + \frac{0.1^3}{6}A^3 = \begin{bmatrix} 1.12716667 & 0.22233333 \\ 0.22233333 & 1.12716667 \end{bmatrix}.$$

And plugging  $t = 0.1$  into the real solution (rounded to 8 decimal places) we get

$$e^{0.1A} = \begin{bmatrix} 1.12734811 & 0.22251069 \\ 0.22251069 & 1.12734811 \end{bmatrix}.$$

This is not bad at all. Although if we take the same approximation for  $t = 1$  we get (using the Taylor series)

$$\begin{bmatrix} 6.66666667 & 6.33333333 \\ 6.33333333 & 6.66666667 \end{bmatrix},$$

while the real value is (again rounded to 8 decimal places)

$$\begin{bmatrix} 10.22670818 & 9.85882874 \\ 9.85882874 & 10.22670818 \end{bmatrix}.$$

So the approximation is not very good once we get up to  $t = 1$ . To get a good approximation at  $t = 1$  (say up to 2 decimal places) we would need to go up to the 11<sup>th</sup> power (exercise).

### 3.8.6 Exercises

**Exercise 3.8.2:** Using the matrix exponential, find a fundamental matrix solution for the system  $x' = 3x + y$ ,  $y' = x + 3y$ .

---

\*C. Moler and C.F. Van Loan, *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*, SIAM Review 45 (1), 2003, 3–49

**Exercise 3.8.3:** Find  $e^{tA}$  for the matrix  $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ .

**Exercise 3.8.4:** Find a fundamental matrix solution for the system  $x'_1 = 7x_1 + 4x_2 + 12x_3$ ,  $x'_2 = x_1 + 2x_2 + x_3$ ,  $x'_3 = -3x_1 - 2x_2 - 5x_3$ . Then find the solution that satisfies  $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ .

**Exercise 3.8.5:** Compute the matrix exponential  $e^A$  for  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ .

**Exercise 3.8.6** (challenging): Suppose  $AB = BA$ . Show that under this assumption,  $e^{A+B} = e^A e^B$ .

**Exercise 3.8.7:** Use Exercise 3.8.6 to show that  $(e^A)^{-1} = e^{-A}$ . In particular this means that  $e^A$  is invertible even if  $A$  is not.

**Exercise 3.8.8:** Suppose  $A$  is a matrix with eigenvalues  $-1$ ,  $1$ , and corresponding eigenvectors  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . a) Find matrix  $A$  with these properties. b) Find the fundamental matrix solution to  $\vec{x}' = A\vec{x}$ . c) Solve the system in with initial conditions  $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

**Exercise 3.8.9:** Suppose that  $A$  is an  $n \times n$  matrix with a repeated eigenvalue  $\lambda$  of multiplicity  $n$ . Suppose that there are  $n$  linearly independent eigenvectors. Show that the matrix is diagonal, in particular  $A = \lambda I$ . Hint: Use diagonalization and the fact that the identity matrix commutes with every other matrix.

**Exercise 3.8.10:** Let  $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ . a) Find  $e^{tA}$ . b) Solve  $\vec{x}' = A\vec{x}$ ,  $\vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ .

**Exercise 3.8.11:** Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ . Approximate  $e^{tA}$  by expanding the power series up to the third order.

**Exercise 3.8.101:** Compute  $e^{tA}$  where  $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ .

**Exercise 3.8.102:** Compute  $e^{tA}$  where  $A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 1 & 2 \\ -1 & -3 & 4 \end{bmatrix}$ .

**Exercise 3.8.103:** a) Compute  $e^{tA}$  where  $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ . b) Solve  $\vec{x}' = A\vec{x}$  for  $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

**Exercise 3.8.104:** Compute the first 3 terms (up to the second degree) of the Taylor expansion of  $e^{tA}$  where  $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$  (Write as a single matrix). Then use it to approximate  $e^{0.1A}$ .

### 3.9 Nonhomogeneous systems

*Note: 3 lectures (may have to skip a little), somewhat different from §5.6 in [EP], §7.9 in [BD]*

#### 3.9.1 First order constant coefficient

##### Integrating factor

Let us first focus on the nonhomogeneous first order equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t),$$

where  $A$  is a constant matrix. The first method we will look at is the *integrating factor method*. For simplicity we rewrite the equation as

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t),$$

where  $P = -A$ . We multiply both sides of the equation by  $e^{tP}$  (being mindful that we are dealing with matrices that may not commute) to obtain

$$e^{tP}\vec{x}'(t) + e^{tP}P\vec{x}(t) = e^{tP}\vec{f}(t).$$

We notice that  $Pe^{tP} = e^{tP}P$ . This fact follows by writing down the series definition of  $e^{tP}$ ,

$$\begin{aligned} Pe^{tP} &= P \left( I + I + tP + \frac{1}{2}(tP)^2 + \cdots \right) = P + tP^2 + \frac{1}{2}t^2P^3 + \cdots = \\ &= \left( I + I + tP + \frac{1}{2}(tP)^2 + \cdots \right) P = Pe^{tP}. \end{aligned}$$

We have already seen that  $\frac{d}{dt}(e^{tP}) = Pe^{tP}$ . Hence,

$$\frac{d}{dt}(e^{tP}\vec{x}(t)) = e^{tP}\vec{f}(t).$$

We can now integrate. That is, we integrate each component of the vector separately

$$e^{tP}\vec{x}(t) = \int e^{tP}\vec{f}(t) dt + \vec{c}.$$

Recall from Exercise 3.8.7 that  $(e^{tP})^{-1} = e^{-tP}$ . Therefore, we obtain

$$\vec{x}(t) = e^{-tP} \int e^{tP}\vec{f}(t) dt + e^{-tP}\vec{c}.$$



Perhaps it is better understood as a definite integral. In this case it will be easy to also solve for the initial conditions as well. Suppose we have the equation with initial conditions

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t), \quad \vec{x}(0) = \vec{b}.$$

The solution can then be written as

$$\boxed{\vec{x}(t) = e^{-tP} \int_0^t e^{sP} \vec{f}(s) ds + e^{-tP} \vec{b}.} \quad (3.5)$$

Again, the integration means that each component of the vector  $e^{sP} \vec{f}(s)$  is integrated separately. It is not hard to see that (3.5) really does satisfy the initial condition  $\vec{x}(0) = \vec{b}$ .

$$\vec{x}(0) = e^{-0P} \int_0^0 e^{sP} \vec{f}(s) ds + e^{-0P} \vec{b} = I\vec{b} = \vec{b}.$$

**Example 3.9.1:** Suppose that we have the system

$$\begin{aligned} x_1' + 5x_1 - 3x_2 &= e^t, \\ x_2' + 3x_1 - x_2 &= 0, \end{aligned}$$

with initial conditions  $x_1(0) = 1, x_2(0) = 0$ .

Let us write the system as

$$\vec{x}' + \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We have previously computed  $e^{tP}$  for  $P = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ . We immediately have  $e^{-tP}$ , simply by negating  $t$ .

$$e^{tP} = \begin{bmatrix} (1+3t)e^{2t} & -3te^{2t} \\ 3te^{2t} & (1-3t)e^{2t} \end{bmatrix}, \quad e^{-tP} = \begin{bmatrix} (1-3t)e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)e^{-2t} \end{bmatrix}.$$

Instead of computing the whole formula at once. Let us do it in stages. First

$$\begin{aligned} \int_0^t e^{sP} \vec{f}(s) ds &= \int_0^t \begin{bmatrix} (1+3s)e^{2s} & -3se^{2s} \\ 3se^{2s} & (1-3s)e^{2s} \end{bmatrix} \begin{bmatrix} e^s \\ 0 \end{bmatrix} ds \\ &= \int_0^t \begin{bmatrix} (1+3s)e^{3s} \\ 3se^{3s} \end{bmatrix} ds \\ &= \begin{bmatrix} te^{3t} \\ \frac{(3t-1)e^{3t}+1}{3} \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned}
 \vec{x}(t) &= e^{-tP} \int_0^t e^{sP} \vec{f}(s) ds + e^{-tP} \vec{b} \\
 &= \begin{bmatrix} (1-3t)e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)e^{-2t} \end{bmatrix} \begin{bmatrix} te^{3t} \\ \frac{(3t-1)e^{3t}+1}{3} \end{bmatrix} + \begin{bmatrix} (1-3t)e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} te^{-2t} \\ -\frac{e^t}{3} + \left(\frac{1}{3} + t\right)e^{-2t} \end{bmatrix} + \begin{bmatrix} (1-3t)e^{-2t} \\ -3te^{-2t} \end{bmatrix} \\
 &= \begin{bmatrix} (1-2t)e^{-2t} \\ -\frac{e^t}{3} + \left(\frac{1}{3} - 2t\right)e^{-2t} \end{bmatrix}.
 \end{aligned}$$

Phew!

Let us check that this really works.

$$x_1' + 5x_1 - 3x_2 = (4te^{-2t} - 4e^{-2t}) + 5(1-2t)e^{-2t} + e^t - (1-6t)e^{-2t} = e^t.$$

Similarly (exercise)  $x_2' + 3x_1 - x_2 = 0$ . The initial conditions are also satisfied as well (exercise).

For systems, the integrating factor method only works if  $P$  does not depend on  $t$ , that is,  $P$  is constant. The problem is that in general

$$\frac{d}{dt} e^{\int P(t) dt} \neq P(t) e^{\int P(t) dt},$$

because matrix multiplication is not commutative.

### Eigenvector decomposition

For the next method, we note that eigenvectors of a matrix give the directions in which the matrix acts like a scalar. If we solve our system along these directions these solutions would be simpler as we can treat the matrix as a scalar. We can put those solutions together to get the general solution.

Take the equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t). \quad (3.6)$$

Assume that  $A$  has  $n$  linearly independent eigenvectors  $\vec{v}_1, \dots, \vec{v}_n$ . Let us write

$$\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \dots + \vec{v}_n \xi_n(t). \quad (3.7)$$

That is, we wish to write our solution as a linear combination of eigenvectors of  $A$ . If we can solve for the scalar functions  $\xi_1$  through  $\xi_n$  we have our solution  $\vec{x}$ . Let us decompose  $\vec{f}$  in terms of the eigenvectors as well. Write

$$\vec{f}(t) = \vec{v}_1 g_1(t) + \vec{v}_2 g_2(t) + \dots + \vec{v}_n g_n(t). \quad (3.8)$$

That is, we wish to find  $g_1$  through  $g_n$  that satisfy (3.8). We note that since all the eigenvectors are independent, the matrix  $E = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$  is invertible. We see that (3.8) can be written as  $\vec{f} = E\vec{g}$ , where the components of  $\vec{g}$  are the functions  $g_1$  through  $g_n$ . Then  $\vec{g} = E^{-1}\vec{f}$ . Hence it is always possible to find  $\vec{g}$  when there are  $n$  linearly independent eigenvectors.

We plug (3.7) into (3.6), and note that  $A\vec{v}_k = \lambda_k\vec{v}_k$ .

$$\begin{aligned}\vec{x}' &= \vec{v}_1 \xi_1' + \vec{v}_2 \xi_2' + \cdots + \vec{v}_n \xi_n' \\ &= A(\vec{v}_1 \xi_1 + \vec{v}_2 \xi_2 + \cdots + \vec{v}_n \xi_n) + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= A\vec{v}_1 \xi_1 + A\vec{v}_2 \xi_2 + \cdots + A\vec{v}_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= \vec{v}_1 \lambda_1 \xi_1 + \vec{v}_2 \lambda_2 \xi_2 + \cdots + \vec{v}_n \lambda_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= \vec{v}_1 (\lambda_1 \xi_1 + g_1) + \vec{v}_2 (\lambda_2 \xi_2 + g_2) + \cdots + \vec{v}_n (\lambda_n \xi_n + g_n).\end{aligned}$$

If we identify the coefficients of the vectors  $\vec{v}_1$  through  $\vec{v}_n$  we get the equations

$$\begin{aligned}\xi_1' &= \lambda_1 \xi_1 + g_1, \\ \xi_2' &= \lambda_2 \xi_2 + g_2, \\ &\vdots \\ \xi_n' &= \lambda_n \xi_n + g_n.\end{aligned}$$

Each one of these equations is independent of the others. They are all linear first order equations and can easily be solved by the standard integrating factor method for single equations. That is, for example for the  $k^{\text{th}}$  equation we write

$$\xi_k'(t) - \lambda_k \xi_k(t) = g_k(t).$$

We use the integrating factor  $e^{-\lambda_k t}$  to find that

$$\frac{d}{dt} [\xi_k(t) e^{-\lambda_k t}] = e^{-\lambda_k t} g_k(t).$$

Now we integrate and solve for  $\xi_k$  to get

$$\xi_k(t) = e^{\lambda_k t} \int e^{-\lambda_k t} g_k(t) dt + C_k e^{\lambda_k t}.$$

Note that if we are looking for just any particular solution, we could set  $C_k$  to be zero. If we leave these constants in, we will get the general solution. Write  $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \cdots + \vec{v}_n \xi_n(t)$ , and we are done.

Again, as always, it is perhaps better to write these integrals as definite integrals. Suppose that we have an initial condition  $\vec{x}(0) = \vec{b}$ . We take  $\vec{c} = E^{-1}\vec{b}$  and note  $\vec{b} = \vec{v}_1 a_1 + \cdots + \vec{v}_n a_n$ , just like before. Then if we write

$$\xi_k(t) = e^{\lambda_k t} \int_0^t e^{-\lambda_k s} g_k(s) ds + a_k e^{\lambda_k t},$$

we will actually get the particular solution  $\vec{x}(t) = \vec{v}_1\xi_1(t) + \vec{v}_2\xi_2(t) + \cdots + \vec{v}_n\xi_n(t)$  satisfying  $\vec{x}(0) = \vec{b}$ , because  $\xi_k(0) = a_k$ .

**Example 3.9.2:** Let  $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ . Solve  $\vec{x}' = A\vec{x} + \vec{f}$  where  $\vec{f}(t) = \begin{bmatrix} 2e^t \\ 2t \end{bmatrix}$  for  $\vec{x}(0) = \begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix}$ .

The eigenvalues of  $A$  are  $-2$  and  $4$  and corresponding eigenvectors are  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  respectively. This calculation is left as an exercise. We write down the matrix  $E$  of the eigenvectors and compute its inverse (using the inverse formula for  $2 \times 2$  matrices)

$$E = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad E^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We are looking for a solution of the form  $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_2$ . We also wish to write  $\vec{f}$  in terms of the eigenvectors. That is we wish to write  $\vec{f} = \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2$ . Thus

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1} \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} e^t - t \\ e^t + t \end{bmatrix}.$$

So  $g_1 = e^t - t$  and  $g_2 = e^t + t$ .

We further want to write  $\vec{x}(0)$  in terms of the eigenvectors. That is, we wish to write  $\vec{x}(0) = \begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} a_2$ . Hence

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = E^{-1} \begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/16 \end{bmatrix}.$$

So  $a_1 = 1/4$  and  $a_2 = -1/16$ . We plug our  $\vec{x}$  into the equation and get that

$$\begin{aligned} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi_1' + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_2' &= A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xi_1 + A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2 \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-2\xi_1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 4\xi_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} (e^t - t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (e^t + t). \end{aligned}$$

We get the two equations

$$\begin{aligned} \xi_1' &= -2\xi_1 + e^t - t, & \text{where } \xi_1(0) &= a_1 = \frac{1}{4}, \\ \xi_2' &= 4\xi_2 + e^t + t, & \text{where } \xi_2(0) &= a_2 = -\frac{1}{16}. \end{aligned}$$

We solve with integrating factor. Computation of the integral is left as an exercise to the student. Note that we will need integration by parts.

$$\xi_1 = e^{-2t} \int e^{2t} (e^t - t) dt + C_1 e^{-2t} = \frac{e^t}{3} - \frac{t}{2} + \frac{1}{4} + C_1 e^{-2t}.$$

$C_1$  is the constant of integration. As  $\xi_1(0) = 1/4$ , then  $1/4 = 1/3 + 1/4 + C_1$  and hence  $C_1 = -1/3$ . Similarly

$$\xi_2 = e^{4t} \int e^{-4t} (e^t + t) dt + C_2 e^{4t} = -\frac{e^t}{3} - \frac{t}{4} - \frac{1}{16} + C_2 e^{4t}.$$

As  $\xi_2(0) = 1/16$  we have that  $-1/16 = -1/3 - 1/16 + C_2$  and hence  $C_2 = 1/3$ . The solution is

$$\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left( \frac{e^t - e^{-2t}}{3} + \frac{1 - 2t}{4} \right) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left( \frac{e^{4t} - e^t}{3} - \frac{4t + 1}{16} \right) = \begin{bmatrix} \frac{e^{4t} - e^{-2t}}{3} + \frac{3 - 12t}{16} \\ \frac{e^{-2t} + e^{4t} + 2e^t}{3} + \frac{4t - 5}{16} \end{bmatrix}.$$

That is,  $x_1 = \frac{e^{4t} - e^{-2t}}{3} + \frac{3 - 12t}{16}$  and  $x_2 = \frac{e^{-2t} + e^{4t} + 2e^t}{3} + \frac{4t - 5}{16}$ .

**Exercise 3.9.1:** Check that  $x_1$  and  $x_2$  solve the problem. Check both that they satisfy the differential equation and that they satisfy the initial conditions.

### Undetermined coefficients

We also have the method of undetermined coefficients for systems. The only difference here is that we will have to take unknown vectors rather than just numbers. Same caveats apply to undetermined coefficients for systems as for single equations. This method does not always work. Furthermore if the right hand side is complicated, we will have to solve for lots of variables. Each element of an unknown vector is an unknown number. So in system of 3 equations if we have say 4 unknown vectors (this would not be uncommon), then we already have 12 unknown numbers that we need to solve for. The method can turn into a lot of tedious work. As this method is essentially the same as it is for single equations, let us just do an example.

**Example 3.9.3:** Let  $A = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ . Find a particular solution of  $\vec{x}' = A\vec{x} + \vec{f}$  where  $\vec{f}(t) = \begin{bmatrix} e^t \\ t \end{bmatrix}$ .

Note that we can solve this system in an easier way (can you see how?), but for the purposes of the example, let us use the eigenvalue method plus undetermined coefficients.

The eigenvalues of  $A$  are  $-1$  and  $1$  and corresponding eigenvectors are  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  respectively. Hence our complementary solution is

$$\vec{x}_c = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t,$$

for some arbitrary constants  $\alpha_1$  and  $\alpha_2$ .

We would want to guess a particular solution of

$$\vec{x} = \vec{a}e^t + \vec{b}t + \vec{c}.$$

However, something of the form  $\vec{a}e^t$  appears in the complementary solution. Because we do not yet know if the vector  $\vec{a}$  is a multiple of  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  we do not know if a conflict arises. It is possible that

there is no conflict, but to be safe we should also try  $\vec{b}te^t$ . Here we find the crux of the difference for systems. We try *both* terms  $\vec{a}e^t$  and  $\vec{b}te^t$  in the solution, not just the term  $\vec{b}te^t$ . Therefore, we try

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d}.$$

Thus we have 8 unknowns. We write  $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ,  $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ,  $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ , and  $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ . We plug  $\vec{x}$  into the equation. First let us compute  $\vec{x}'$ .

$$\vec{x}' = (\vec{a} + \vec{b})e^t + \vec{b}te^t + \vec{c} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}te^t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Now  $\vec{x}'$  must equal  $A\vec{x} + \vec{f}$ , which is

$$\begin{aligned} A\vec{x} + \vec{f} &= A\vec{a}e^t + A\vec{b}te^t + A\vec{c}t + A\vec{d} + \vec{f} = \\ &= \begin{bmatrix} -a_1 \\ -2a_1 + a_2 \end{bmatrix}e^t + \begin{bmatrix} -b_1 \\ -2b_1 + b_2 \end{bmatrix}te^t + \begin{bmatrix} -c_1 \\ -2c_1 + c_2 \end{bmatrix}t + \begin{bmatrix} -d_1 \\ -2d_1 + d_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}t. \end{aligned}$$

We identify the coefficients of  $e^t$ ,  $te^t$ ,  $t$  and any constant vectors.

$$\begin{aligned} a_1 + b_1 &= -a_1 + 1, \\ a_2 + b_2 &= -2a_1 + a_2, \\ b_1 &= -b_1, \\ b_2 &= -2b_1 + b_2, \\ 0 &= -c_1, \\ 0 &= -2c_1 + c_2 + 1, \\ c_1 &= -d_1, \\ c_2 &= -2d_1 + d_2. \end{aligned}$$

We could write the  $8 \times 9$  augmented matrix and start row reduction, but it is easier to just solve the equations in an ad hoc manner. Immediately we see that  $b_1 = 0$ ,  $c_1 = 0$ ,  $d_1 = 0$ . Plugging these back in, we get that  $c_2 = -1$  and  $d_2 = -1$ . The remaining equations that tell us something are

$$\begin{aligned} a_1 &= -a_1 + 1, \\ a_2 + b_2 &= -2a_1 + a_2. \end{aligned}$$

So  $a_1 = 1/2$  and  $b_2 = -1$ . Finally,  $a_2$  can be arbitrary and still satisfy the equations. We are looking for just a single solution so presumably the simplest one is when  $a_2 = 0$ . Therefore,

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}e^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t \\ -te^t - t - 1 \end{bmatrix}.$$

That is,  $x_1 = \frac{1}{2}e^t$ ,  $x_2 = -te^t - t - 1$ . We would add this to the complementary solution to get the general solution of the problem. Notice also that both  $\vec{a}e^t$  and  $\vec{b}te^t$  were really needed.

**Exercise 3.9.2:** Check that  $x_1$  and  $x_2$  solve the problem. Also try setting  $a_2 = 1$  and again check these solutions. What is the difference between the two solutions we can obtain in this way?

As you can see, other than the handling of conflicts, undetermined coefficients works exactly the same as it did for single equations. However, the computations can get out of hand pretty quickly for systems. The equation we had done was very simple.

### 3.9.2 First order variable coefficient

Just as for a single equation, there is the method of variation of parameters. In fact for constant coefficient systems, this is essentially the same thing as the integrating factor method we discussed earlier. However, this method will work for any linear system, even if it is not constant coefficient, provided we can somehow solve the associated homogeneous problem.

Suppose we have the equation

$$\vec{x}' = A(t) \vec{x} + \vec{f}(t). \quad (3.9)$$

Further, suppose that we have solved the associated homogeneous equation  $\vec{x}' = A(t) \vec{x}$  and found the fundamental matrix solution  $X(t)$ . The general solution to the associated homogeneous equation is  $X(t)\vec{c}$  for a constant vector  $\vec{c}$ . Just like for variation of parameters for single equation we try the solution to the nonhomogeneous equation of the form

$$\vec{x}_p = X(t) \vec{u}(t),$$

where  $\vec{u}(t)$  is a vector valued function instead of a constant. Now we substitute into (3.9) to obtain

$$\vec{x}_p'(t) = X'(t) \vec{u}(t) + X(t) \vec{u}'(t) = A(t) X(t) \vec{u}(t) + \vec{f}(t).$$

But  $X(t)$  is the fundamental matrix solution to the homogeneous problem, so  $X'(t) = A(t)X(t)$ , and

$$X'(t) \vec{u}(t) + X(t) \vec{u}'(t) = X'(t) \vec{u}(t) + \vec{f}(t).$$

Hence  $X(t) \vec{u}'(t) = \vec{f}(t)$ . If we compute  $[X(t)]^{-1}$ , then  $\vec{u}'(t) = [X(t)]^{-1} \vec{f}(t)$ . We integrate to obtain  $\vec{u}$  and we have the particular solution  $\vec{x}_p = X(t) \vec{u}(t)$ . Let us write this as a formula

$$\boxed{\vec{x}_p = X(t) \int [X(t)]^{-1} \vec{f}(t) dt.}$$

Note that if  $A$  is constant and we let  $X(t) = e^{tA}$ , then  $[X(t)]^{-1} = e^{-tA}$  and hence we get a solution  $\vec{x}_p = e^{tA} \int e^{-tA} \vec{f}(t) dt$ , which is precisely what we got using the integrating factor method.

**Example 3.9.4:** Find a particular solution to

$$\vec{x}' = \frac{1}{t^2 + 1} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ 1 \end{bmatrix} (t^2 + 1). \quad (3.10)$$

Here  $A = \frac{1}{t^2+1} \begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}$  is most definitely not constant. Perhaps by a lucky guess, we find that  $X = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$  solves  $X'(t) = A(t)X(t)$ . Once we know the complementary solution we can easily find a solution to (3.10). First we find

$$[X(t)]^{-1} = \frac{1}{t^2+1} \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}.$$

Next we know a particular solution to (3.10) is

$$\begin{aligned} \vec{x}_p &= X(t) \int [X(t)]^{-1} \vec{f}(t) dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \int \frac{1}{t^2+1} \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} (t^2+1) dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \int \begin{bmatrix} 2t \\ -t^2+1 \end{bmatrix} dt \\ &= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \begin{bmatrix} t^2 \\ -\frac{1}{3}t^3+t \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3+t \end{bmatrix}. \end{aligned}$$

Adding the complementary solution we have that the general solution to (3.10).

$$\vec{x} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3+t \end{bmatrix} = \begin{bmatrix} c_1 - c_2t + \frac{1}{3}t^4 \\ c_2 + (c_1+1)t + \frac{2}{3}t^3 \end{bmatrix}.$$

**Exercise 3.9.3:** Check that  $x_1 = \frac{1}{3}t^4$  and  $x_2 = \frac{2}{3}t^3 + t$  really solve (3.10).

In the variation of parameters, just like in the integrating factor method we can obtain the general solution by adding in constants of integration. That is, we will add  $X(t)\vec{c}$  for a vector of arbitrary constants. But that is precisely the complementary solution.

### 3.9.3 Second order constant coefficients

#### Undetermined coefficients

We have already previously did a simple example of the method of undetermined coefficients for second order systems in § 3.6. This method is essentially the same as undetermined coefficients for first order systems. There are some simplifications that we can make, as we did in § 3.6. Let the equation be

$$\vec{x}'' = A\vec{x} + \vec{F}(t),$$

where  $A$  is a constant matrix. If  $\vec{F}(t)$  is of the form  $\vec{F}_0 \cos(\omega t)$ , then we can try a solution of the form

$$\vec{x}_p = \vec{c} \cos(\omega t),$$



and we do not need to introduce sines.

If the  $\vec{F}$  is a sum of cosines, note that we still have the superposition principle. If  $\vec{F}(t) = \vec{F}_0 \cos(\omega_0 t) + \vec{F}_1 \cos(\omega_1 t)$ , then we would try  $\vec{d} \cos(\omega_0 t)$  for the problem  $\vec{x}'' = A\vec{x} + \vec{F}_0 \cos(\omega_0 t)$ , and we would try  $\vec{b} \cos(\omega_1 t)$  for the problem  $\vec{x}'' = A\vec{x} + \vec{F}_1 \cos(\omega_1 t)$ . Then we sum the solutions.

However, if there is duplication with the complementary solution, or the equation is of the form  $\vec{x}'' = A\vec{x}' + B\vec{x} + \vec{F}(t)$ , then we need to do the same thing as we do for first order systems.

You will never go wrong with putting in more terms than needed into your guess. You will find that the extra coefficients will turn out to be zero. But it is useful to save some time and effort.

### Eigenvector decomposition

If we have the system

$$\vec{x}'' = A\vec{x} + \vec{F}(t),$$

we can do *eigenvector decomposition*, just like for first order systems.

Let  $\lambda_1, \dots, \lambda_n$  be the eigenvalues and  $\vec{v}_1, \dots, \vec{v}_n$  be eigenvectors. Again form the matrix  $E = [\vec{v}_1 \cdots \vec{v}_n]$ . We write

$$\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \cdots + \vec{v}_n \xi_n(t).$$

We decompose  $\vec{F}$  in terms of the eigenvectors

$$\vec{F}(t) = \vec{v}_1 g_1(t) + \vec{v}_2 g_2(t) + \cdots + \vec{v}_n g_n(t).$$

And again  $\vec{g} = E^{-1}\vec{F}$ .

Now we plug in and doing the same thing as before we obtain

$$\begin{aligned} \vec{x}'' &= \vec{v}_1 \xi_1'' + \vec{v}_2 \xi_2'' + \cdots + \vec{v}_n \xi_n'' \\ &= A(\vec{v}_1 \xi_1 + \vec{v}_2 \xi_2 + \cdots + \vec{v}_n \xi_n) + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= A\vec{v}_1 \xi_1 + A\vec{v}_2 \xi_2 + \cdots + A\vec{v}_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= \vec{v}_1 \lambda_1 \xi_1 + \vec{v}_2 \lambda_2 \xi_2 + \cdots + \vec{v}_n \lambda_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\ &= \vec{v}_1 (\lambda_1 \xi_1 + g_1) + \vec{v}_2 (\lambda_2 \xi_2 + g_2) + \cdots + \vec{v}_n (\lambda_n \xi_n + g_n). \end{aligned}$$

We identify the coefficients of the eigenvectors to get the equations

$$\begin{aligned} \xi_1'' &= \lambda_1 \xi_1 + g_1, \\ \xi_2'' &= \lambda_2 \xi_2 + g_2, \\ &\vdots \\ \xi_n'' &= \lambda_n \xi_n + g_n. \end{aligned}$$

Each one of these equations is independent of the others. We solve each one of these using the methods of chapter 2. We write  $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \cdots + \vec{v}_n \xi_n(t)$ , and we are done; we have a particular solution. If we have found the general solution for  $\xi_1$  through  $\xi_n$ , then again  $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \cdots + \vec{v}_n \xi_n(t)$  is the general solution (and not just a particular solution).

**Example 3.9.5:** Let us do the example from § 3.6 using this method. The equation is

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t).$$

The eigenvalues were  $-1$  and  $-4$ , with eigenvectors  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ . Therefore  $E = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$  and  $E^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ . Therefore,

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1} \vec{F}(t) = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \cos(3t) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \cos(3t) \\ -\frac{2}{3} \cos(3t) \end{bmatrix}.$$

So after the whole song and dance of plugging in, the equations we get are

$$\begin{aligned} \xi_1'' &= -\xi_1 + \frac{2}{3} \cos(3t), \\ \xi_2'' &= -4\xi_2 - \frac{2}{3} \cos(3t). \end{aligned}$$

For each we can try the method of undetermined coefficients and try  $C_1 \cos(3t)$  for the first equation and  $C_2 \cos(3t)$  for the second equation. We plug in to get

$$\begin{aligned} -9C_1 \cos(3t) &= -C_1 \cos(3t) + \frac{2}{3} \cos(3t), \\ -9C_2 \cos(3t) &= -4C_2 \cos(3t) - \frac{2}{3} \cos(3t). \end{aligned}$$

Each of these equations we solve separately. We get  $-9C_1 = -C_1 + 2/3$  and  $-9C_2 = -4C_2 - 2/3$ . And hence  $C_1 = 1/12$  and  $C_2 = 2/15$ . So our particular solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left( \frac{-1}{12} \cos(3t) \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left( \frac{2}{15} \cos(3t) \right) = \begin{bmatrix} 1/20 \\ -3/10 \end{bmatrix} \cos(3t).$$

This solution matches what we got previously in § 3.6.

### 3.9.4 Exercises

**Exercise 3.9.4:** Find a particular solution to  $x' = x + 2y + 2t$ ,  $y' = 3x + 2y - 4$ , a) using integrating factor method, b) using eigenvector decomposition, c) using undetermined coefficients.

**Exercise 3.9.5:** Find the general solution to  $x' = 4x + y - 1$ ,  $y' = x + 4y - e^t$ , a) using integrating factor method, b) using eigenvector decomposition, c) using undetermined coefficients.

**Exercise 3.9.6:** Find the general solution to  $x_1'' = -6x_1 + 3x_2 + \cos(t)$ ,  $x_2'' = 2x_1 - 7x_2 + 3 \cos(t)$ , a) using eigenvector decomposition, b) using undetermined coefficients.

**Exercise 3.9.7:** Find the general solution to  $x_1'' = -6x_1 + 3x_2 + \cos(2t)$ ,  $x_2'' = 2x_1 - 7x_2 + 3\cos(2t)$ , a) using eigenvector decomposition, b) using undetermined coefficients.

**Exercise 3.9.8:** Take the equation

$$\vec{x}' = \begin{bmatrix} \frac{1}{t} & -1 \\ 1 & \frac{1}{t} \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ -t \end{bmatrix}.$$

a) Check that

$$\vec{x}_c = c_1 \begin{bmatrix} t \sin t \\ -t \cos t \end{bmatrix} + c_2 \begin{bmatrix} t \cos t \\ t \sin t \end{bmatrix}$$

is the complementary solution. b) Use variation of parameters to find a particular solution.

**Exercise 3.9.101:** Find a particular solution to  $x' = 5x + 4y + t$ ,  $y' = x + 8y - t$ , a) using integrating factor method, b) using eigenvector decomposition, c) using undetermined coefficients.

**Exercise 3.9.102:** Find a particular solution to  $x' = y + e^t$ ,  $y' = x + e^t$ , a) using integrating factor method, b) using eigenvector decomposition, c) using undetermined coefficients.

**Exercise 3.9.103:** Solve  $x_1' = x_2 + t$ ,  $x_2' = x_1 + t$  with initial conditions  $x_1(0) = 1$ ,  $x_2(0) = 2$ , using eigenvector decomposition.

**Exercise 3.9.104:** Solve  $x_1'' = -3x_1 + x_2 + t$ ,  $x_2'' = 9x_1 + 5x_2 + \cos(t)$  with initial conditions  $x_1(0) = 0$ ,  $x_2(0) = 0$ ,  $x_1'(0) = 0$ ,  $x_2'(0) = 0$ , using eigenvector decomposition.



# Chapter 4

## Fourier series and PDEs

### 4.1 Boundary value problems

*Note: 2 lectures, similar to §3.8 in [EP], §10.1 and §11.1 in [BD]*

#### 4.1.1 Boundary value problems

Before we tackle the Fourier series, we need to study the so-called *boundary value problems* (or *endpoint problems*). For example, suppose we have

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0,$$

for some constant  $\lambda$ , where  $x(t)$  is defined for  $t$  in the interval  $[a, b]$ . Unlike before, when we specified the value of the solution and its derivative at a single point, we now specify the value of the solution at two different points. Note that  $x = 0$  is a solution to this equation, so existence of solutions is not an issue here. Uniqueness of solutions is another issue. The general solution to  $x'' + \lambda x = 0$  will have two arbitrary constants present. It is, therefore, natural (but wrong) to believe that requiring two conditions will guarantee a unique solution.

**Example 4.1.1:** Take  $\lambda = 1$ ,  $a = 0$ ,  $b = \pi$ . That is,

$$x'' + x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

Then  $x = \sin t$  is another solution (besides  $x = 0$ ) satisfying both boundary conditions. There are more. Write down the general solution of the differential equation, which is  $x = A \cos t + B \sin t$ . The condition  $x(0) = 0$  forces  $A = 0$ . Letting  $x(\pi) = 0$  does not give us any more information as  $x = B \sin t$  already satisfies both boundary conditions. Hence, there are infinitely many solutions of the form  $x = B \sin t$ , where  $B$  is an arbitrary constant.

**Example 4.1.2:** On the other hand, change to  $\lambda = 2$ .

$$x'' + 2x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

Then the general solution is  $x = A \cos(\sqrt{2} t) + B \sin(\sqrt{2} t)$ . Letting  $x(0) = 0$  still forces  $A = 0$ . We apply the second condition to find  $0 = x(\pi) = B \sin(\sqrt{2} \pi)$ . As  $\sin(\sqrt{2} \pi) \neq 0$  we obtain  $B = 0$ . Therefore  $x = 0$  is the unique solution to this problem.

What is going on? We will be interested in finding which constants  $\lambda$  allow a nonzero solution, and we will be interested in finding those solutions. This problem is an analogue of finding eigenvalues and eigenvectors of matrices.

### 4.1.2 Eigenvalue problems

For basic Fourier series theory we will need the following three eigenvalue problems. We will consider more general equations, but we will postpone this until chapter 5.

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0, \tag{4.1}$$

$$x'' + \lambda x = 0, \quad x'(a) = 0, \quad x'(b) = 0, \tag{4.2}$$

and

$$x'' + \lambda x = 0, \quad x(a) = x(b), \quad x'(a) = x'(b), \tag{4.3}$$

A number  $\lambda$  is called an *eigenvalue* of (4.1) (resp. (4.2) or (4.3)) if and only if there exists a nonzero (not identically zero) solution to (4.1) (resp. (4.2) or (4.3)) given that specific  $\lambda$ . The nonzero solution we found is called the corresponding *eigenfunction*.

Note the similarity to eigenvalues and eigenvectors of matrices. The similarity is not just coincidental. If we think of the equations as differential operators, then we are doing the same exact thing. For example, let  $L = -\frac{d^2}{dt^2}$ . We are looking for nonzero functions  $f$  satisfying certain endpoint conditions that solve  $(L - \lambda)f = 0$ . A lot of the formalism from linear algebra can still apply here, though we will not pursue this line of reasoning too far.

**Example 4.1.3:** Let us find the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x(0) = 0, \quad x(\pi) = 0.$$

For reasons that will be clear from the computations, we will have to handle the cases  $\lambda > 0$ ,  $\lambda = 0$ ,  $\lambda < 0$  separately. First suppose that  $\lambda > 0$ , then the general solution to  $x'' + \lambda x = 0$  is

$$x = A \cos(\sqrt{\lambda} t) + B \sin(\sqrt{\lambda} t).$$

The condition  $x(0) = 0$  implies immediately  $A = 0$ . Next

$$0 = x(\pi) = B \sin(\sqrt{\lambda} \pi).$$

If  $B$  is zero, then  $x$  is not a nonzero solution. So to get a nonzero solution we must have that  $\sin(\sqrt{\lambda}\pi) = 0$ . Hence,  $\sqrt{\lambda}\pi$  must be an integer multiple of  $\pi$ . In other words,  $\sqrt{\lambda} = k$  for a positive integer  $k$ . Hence the positive eigenvalues are  $k^2$  for all integers  $k \geq 1$ . The corresponding eigenfunctions can be taken as  $x = \sin(kt)$ . Just like for eigenvectors, we get all the multiples of an eigenfunction, so we only need to pick one.

Now suppose that  $\lambda = 0$ . In this case the equation is  $x'' = 0$  and the general solution is  $x = At + B$ . The condition  $x(0) = 0$  implies that  $B = 0$ , and  $x(\pi) = 0$  implies that  $A = 0$ . This means that  $\lambda = 0$  is *not* an eigenvalue.

Finally, suppose that  $\lambda < 0$ . In this case we have the general solution

$$x = A \cosh(\sqrt{-\lambda} t) + B \sinh(\sqrt{-\lambda} t).$$

Letting  $x(0) = 0$  implies that  $A = 0$  (recall  $\cosh 0 = 1$  and  $\sinh 0 = 0$ ). So our solution must be  $x = B \sinh(\sqrt{-\lambda} t)$  and satisfy  $x(\pi) = 0$ . This is only possible if  $B$  is zero. Why? Because  $\sinh \xi$  is only zero for  $\xi = 0$ , you should plot  $\sinh$  to see this. We can also see this from the definition of  $\sinh$ . We get  $0 = \sinh t = \frac{e^t - e^{-t}}{2}$ . Hence  $e^t = e^{-t}$ , which implies  $t = -t$  and that is only true if  $t = 0$ . So there are no negative eigenvalues.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k^2 \quad \text{with an eigenfunction} \quad x_k = \sin(kt) \quad \text{for all integers } k \geq 1.$$

**Example 4.1.4:** Let us compute the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x'(0) = 0, \quad x'(\pi) = 0.$$

Again we will have to handle the cases  $\lambda > 0$ ,  $\lambda = 0$ ,  $\lambda < 0$  separately. First suppose that  $\lambda > 0$ . The general solution to  $x'' + \lambda x = 0$  is  $x = A \cos(\sqrt{\lambda} t) + B \sin(\sqrt{\lambda} t)$ . So

$$x' = -A \sqrt{\lambda} \sin(\sqrt{\lambda} t) + B \sqrt{\lambda} \cos(\sqrt{\lambda} t).$$

The condition  $x'(0) = 0$  implies immediately  $B = 0$ . Next

$$0 = x'(\pi) = -A \sqrt{\lambda} \sin(\sqrt{\lambda} \pi).$$

Again  $A$  cannot be zero if  $\lambda$  is to be an eigenvalue, and  $\sin(\sqrt{\lambda} \pi)$  is only zero if  $\sqrt{\lambda} = k$  for a positive integer  $k$ . Hence the positive eigenvalues are again  $k^2$  for all integers  $k \geq 1$ . And the corresponding eigenfunctions can be taken as  $x = \cos(kt)$ .

Now suppose that  $\lambda = 0$ . In this case the equation is  $x'' = 0$  and the general solution is  $x = At + B$  so  $x' = A$ . The condition  $x'(0) = 0$  implies that  $A = 0$ . Now  $x'(\pi) = 0$  also simply implies  $A = 0$ . This means that  $B$  could be anything (let us take it to be 1). So  $\lambda = 0$  is an eigenvalue and  $x = 1$  is a corresponding eigenfunction.

Finally, let  $\lambda < 0$ . In this case we have the general solution  $x = A \cosh(\sqrt{-\lambda} t) + B \sinh(\sqrt{-\lambda} t)$  and hence

$$x' = A \sqrt{-\lambda} \sinh(\sqrt{-\lambda} t) + B \sqrt{-\lambda} \cosh(\sqrt{-\lambda} t).$$

We have already seen (with roles of  $A$  and  $B$  switched) that for this to be zero at  $t = 0$  and  $t = \pi$  it implies that  $A = B = 0$ . Hence there are no negative eigenvalues.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\lambda_k = k^2 \quad \text{with an eigenfunction} \quad x_k = \cos(kt) \quad \text{for all integers } k \geq 1,$$

and there is another eigenvalue

$$\lambda_0 = 0 \quad \text{with an eigenfunction} \quad x_0 = 1.$$

The following problem is the one that leads to the general Fourier series.

**Example 4.1.5:** Let us compute the eigenvalues and eigenfunctions of

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi).$$

Notice that we have not specified the values or the derivatives at the endpoints, but rather that they are the same at the beginning and at the end of the interval.

Let us skip  $\lambda < 0$ . The computations are the same as before, and again we find that there are no negative eigenvalues.

For  $\lambda = 0$ , the general solution is  $x = At + B$ . The condition  $x(-\pi) = x(\pi)$  implies that  $A = 0$  ( $A\pi + B = -A\pi + B$  implies  $A = 0$ ). The second condition  $x'(-\pi) = x'(\pi)$  says nothing about  $B$  and hence  $\lambda = 0$  is an eigenvalue with a corresponding eigenfunction  $x = 1$ .

For  $\lambda > 0$  we get that  $x = A \cos(\sqrt{\lambda} t) + B \sin(\sqrt{\lambda} t)$ . Now

$$A \cos(-\sqrt{\lambda} \pi) + B \sin(-\sqrt{\lambda} \pi) = A \cos(\sqrt{\lambda} \pi) + B \sin(\sqrt{\lambda} \pi).$$

We remember that  $\cos(-\theta) = \cos(\theta)$  and  $\sin(-\theta) = -\sin(\theta)$ . Therefore,

$$A \cos(\sqrt{\lambda} \pi) - B \sin(\sqrt{\lambda} \pi) = A \cos(\sqrt{\lambda} \pi) + B \sin(\sqrt{\lambda} \pi).$$

Hence either  $B = 0$  or  $\sin(\sqrt{\lambda} \pi) = 0$ . Similarly (exercise) if we differentiate  $x$  and plug in the second condition we find that  $A = 0$  or  $\sin(\sqrt{\lambda} \pi) = 0$ . Therefore, unless we want  $A$  and  $B$  to both be zero (which we do not) we must have  $\sin(\sqrt{\lambda} \pi) = 0$ . Hence,  $\sqrt{\lambda}$  is an integer and the eigenvalues are yet again  $\lambda = k^2$  for an integer  $k \geq 1$ . In this case, however,  $x = A \cos(kt) + B \sin(kt)$  is an eigenfunction for any  $A$  and any  $B$ . So we have two linearly independent eigenfunctions  $\sin(kt)$  and  $\cos(kt)$ . Remember that for a matrix we could also have had two eigenvectors corresponding to a single eigenvalue if the eigenvalue was repeated.

In summary, the eigenvalues and corresponding eigenfunctions are

$$\begin{aligned} \lambda_k &= k^2 & \text{with the eigenfunctions} & \quad \cos(kt) \quad \text{and} \quad \sin(kt) & \quad \text{for all integers } k \geq 1, \\ \lambda_0 &= 0 & \text{with an eigenfunction} & \quad x_0 = 1. \end{aligned}$$



### 4.1.3 Orthogonality of eigenfunctions

Something that will be very useful in the next section is the *orthogonality* property of the eigenfunctions. This is an analogue of the following fact about eigenvectors of a matrix. A matrix is called *symmetric* if  $A = A^T$ . *Eigenvectors for two distinct eigenvalues of a symmetric matrix are orthogonal*. That symmetry is required. We will not prove this fact here. The differential operators we are dealing with act much like a symmetric matrix. We, therefore, get the following theorem.

**Theorem 4.1.1.** *Suppose that  $x_1(t)$  and  $x_2(t)$  are two eigenfunctions of the problem (4.1), (4.2) or (4.3) for two different eigenvalues  $\lambda_1$  and  $\lambda_2$ . Then they are orthogonal in the sense that*

$$\int_a^b x_1(t)x_2(t) dt = 0.$$

Note that the terminology comes from the fact that the integral is a type of inner product. We will expand on this in the next section. The theorem has a very short, elegant, and illuminating proof so let us give it here. First note that we have the following two equations.

$$x_1'' + \lambda_1 x_1 = 0 \quad \text{and} \quad x_2'' + \lambda_2 x_2 = 0.$$

Multiply the first by  $x_2$  and the second by  $x_1$  and subtract to get

$$(\lambda_1 - \lambda_2)x_1x_2 = x_2''x_1 - x_2x_1''.$$

Now integrate both sides of the equation.

$$\begin{aligned} (\lambda_1 - \lambda_2) \int_a^b x_1x_2 dt &= \int_a^b x_2''x_1 - x_2x_1'' dt \\ &= \int_a^b \frac{d}{dt} (x_2'x_1 - x_2x_1') dt \\ &= \left[ x_2'x_1 - x_2x_1' \right]_{t=a}^b = 0. \end{aligned}$$

The last equality holds because of the boundary conditions. For example, if we consider (4.1) we have  $x_1(a) = x_1(b) = x_2(a) = x_2(b) = 0$  and so  $x_2'x_1 - x_2x_1'$  is zero at both  $a$  and  $b$ . As  $\lambda_1 \neq \lambda_2$ , the theorem follows.

**Exercise 4.1.1** (easy): *Finish the theorem (check the last equality in the proof) for the cases (4.2) and (4.3).*

We have seen previously that  $\sin(nt)$  was an eigenfunction for the problem  $x'' + \lambda x = 0$ ,  $x(0) = 0$ ,  $x(\pi) = 0$ . Hence we have the integral

$$\int_0^\pi \sin(mt) \sin(nt) dt = 0, \quad \text{when } m \neq n.$$

Similarly

$$\int_0^\pi \cos(mt) \cos(nt) dt = 0, \quad \text{when } m \neq n.$$

And finally we also get

$$\int_{-\pi}^\pi \sin(mt) \sin(nt) dt = 0, \quad \text{when } m \neq n,$$

$$\int_{-\pi}^\pi \cos(mt) \cos(nt) dt = 0, \quad \text{when } m \neq n,$$

and

$$\int_{-\pi}^\pi \cos(mt) \sin(nt) dt = 0.$$

#### 4.1.4 Fredholm alternative

We now touch on a very useful theorem in the theory of differential equations. The theorem holds in a more general setting than we are going to state it, but for our purposes the following statement is sufficient. We will give a slightly more general version in chapter 5.

**Theorem 4.1.2** (Fredholm alternative\*). *Exactly one of the following statements holds. Either*

$$x'' + \lambda x = 0, \quad x(a) = 0, \quad x(b) = 0 \tag{4.4}$$

*has a nonzero solution, or*

$$x'' + \lambda x = f(t), \quad x(a) = 0, \quad x(b) = 0 \tag{4.5}$$

*has a unique solution for every function  $f$  continuous on  $[a, b]$ .*

The theorem is also true for the other types of boundary conditions we considered. The theorem means that if  $\lambda$  is not an eigenvalue, the nonhomogeneous equation (4.5) has a unique solution for every right hand side. On the other hand if  $\lambda$  is an eigenvalue, then (4.5) need not have a solution for every  $f$ , and furthermore, even if it happens to have a solution, the solution is not unique.

We also want to reinforce the idea here that linear differential operators have much in common with matrices. So it is no surprise that there is a finite dimensional version of Fredholm alternative for matrices as well. Let  $A$  be an  $n \times n$  matrix. The Fredholm alternative then states that either  $(A - \lambda I)\vec{x} = \vec{0}$  has a nontrivial solution, or  $(A - \lambda I)\vec{x} = \vec{b}$  has a solution for every  $\vec{b}$ .

A lot of intuition from linear algebra can be applied for linear differential operators, but one must be careful of course. For example, one difference we have already seen is that in general a differential operator will have infinitely many eigenvalues, while a matrix has only finitely many.

---

\*Named after the Swedish mathematician Erik Ivar Fredholm (1866 – 1927).

### 4.1.5 Application

Let us consider a physical application of an endpoint problem. Suppose we have a tightly stretched quickly spinning elastic string or rope of uniform linear density  $\rho$ . Let us put this problem into the  $xy$ -plane. The  $x$  axis represents the position on the string. The string rotates at angular velocity  $\omega$ , so we will assume that the whole  $xy$ -plane rotates at angular velocity  $\omega$ . We will assume that the string stays in this  $xy$ -plane and  $y$  will measure its deflection from the equilibrium position,  $y = 0$ , on the  $x$  axis. Hence, we will find a graph giving the shape of the string. We will idealize the string to have no volume to just be a mathematical curve. If we take a small segment and we look at the tension at the endpoints, we see that this force is tangential and we will assume that the magnitude is the same at both end points. Hence the magnitude is constant everywhere and we will call its magnitude  $T$ . If we assume that the deflection is small, then we can use Newton's second law to get an equation

$$Ty'' + \rho\omega^2 y = 0.$$

Let  $L$  be the length of the string and the string is fixed at the beginning and end points. Hence,  $y(0) = 0$  and  $y(L) = 0$ . See Figure 4.1.

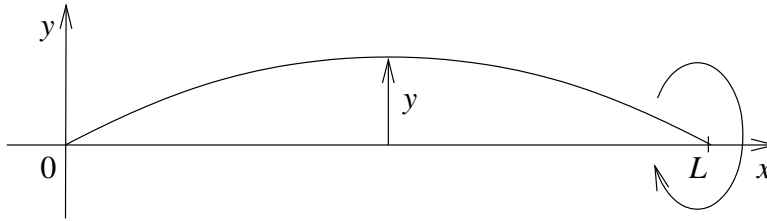


Figure 4.1: Whirling string.

We rewrite the equation as  $y'' + \frac{\rho\omega^2}{T}y = 0$ . The setup is similar to Example 4.1.3 on page 150, except for the interval length being  $L$  instead of  $\pi$ . We are looking for eigenvalues of  $y'' + \lambda y = 0$ ,  $y(0) = 0$ ,  $y(L) = 0$  where  $\lambda = \frac{\rho\omega^2}{T}$ . As before there are no nonpositive eigenvalues. With  $\lambda > 0$ , the general solution to the equation is  $y = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$ . The condition  $y(0) = 0$  implies that  $A = 0$  as before. The condition  $y(L) = 0$  implies that  $\sin(\sqrt{\lambda}L) = 0$  and hence  $\sqrt{\lambda}L = k\pi$  for some integer  $k > 0$ , so

$$\frac{\rho\omega^2}{T} = \lambda = \frac{k^2\pi^2}{L^2}.$$

What does this say about the shape of the string? It says that for all parameters  $\rho$ ,  $\omega$ ,  $T$  not satisfying the above equation, the string is in the equilibrium position,  $y = 0$ . When  $\frac{\rho\omega^2}{T} = \frac{k^2\pi^2}{L^2}$ , then the string will “pop out” some distance  $B$  at the midpoint. We cannot compute  $B$  with the information we have.

Let us assume that  $\rho$  and  $T$  are fixed and we are changing  $\omega$ . For most values of  $\omega$  the string is in the equilibrium state. When the angular velocity  $\omega$  hits a value  $\omega = \frac{k\pi\sqrt{T}}{L\sqrt{\rho}}$ , then the string will pop

out and will have the shape of a sin wave crossing the  $x$  axis  $k$  times. When  $\omega$  changes again, the string returns to the equilibrium position. You can see that the higher the angular velocity the more times it crosses the  $x$  axis when it is popped out.

### 4.1.6 Exercises

Hint for the following exercises: Note that when  $\lambda > 0$ , then  $\cos(\sqrt{\lambda}(t - a))$  and  $\sin(\sqrt{\lambda}(t - a))$  are also solutions of the homogeneous equation.

**Exercise 4.1.2:** Compute all eigenvalues and eigenfunctions of  $x'' + \lambda x = 0$ ,  $x(a) = 0$ ,  $x(b) = 0$  (assume  $a < b$ ).

**Exercise 4.1.3:** Compute all eigenvalues and eigenfunctions of  $x'' + \lambda x = 0$ ,  $x'(a) = 0$ ,  $x'(b) = 0$  (assume  $a < b$ ).

**Exercise 4.1.4:** Compute all eigenvalues and eigenfunctions of  $x'' + \lambda x = 0$ ,  $x'(a) = 0$ ,  $x(b) = 0$  (assume  $a < b$ ).

**Exercise 4.1.5:** Compute all eigenvalues and eigenfunctions of  $x'' + \lambda x = 0$ ,  $x(a) = x(b)$ ,  $x'(a) = x'(b)$  (assume  $a < b$ ).

**Exercise 4.1.6:** We have skipped the case of  $\lambda < 0$  for the boundary value problem  $x'' + \lambda x = 0$ ,  $x(-\pi) = x(\pi)$ ,  $x'(-\pi) = x'(\pi)$ . Finish the calculation and show that there are no negative eigenvalues.

**Exercise 4.1.101:** Consider a spinning string of length 2 and linear density 0.1 and tension 3. Find smallest angular velocity when the string pops out.

**Exercise 4.1.102:** Suppose  $x'' + \lambda x = 0$  and  $x(0) = 1$ ,  $x(1) = 1$ . Find all  $\lambda$  for which there is more than one solution. Also find the corresponding solutions (only for the eigenvalues).

**Exercise 4.1.103:** Suppose  $x'' + x = 0$  and  $x(0) = 0$ ,  $x'(\pi) = 1$ . Find all the solution(s) if any exist.

**Exercise 4.1.104:** Consider  $x' + \lambda x = 0$  and  $x(0) = 0$ ,  $x(1) = 0$ . Why does it not have any eigenvalues? Why does any first order equation with two endpoint conditions such as above have no eigenvalues?

**Exercise 4.1.105** (challenging): Suppose  $x''' + \lambda x = 0$  and  $x(0) = 0$ ,  $x'(0) = 0$ ,  $x(1) = 0$ . Suppose that  $\lambda > 0$ . Find an equation that all such eigenvalues must satisfy. Hint: Note that  $-\sqrt[3]{\lambda}$  is a root of  $r^3 + \lambda = 0$ .

## 4.2 The trigonometric series

*Note: 2 lectures, §9.1 in [EP], §10.2 in [BD]*

### 4.2.1 Periodic functions and motivation

As motivation for studying Fourier series, suppose we have the problem

$$x'' + \omega_0^2 x = f(t), \quad (4.6)$$

for some periodic function  $f(t)$ . We have already solved

$$x'' + \omega_0^2 x = F_0 \cos(\omega t). \quad (4.7)$$

One way to solve (4.6) is to decompose  $f(t)$  as a sum of cosines (and sines) and then solve many problems of the form (4.7). We then use the principle of superposition, to sum up all the solutions we got to get a solution to (4.6).

Before we proceed, let us talk a little bit more in detail about periodic functions. A function is said to be *periodic* with period  $P$  if  $f(t) = f(t + P)$  for all  $t$ . For brevity we will say  $f(t)$  is  $P$ -periodic. Note that a  $P$ -periodic function is also  $2P$ -periodic,  $3P$ -periodic and so on. For example,  $\cos(t)$  and  $\sin(t)$  are  $2\pi$ -periodic. So are  $\cos(kt)$  and  $\sin(kt)$  for all integers  $k$ . The constant functions are an extreme example. They are periodic for any period (exercise).

Normally we will start with a function  $f(t)$  defined on some interval  $[-L, L]$  and we will want to *extend periodically* to make it a  $2L$ -periodic function. We do this extension by defining a new function  $F(t)$  such that for  $t$  in  $[-L, L]$ ,  $F(t) = f(t)$ . For  $t$  in  $[L, 3L]$ , we define  $F(t) = f(t - 2L)$ , for  $t$  in  $[-3L, -L]$ ,  $F(t) = f(t + 2L)$ , and so on. We assumed that  $f(-L) = f(L)$ . We could have also started with  $f$  defined only on the half-open interval  $(-L, L]$  and then define  $f(-L) = f(L)$ .

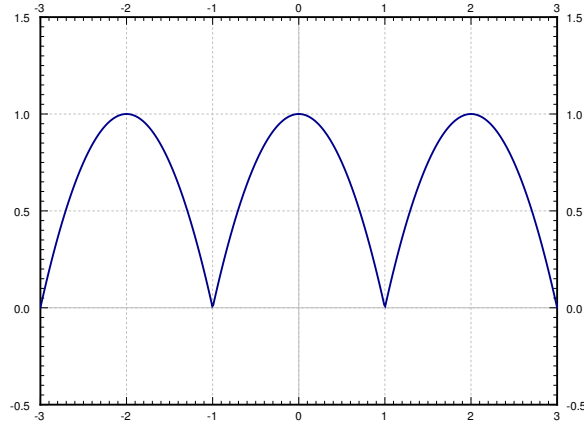
**Example 4.2.1:** Define  $f(t) = 1 - t^2$  on  $[-1, 1]$ . Now extend periodically to a 2-periodic function. See Figure 4.2 on the following page.

You should be careful to distinguish between  $f(t)$  and its extension. A common mistake is to assume that a formula for  $f(t)$  holds for its extension. It can be confusing when the formula for  $f(t)$  is periodic, but with perhaps a different period.

**Exercise 4.2.1:** Define  $f(t) = \cos t$  on  $[-\pi/2, \pi/2]$ . Now take the  $\pi$ -periodic extension and sketch its graph. How does it compare to the graph of  $\cos t$ .

### 4.2.2 Inner product and eigenvector decomposition

Suppose we have a *symmetric matrix*, that is  $A^T = A$ . We have said before that the eigenvectors of  $A$  are then orthogonal. Here the word *orthogonal* means that if  $\vec{v}$  and  $\vec{w}$  are two distinct (and not

Figure 4.2: Periodic extension of the function  $1 - t^2$ .

multiples of each other) eigenvectors of  $A$ , then  $\langle \vec{v}, \vec{w} \rangle = 0$ . In this case the inner product  $\langle \vec{v}, \vec{w} \rangle$  is the *dot product*, which can be computed as  $\vec{v}^T \vec{w}$ .

To decompose a vector  $\vec{v}$  in terms of mutually orthogonal vectors  $\vec{w}_1$  and  $\vec{w}_2$  we write

$$\vec{v} = a_1 \vec{w}_1 + a_2 \vec{w}_2.$$

Let us find the formula for  $a_1$  and  $a_2$ . First let us compute

$$\langle \vec{v}, \vec{w}_1 \rangle = \langle a_1 \vec{w}_1 + a_2 \vec{w}_2, \vec{w}_1 \rangle = a_1 \langle \vec{w}_1, \vec{w}_1 \rangle + a_2 \langle \vec{w}_2, \vec{w}_1 \rangle = a_1 \langle \vec{w}_1, \vec{w}_1 \rangle.$$

Therefore,

$$a_1 = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}.$$

Similarly

$$a_2 = \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle}.$$

You probably remember this formula from vector calculus.

**Example 4.2.2:** Write  $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  as a linear combination of  $\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

First note that  $\vec{w}_1$  and  $\vec{w}_2$  are orthogonal as  $\langle \vec{w}_1, \vec{w}_2 \rangle = 1(1) + (-1)1 = 0$ . Then

$$a_1 = \frac{\langle \vec{v}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} = \frac{2(1) + 3(-1)}{1(1) + (-1)(-1)} = \frac{-1}{2},$$

$$a_2 = \frac{\langle \vec{v}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} = \frac{2 + 3}{1 + 1} = \frac{5}{2}.$$

Hence

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

### 4.2.3 The trigonometric series

Instead of decomposing a vector in terms of eigenvectors of a matrix, we will decompose a function in terms of eigenfunctions of a certain eigenvalue problem. The eigenvalue problem we will use for the Fourier series is

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi).$$

We have previously computed that the eigenfunctions are  $1, \cos(kt), \sin(kt)$ . That is, we will want to find a representation of a  $2\pi$ -periodic function  $f(t)$  as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

This series is called the *Fourier series*<sup>†</sup> or the *trigonometric series* for  $f(t)$ . We write the coefficient of the eigenfunction 1 as  $\frac{a_0}{2}$  for convenience. We could also think of  $1 = \cos(0t)$ , so that we only need to look at  $\cos(kt)$  and  $\sin(kt)$ .

As for matrices we will want to find a *projection* of  $f(t)$  onto the subspace generated by the eigenfunctions. So we will want to define an *inner product of functions*. For example, to find  $a_n$  we want to compute  $\langle f(t), \cos(nt) \rangle$ . We define the inner product as

$$\langle f(t), g(t) \rangle \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} f(t) g(t) dt.$$

With this definition of the inner product, we have seen in the previous section that the eigenfunctions  $\cos(kt)$  (including the constant eigenfunction), and  $\sin(kt)$  are *orthogonal* in the sense that

$$\begin{aligned} \langle \cos(mt), \cos(nt) \rangle &= 0 && \text{for } m \neq n, \\ \langle \sin(mt), \sin(nt) \rangle &= 0 && \text{for } m \neq n, \\ \langle \sin(mt), \cos(nt) \rangle &= 0 && \text{for all } m \text{ and } n. \end{aligned}$$

By elementary calculus for  $n = 1, 2, 3, \dots$  we have  $\langle \cos(nt), \cos(nt) \rangle = \pi$  and  $\langle \sin(nt), \sin(nt) \rangle = \pi$ . For the constant we get that  $\langle 1, 1 \rangle = 2\pi$ . The coefficients are given by

$$\begin{aligned} a_n &= \frac{\langle f(t), \cos(nt) \rangle}{\langle \cos(nt), \cos(nt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \\ b_n &= \frac{\langle f(t), \sin(nt) \rangle}{\langle \sin(nt), \sin(nt) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt. \end{aligned}$$

Compare these expressions with the finite-dimensional example. For  $a_0$  we get a similar formula

$$a_0 = 2 \frac{\langle f(t), 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt.$$

<sup>†</sup>Named after the French mathematician Jean Baptiste Joseph Fourier (1768 – 1830).

Let us check the formulas using the orthogonality properties. Suppose for a moment that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Then for  $m \geq 1$  we have

$$\begin{aligned} \langle f(t), \cos(mt) \rangle &= \left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt), \cos(mt) \right\rangle \\ &= \frac{a_0}{2} \langle 1, \cos(mt) \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(nt), \cos(mt) \rangle + b_n \langle \sin(nt), \cos(mt) \rangle \\ &= a_m \langle \cos(mt), \cos(mt) \rangle. \end{aligned}$$

And hence  $a_m = \frac{\langle f(t), \cos(mt) \rangle}{\langle \cos(mt), \cos(mt) \rangle}$ .

**Exercise 4.2.2:** Carry out the calculation for  $a_0$  and  $b_m$ .

**Example 4.2.3:** Take the function

$$f(t) = t$$

for  $t$  in  $(-\pi, \pi]$ . Extend  $f(t)$  periodically and write it as a Fourier series. This function is called the *sawtooth*.

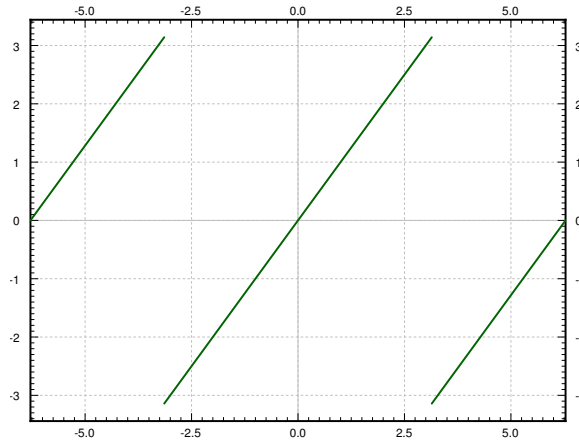


Figure 4.3: The graph of the sawtooth function.

The plot of the extended periodic function is given in Figure 4.3. Let us compute the coefficients. We start with  $a_0$ ,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} t \, dt = 0.$$



We will often use the result from calculus that says that the integral of an odd function over a symmetric interval is zero. Recall that an *odd function* is a function  $\varphi(t)$  such that  $\varphi(-t) = -\varphi(t)$ . For example the functions  $t$ ,  $\sin t$ , or (importantly for us)  $t \cos(nt)$  are all odd functions. Thus

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos(nt) dt = 0.$$

Let us move to  $b_n$ . Another useful fact from calculus is that the integral of an even function over a symmetric interval is twice the integral of the same function over half the interval. Recall an *even function* is a function  $\varphi(t)$  such that  $\varphi(-t) = \varphi(t)$ . For example  $t \sin(nt)$  is even.

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin(nt) dt \\ &= \frac{2}{\pi} \int_0^{\pi} t \sin(nt) dt \\ &= \frac{2}{\pi} \left( \left[ \frac{-t \cos(nt)}{n} \right]_{t=0}^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nt) dt \right) \\ &= \frac{2}{\pi} \left( \frac{-\pi \cos(n\pi)}{n} + 0 \right) \\ &= \frac{-2 \cos(n\pi)}{n} = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

We have used the fact that

$$\cos(n\pi) = (-1)^n = \begin{cases} 1 & \text{if } n \text{ even,} \\ -1 & \text{if } n \text{ odd.} \end{cases}$$

The series, therefore, is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nt).$$

Let us write out the first 3 harmonics of the series for  $f(t)$ .

$$2 \sin(t) - \sin(2t) + \frac{2}{3} \sin(3t) + \cdots$$

The plot of these first three terms of the series, along with a plot of the first 20 terms is given in Figure 4.4 on the following page.

**Example 4.2.4:** Take the function

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ \pi & \text{if } 0 < t \leq \pi. \end{cases}$$

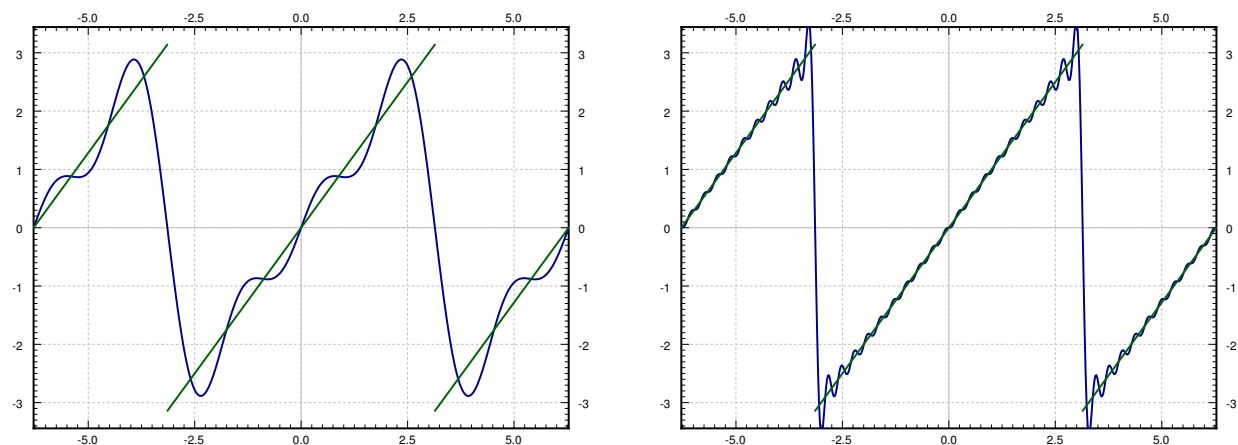


Figure 4.4: First 3 (left graph) and 20 (right graph) harmonics of the sawtooth function.

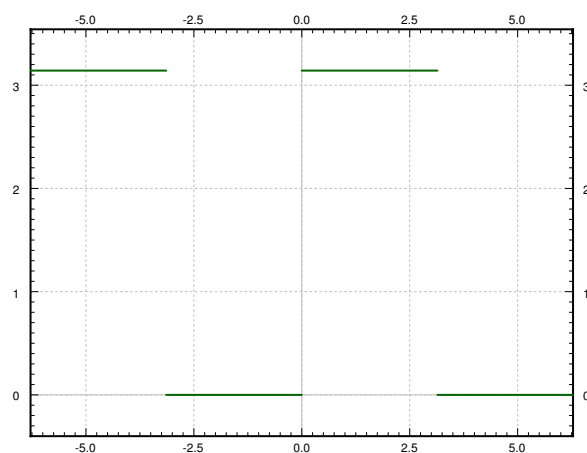


Figure 4.5: The graph of the square wave function.

Extend  $f(t)$  periodically and write it as a Fourier series. This function or its variants appear often in applications and the function is called the *square wave*.

The plot of the extended periodic function is given in Figure 4.5. Now we compute the coefficients. Let us start with  $a_0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} \int_0^{\pi} \pi dt = \pi.$$

Next,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \frac{1}{\pi} \int_0^{\pi} \pi \cos(nt) dt = 0.$$

And finally

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\
 &= \frac{1}{\pi} \int_0^{\pi} \pi \sin(nt) dt \\
 &= \left[ \frac{-\cos(nt)}{n} \right]_{t=0}^{\pi} \\
 &= \frac{1 - \cos(\pi n)}{n} = \frac{1 - (-1)^n}{n} = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}$$

The Fourier series is

$$\frac{\pi}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{n} \sin(nt) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin((2k-1)t).$$

Let us write out the first 3 harmonics of the series for  $f(t)$ .

$$\frac{\pi}{2} + 2 \sin(t) + \frac{2}{3} \sin(3t) + \cdots$$

The plot of these first three and also of the first 20 terms of the series is given in Figure 4.6.

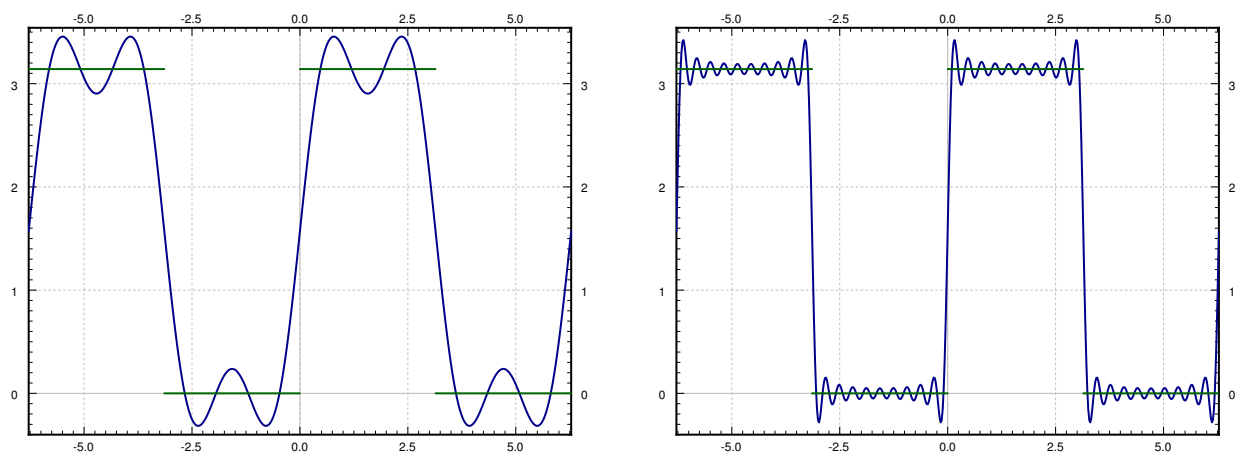


Figure 4.6: First 3 (left graph) and 20 (right graph) harmonics of the square wave function.

We have so far skirted the issue of convergence. For example, if  $f(t)$  is the square wave function, the equation

$$f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin((2k-1)t).$$

is only an equality for such  $t$  where  $f(t)$  is continuous. That is, we do not get an equality for  $t = -\pi, 0, \pi$  and all the other discontinuities of  $f(t)$ . It is not hard to see that when  $t$  is an integer multiple of  $\pi$  (which includes all the discontinuities), then

$$\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{2}{2k-1} \sin((2k-1)t) = \frac{\pi}{2}.$$

We redefine  $f(t)$  on  $[-\pi, \pi]$  as

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t < 0, \\ \pi & \text{if } 0 < t < \pi, \\ \pi/2 & \text{if } t = -\pi, t = 0, \text{ or } t = \pi, \end{cases}$$

and extend periodically. The series equals this extended  $f(t)$  everywhere, including the discontinuities. We will generally not worry about changing the function values at several (finitely many) points.

We will say more about convergence in the next section. Let us however mention briefly an effect of the discontinuity. Let us zoom in near the discontinuity in the square wave. Further, let us plot the first 100 harmonics, see Figure 4.7. You will notice that while the series is a very good approximation away from the discontinuities, the error (the overshoot) near the discontinuity at  $t = \pi$  does not seem to be getting any smaller. This behavior is known as the *Gibbs phenomenon*. The region where the error is large does get smaller, however, the more terms in the series we take.

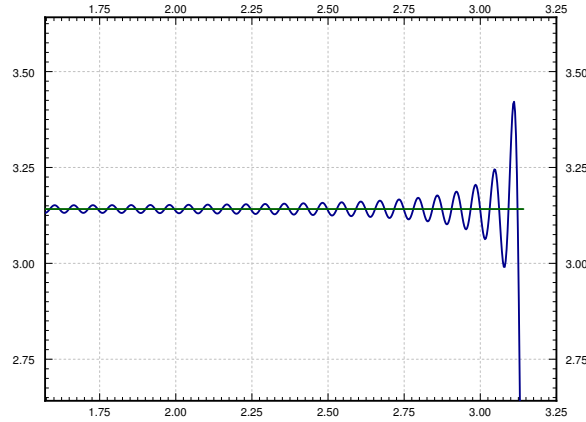


Figure 4.7: Gibbs phenomenon in action.

We can think of a periodic function as a “signal” being a superposition of many signals of pure frequency. For example, we could think of the square wave as a tone of certain base frequency. It will be, in fact, a superposition of many different pure tones of frequencies that are multiples of the

base frequency. On the other hand a simple sine wave is only the pure tone. The simplest way to make sound using a computer is the square wave, and the sound will be a very different from nice pure tones. If you have played video games from the 1980s or so, then you have heard what square waves sound like.

#### 4.2.4 Exercises

**Exercise 4.2.3:** Suppose  $f(t)$  is defined on  $[-\pi, \pi]$  as  $\sin(5t) + \cos(3t)$ . Extend periodically and compute the Fourier series of  $f(t)$ .

**Exercise 4.2.4:** Suppose  $f(t)$  is defined on  $[-\pi, \pi]$  as  $|t|$ . Extend periodically and compute the Fourier series of  $f(t)$ .

**Exercise 4.2.5:** Suppose  $f(t)$  is defined on  $[-\pi, \pi]$  as  $|t|^3$ . Extend periodically and compute the Fourier series of  $f(t)$ .

**Exercise 4.2.6:** Suppose  $f(t)$  is defined on  $(-\pi, \pi]$  as

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t \leq 0, \\ 1 & \text{if } 0 < t \leq \pi. \end{cases}$$

Extend periodically and compute the Fourier series of  $f(t)$ .

**Exercise 4.2.7:** Suppose  $f(t)$  is defined on  $(-\pi, \pi]$  as  $t^3$ . Extend periodically and compute the Fourier series of  $f(t)$ .

**Exercise 4.2.8:** Suppose  $f(t)$  is defined on  $[-\pi, \pi]$  as  $t^2$ . Extend periodically and compute the Fourier series of  $f(t)$ .

There is another form of the Fourier series using complex exponentials that is sometimes easier to work with.

**Exercise 4.2.9:** Let

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Use Euler's formula  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$  to show that there exist complex numbers  $c_m$  such that

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{imt}.$$

Note that the sum now ranges over all the integers including negative ones. Do not worry about convergence in this calculation. Hint: It may be better to start from the complex exponential form and write the series as

$$c_0 + \sum_{m=1}^{\infty} c_m e^{imt} + c_{-m} e^{-imt}.$$

**Exercise 4.2.101:** Suppose  $f(t)$  is defined on  $[-\pi, \pi]$  as  $f(t) = \sin(t)$ . Extend periodically and compute the Fourier series.

**Exercise 4.2.102:** Suppose  $f(t)$  is defined on  $(-\pi, \pi]$  as  $f(t) = \sin(\pi t)$ . Extend periodically and compute the Fourier series.

**Exercise 4.2.103:** Suppose  $f(t)$  is defined on  $(-\pi, \pi]$  as  $f(t) = \sin^2(t)$ . Extend periodically and compute the Fourier series.

**Exercise 4.2.104:** Suppose  $f(t)$  is defined on  $(-\pi, \pi]$  as  $f(t) = t^4$ . Extend periodically and compute the Fourier series.

## 4.3 More on the Fourier series

*Note: 2 lectures, §9.2 – §9.3 in [EP], §10.3 in [BD]*

Before reading the lecture, it may be good to first try Project IV (Fourier series) from the IODE website: <http://www.math.uiuc.edu/iode/>. After reading the lecture it may be good to continue with Project V (Fourier series again).

### 4.3.1 $2L$ -periodic functions

We have computed the Fourier series for a  $2\pi$ -periodic function, but what about functions of different periods. Well, fear not, the computation is a simple case of change of variables. We can just rescale the independent axis. Suppose that you have the  $2L$ -periodic function  $f(t)$  ( $L$  is called the *half period*). Let  $s = \frac{\pi}{L} t$ . Then the function

$$g(s) = f\left(\frac{L}{\pi} s\right)$$

is  $2\pi$ -periodic. We want to also rescale all our sines and cosines. We want to write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).$$

If we change variables to  $s$  we see that

$$g(s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(ns) + b_n \sin(ns).$$

We can compute  $a_n$  and  $b_n$  as before. After we write down the integrals we change variables from  $s$  back to  $t$ .

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(s) ds = \frac{1}{L} \int_{-L}^L f(t) dt, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(s) \cos(ns) ds = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi}{L} t\right) dt, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(s) \sin(ns) ds = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt. \end{aligned}$$

The two most common half periods that show up in examples are  $\pi$  and 1 because of the simplicity. We should stress that we have done no new mathematics, we have only changed variables. If you understand the Fourier series for  $2\pi$ -periodic functions, you understand it for  $2L$ -periodic functions. All that we are doing is moving some constants around, but all the mathematics is the same.

**Example 4.3.1:** Let

$$f(t) = |t| \quad \text{for } -1 < t \leq 1,$$

extended periodically. The plot of the periodic extension is given in Figure 4.8. Compute the Fourier series of  $f(t)$ .

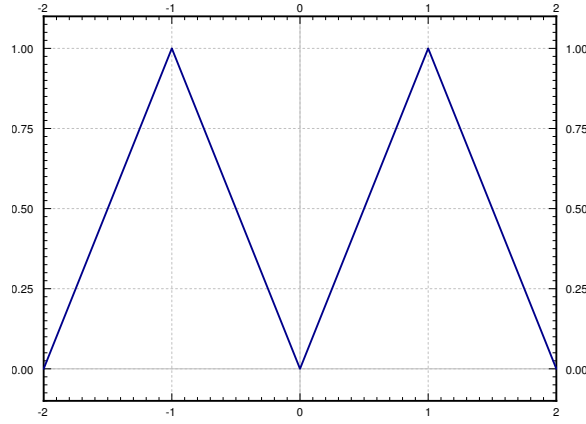


Figure 4.8: Periodic extension of the function  $f(t)$ .

We want to write  $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t)$ . For  $n \geq 1$  we note that  $|t| \cos(n\pi t)$  is even and hence

$$\begin{aligned} a_n &= \int_{-1}^1 f(t) \cos(n\pi t) dt \\ &= 2 \int_0^1 t \cos(n\pi t) dt \\ &= 2 \left[ \frac{t}{n\pi} \sin(n\pi t) \right]_{t=0}^1 - 2 \int_0^1 \frac{1}{n\pi} \sin(n\pi t) dt \\ &= 0 + \frac{1}{n^2 \pi^2} [\cos(n\pi t)]_{t=0}^1 = \frac{2((-1)^n - 1)}{n^2 \pi^2} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{-4}{n^2 \pi^2} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Next we find  $a_0$

$$a_0 = \int_{-1}^1 |t| dt = 1.$$

You should be able to find this integral by thinking about the integral as the area under the graph without doing any computation at all. Finally we can find  $b_n$ . Here, we notice that  $|t| \sin(n\pi t)$  is odd and, therefore,

$$b_n = \int_{-1}^1 f(t) \sin(n\pi t) dt = 0.$$



Hence, the series is

$$\frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2 \pi^2} \cos(n\pi t).$$

Let us explicitly write down the first few terms of the series up to the 3<sup>rd</sup> harmonic.

$$\frac{1}{2} - \frac{4}{\pi^2} \cos(\pi t) - \frac{4}{9\pi^2} \cos(3\pi t) - \dots$$

The plot of these few terms and also a plot up to the 20<sup>th</sup> harmonic is given in Figure 4.9. You should notice how close the graph is to the real function. You should also notice that there is no “Gibbs phenomenon” present as there are no discontinuities.

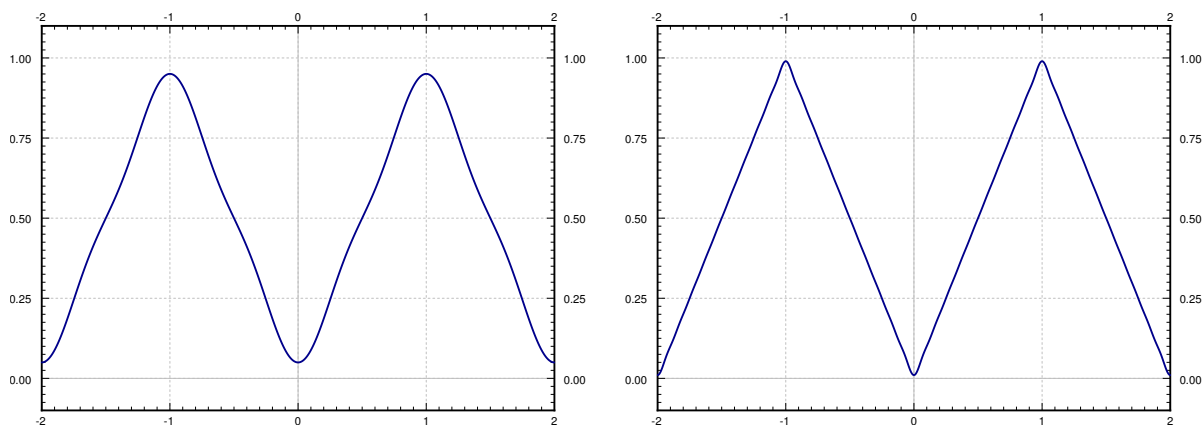


Figure 4.9: Fourier series of  $f(t)$  up to the 3<sup>rd</sup> harmonic (left graph) and up to the 20<sup>th</sup> harmonic (right graph).

### 4.3.2 Convergence

We will need the one sided limits of functions. We will use the following notation

$$f(c-) = \lim_{t \uparrow c} f(t), \quad \text{and} \quad f(c+) = \lim_{t \downarrow c} f(t).$$

If you are unfamiliar with this notation,  $\lim_{t \uparrow c} f(t)$  means we are taking a limit of  $f(t)$  as  $t$  approaches  $c$  from below (i.e.  $t < c$ ) and  $\lim_{t \downarrow c} f(t)$  means we are taking a limit of  $f(t)$  as  $t$  approaches  $c$  from above (i.e.  $t > c$ ). For example, for the square wave function

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \leq 0, \\ \pi & \text{if } 0 < t \leq \pi, \end{cases} \quad (4.8)$$

we have  $f(0-) = 0$  and  $f(0+) = \pi$ .

Let  $f(t)$  be a function defined on an interval  $[a, b]$ . Suppose that we find finitely many points  $a = t_0, t_1, t_2, \dots, t_k = b$  in the interval, such that  $f(t)$  is continuous on the intervals  $(t_0, t_1), (t_1, t_2), \dots, (t_{k-1}, t_k)$ . Also suppose that all the one sided limits exist, that is, all of  $f(t_0+), f(t_1-), f(t_1+), f(t_2-), f(t_2+), \dots, f(t_k-)$  exist and are finite. Then we say  $f(t)$  is *piecewise continuous*.

If moreover,  $f(t)$  is differentiable at all but finitely many points, and  $f'(t)$  is piecewise continuous, then  $f(t)$  is said to be *piecewise smooth*.

**Example 4.3.2:** The square wave function (4.8) is piecewise smooth on  $[-\pi, \pi]$  or any other interval. In such a case we simply say that the function is piecewise smooth.

**Example 4.3.3:** The function  $f(t) = |t|$  is piecewise smooth.

**Example 4.3.4:** The function  $f(t) = \frac{1}{t}$  is not piecewise smooth on  $[-1, 1]$  (or any other interval containing zero). In fact, it is not even piecewise continuous.

**Example 4.3.5:** The function  $f(t) = \sqrt[3]{t}$  is not piecewise smooth on  $[-1, 1]$  (or any other interval containing zero).  $f(t)$  is continuous, but the derivative of  $f(t)$  is unbounded near zero and hence not piecewise continuous.

Piecewise smooth functions have an easy answer on the convergence of the Fourier series.

**Theorem 4.3.1.** Suppose  $f(t)$  is a  $2L$ -periodic piecewise smooth function. Let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

be the Fourier series for  $f(t)$ . Then the series converges for all  $t$ . If  $f(t)$  is continuous near  $t$ , then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).$$

Otherwise

$$\frac{f(t-) + f(t+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).$$

If we happen to have that  $f(t) = \frac{f(t-)+f(t+)}{2}$  at all the discontinuities, the Fourier series converges to  $f(t)$  everywhere. We can always just redefine  $f(t)$  by changing the value at each discontinuity appropriately. Then we can write an equals sign between  $f(t)$  and the series without any worry. We mentioned this fact briefly at the end last section.

Note that the theorem does not say how fast the series converges. Think back the discussion of the Gibbs phenomenon in last section. The closer you get to the discontinuity, the more terms you need to take to get an accurate approximation to the function.

### 4.3.3 Differentiation and integration of Fourier series

Not only does Fourier series converge nicely, but it is easy to differentiate and integrate the series. We can do this just by differentiating or integrating term by term.

**Theorem 4.3.2.** *Suppose*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

*is a piecewise smooth continuous function and the derivative  $f'(t)$  is piecewise smooth. Then the derivative can be obtained by differentiating term by term,*

$$f'(t) = \sum_{n=1}^{\infty} \frac{-a_n n\pi}{L} \sin\left(\frac{n\pi}{L} t\right) + \frac{b_n n\pi}{L} \cos\left(\frac{n\pi}{L} t\right).$$

It is important that the function is continuous. It can have corners, but no jumps. Otherwise the differentiated series will fail to converge. For an exercise, take the series obtained for the square wave and try to differentiate the series. Similarly, we can also integrate a Fourier series.

**Theorem 4.3.3.** *Suppose*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

*is a piecewise smooth function. Then the antiderivative is obtained by antidifferentiating term by term and so*

$$F(t) = \frac{a_0 t}{2} + C + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\left(\frac{n\pi}{L} t\right) + \frac{-b_n L}{n\pi} \cos\left(\frac{n\pi}{L} t\right),$$

*where  $F'(t) = f(t)$  and  $C$  is an arbitrary constant.*

Note that the series for  $F(t)$  is no longer a Fourier series as it contains the  $\frac{a_0 t}{2}$  term. The antiderivative of a periodic function need no longer be periodic and so we should not expect a Fourier series.

### 4.3.4 Rates of convergence and smoothness

Let us do an example of a periodic function with one derivative everywhere.

**Example 4.3.6:** Take the function

$$f(t) = \begin{cases} (t+1)t & \text{if } -1 < t \leq 0, \\ (1-t)t & \text{if } 0 < t \leq 1, \end{cases}$$

and extend to a 2-periodic function. The plot is given in Figure 4.10 on the following page.

Note that this function has one derivative everywhere, but it does not have a second derivative whenever  $t$  is an integer.

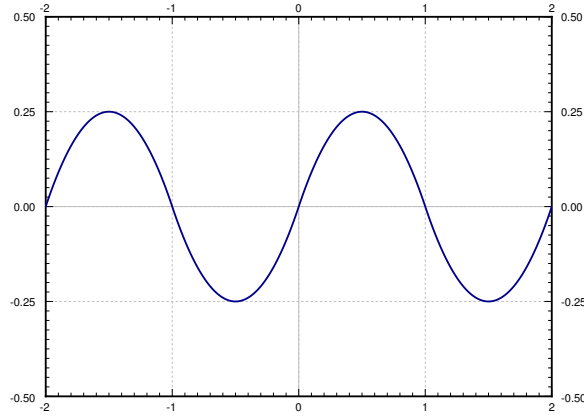


Figure 4.10: Smooth 2-periodic function.

**Exercise 4.3.1:** Compute  $f''(0+)$  and  $f''(0-)$ .

Let us compute the Fourier series coefficients. The actual computation involves several integration by parts and is left to student.

$$\begin{aligned}
 a_0 &= \int_{-1}^1 f(t) dt = \int_{-1}^0 (t+1)t dt + \int_0^1 (1-t)t dt = 0, \\
 a_n &= \int_{-1}^1 f(t) \cos(n\pi t) dt = \int_{-1}^0 (t+1)t \cos(n\pi t) dt + \int_0^1 (1-t)t \cos(n\pi t) dt = 0 \\
 b_n &= \int_{-1}^1 f(t) \sin(n\pi t) dt = \int_{-1}^0 (t+1)t \sin(n\pi t) dt + \int_0^1 (1-t)t \sin(n\pi t) dt \\
 &= \frac{4(1 - (-1)^n)}{\pi^3 n^3} = \begin{cases} \frac{8}{\pi^3 n^3} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}$$

That is, the series is

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{\pi^3 n^3} \sin(n\pi t).$$

This series converges very fast. If you plot up to the third harmonic, that is the function

$$\frac{8}{\pi^3} \sin(\pi t) + \frac{8}{27\pi^3} \sin(3\pi t),$$

it is almost indistinguishable from the plot of  $f(t)$  in Figure 4.10. In fact, the coefficient  $\frac{8}{27\pi^3}$  is already just 0.0096 (approximately). The reason for this behavior is the  $n^3$  term in the denominator. The coefficients  $b_n$  in this case go to zero as fast as  $\frac{1}{n^3}$  goes to zero.

It is a general fact that if you have one derivative, the Fourier coefficients will go to zero approximately like  $\frac{1}{n^3}$ . If you have only a continuous function, then the Fourier coefficients will go to zero as  $\frac{1}{n^2}$ . If you have discontinuities, then the Fourier coefficients will go to zero approximately as  $\frac{1}{n}$ . Therefore, we can tell a lot about the smoothness of a function by looking at its Fourier coefficients.

To justify this behavior, take for example the function defined by the Fourier series

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin(nt).$$

When we differentiate term by term we notice

$$f'(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nt).$$

Therefore, the coefficients now go down like  $\frac{1}{n^2}$ , which we said means that we have a continuous function. The derivative of  $f'(t)$  is defined at most points, but there are points where  $f'(t)$  is not differentiable. It has corners, but no jumps. If we differentiate again (where we can) we find that the function  $f''(t)$ , now fails to be continuous (has jumps)

$$f''(t) = \sum_{n=1}^{\infty} \frac{-1}{n} \sin(nt).$$

This function is similar to the sawtooth. If we tried to differentiate again we would obtain

$$\sum_{n=1}^{\infty} -\cos(nt),$$

which does not converge!

**Exercise 4.3.2:** Use a computer to plot  $f(t)$ ,  $f'(t)$  and  $f''(t)$ . That is, plot say the first 5 harmonics of the functions. At what points does  $f''(t)$  have the discontinuities.

### 4.3.5 Exercises

**Exercise 4.3.3:** Let

$$f(t) = \begin{cases} 0 & \text{if } -1 < t \leq 0, \\ t & \text{if } 0 < t \leq 1, \end{cases}$$

extended periodically. a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic.

**Exercise 4.3.4:** Let

$$f(t) = \begin{cases} -t & \text{if } -1 < t \leq 0, \\ t^2 & \text{if } 0 < t \leq 1, \end{cases}$$

extended periodically. a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic.

**Exercise 4.3.5:** Let

$$f(t) = \begin{cases} \frac{-t}{10} & \text{if } -10 < t \leq 0, \\ \frac{t}{10} & \text{if } 0 < t \leq 10, \end{cases}$$

extended periodically (period is 20). a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic.

**Exercise 4.3.6:** Let  $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nt)$ . Is  $f(t)$  continuous and differentiable everywhere? Find the derivative (if it exists everywhere) or justify why  $f(t)$  is not differentiable everywhere.

**Exercise 4.3.7:** Let  $f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$ . Is  $f(t)$  differentiable everywhere? Find the derivative (if it exists everywhere) or justify why  $f(t)$  is not differentiable everywhere.

**Exercise 4.3.8:** Let

$$f(t) = \begin{cases} 0 & \text{if } -2 < t \leq 0, \\ t & \text{if } 0 < t \leq 1, \\ -t + 2 & \text{if } 1 < t \leq 2, \end{cases}$$

extended periodically. a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic.

**Exercise 4.3.9:** Let

$$f(t) = e^t \quad \text{for } -1 < t < 1$$

extended periodically. a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic. c) What does the series converge to at  $t = 1$ .

**Exercise 4.3.101:** Let

$$f(t) = t^2 \quad \text{for } -2 < t \leq 2$$

extended periodically. a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic.

**Exercise 4.3.102:** Let

$$f(t) = t \quad \text{for } \lambda < t \leq \lambda \text{ (for some } \lambda \text{)}$$

extended periodically. a) Compute the Fourier series for  $f(t)$ . b) Write out the series explicitly up to the 3<sup>rd</sup> harmonic.

**Exercise 4.3.103:** Let

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sin(n\pi t).$$

Compute  $f'(t)$ .

**Exercise 4.3.104:** Let

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nt).$$

a) Find the antiderivative. b) Is the antiderivative periodic?

## 4.4 Sine and cosine series

*Note: 2 lectures, §9.3 in [EP], §10.4 in [BD]*

### 4.4.1 Odd and even periodic functions

You may have noticed by now that an odd function has no cosine terms in the Fourier series and an even function has no sine terms in the Fourier series. This observation is not a coincidence. Let us look at even and odd periodic function in more detail.

Recall that a function  $f(t)$  is *odd* if  $f(-t) = -f(t)$ . A function  $f(t)$  is *even* if  $f(-t) = f(t)$ . For example,  $\cos(nt)$  is even and  $\sin(nt)$  is odd. Similarly the function  $t^k$  is even if  $k$  is even and odd when  $k$  is odd.

**Exercise 4.4.1:** Take two functions  $f(t)$  and  $g(t)$  and define their product  $h(t) = f(t)g(t)$ . a) Suppose both are odd, is  $h(t)$  odd or even? b) Suppose one is even and one is odd, is  $h(t)$  odd or even? c) Suppose both are even, is  $h(t)$  odd or even?

If  $f(t)$  and  $g(t)$  are both odd, then  $f(t) + g(t)$  is odd. Similarly for even functions. On the other hand, if  $f(t)$  is odd and  $g(t)$  even, then we cannot say anything about the sum  $f(t) + g(t)$ . In fact, the Fourier series of any function is a sum of an odd (the sine terms) and an even (the cosine terms) function.

In this section we are interested in odd and even periodic functions. We have previously defined the  $2L$ -periodic extension of a function defined on the interval  $[-L, L]$ . Sometimes we are only interested in the function on the range  $[0, L]$  and it would be convenient to have an odd (resp. even) function. If the function is odd (resp. even), all the cosine (resp. sine) terms will disappear. What we will do is take the odd (resp. even) extension of the function to  $[-L, L]$  and then we extend periodically to a  $2L$ -periodic function.

Take a function  $f(t)$  defined on  $[0, L]$ . On  $(-L, L]$  define the functions

$$F_{\text{odd}}(t) \stackrel{\text{def}}{=} \begin{cases} f(t) & \text{if } 0 \leq t \leq L, \\ -f(-t) & \text{if } -L < t < 0, \end{cases}$$

$$F_{\text{even}}(t) \stackrel{\text{def}}{=} \begin{cases} f(t) & \text{if } 0 \leq t \leq L, \\ f(-t) & \text{if } -L < t < 0. \end{cases}$$

Extend  $F_{\text{odd}}(t)$  and  $F_{\text{even}}(t)$  to be  $2L$ -periodic. Then  $F_{\text{odd}}(t)$  is called the *odd periodic extension* of  $f(t)$ , and  $F_{\text{even}}(t)$  is called the *even periodic extension* of  $f(t)$ .

**Exercise 4.4.2:** Check that  $F_{\text{odd}}(t)$  is odd and that  $F_{\text{even}}(t)$  is even.

**Example 4.4.1:** Take the function  $f(t) = t(1 - t)$  defined on  $[0, 1]$ . Figure 4.11 on the facing page shows the plots of the odd and even extensions of  $f(t)$ .



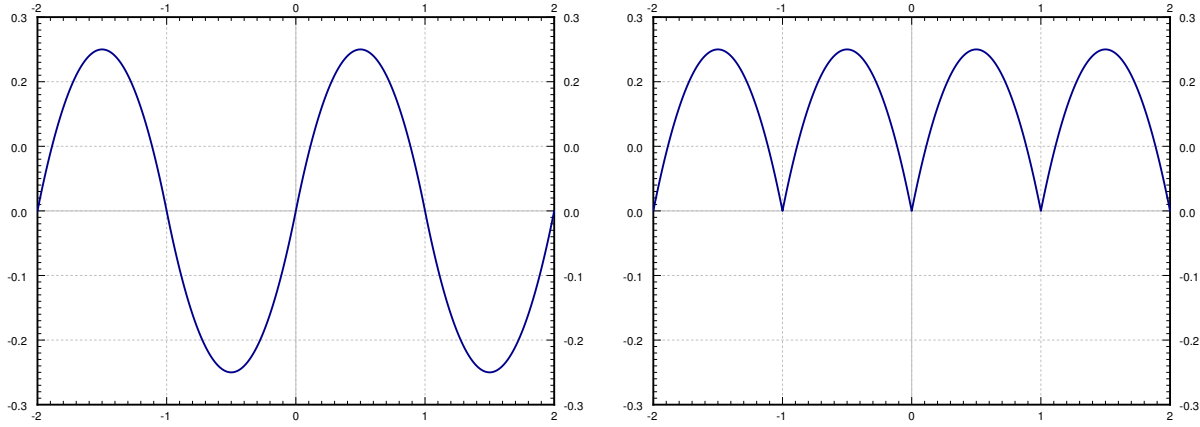


Figure 4.11: Odd and even 2-periodic extension of  $f(t) = t(1-t)$ ,  $0 \leq t \leq 1$ .

#### 4.4.2 Sine and cosine series

Let  $f(t)$  be an odd  $2L$ -periodic function. We write the Fourier series for  $f(t)$ . We compute the coefficients  $a_n$  (including  $n = 0$ ) and get

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi}{L} t\right) dt = 0.$$

That is, there are no cosine terms in the Fourier series of an odd function. The integral is zero because  $f(t) \cos(n\pi L t)$  is an odd function (product of an odd and an even function is odd) and the integral of an odd function over a symmetric interval is always zero. Furthermore, the integral of an even function over a symmetric interval  $[-L, L]$  is twice the integral of the function over the interval  $[0, L]$ . The function  $f(t) \sin\left(\frac{n\pi}{L} t\right)$  is the product of two odd functions and hence even.

$$b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt.$$

We can now write the Fourier series of  $f(t)$  as

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right).$$

Similarly, if  $f(t)$  is an even  $2L$ -periodic function. For the same exact reasons as above, we find that  $b_n = 0$  and

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi}{L} t\right) dt.$$

The formula still works for  $n = 0$  in which case it becomes

$$a_0 = \frac{2}{L} \int_0^L f(t) dt.$$

The Fourier series is then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right).$$

An interesting consequence is that the coefficients of the Fourier series of an odd (or even) function can be computed by just integrating over the half interval  $[0, L]$ . Therefore, we can compute the Fourier series of the odd (or even) extension of a function by computing certain integrals over the interval where the original function is defined.

**Theorem 4.4.1.** *Let  $f(t)$  be a piecewise smooth function defined on  $[0, L]$ . Then the odd extension of  $f(t)$  has the Fourier series*

$$F_{\text{odd}}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right),$$

where

$$b_n = \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi}{L} t\right) dt.$$

The even extension of  $f(t)$  has the Fourier series

$$F_{\text{even}}(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right),$$

where

$$a_n = \frac{2}{L} \int_0^L f(t) \cos\left(\frac{n\pi}{L} t\right) dt.$$

The series  $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} t\right)$  is called the *sine series* of  $f(t)$  and the series  $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right)$  is called the *cosine series* of  $f(t)$ . It is often the case that we do not actually care what happens outside of  $[0, L]$ . In this case, we can pick whichever series fits our problem better.

It is not necessary to start with the full Fourier series to obtain the sine and cosine series. The sine series is really the eigenfunction expansion of  $f(t)$  using the eigenfunctions of the eigenvalue problem  $x'' + \lambda x = 0$ ,  $x(0) = 0$ ,  $x(L) = 0$ . The cosine series is the eigenfunction expansion of  $f(t)$  using the eigenfunctions of the eigenvalue problem  $x'' + \lambda x = 0$ ,  $x'(0) = 0$ ,  $x'(L) = 0$ . We could have, therefore, have gotten the same formulas by defining the inner product

$$\langle f(t), g(t) \rangle = \int_0^L f(t)g(t) dt,$$

and following the procedure of § 4.2. This point of view is useful because many times we use a specific series because our underlying question will lead to a certain eigenvalue problem. If the eigenvalue problem is not one of the three we covered so far, you can still do an eigenfunction expansion, generalizing the results of this chapter. We will deal with such a generalization in chapter 5.

**Example 4.4.2:** Find the Fourier series of the even periodic extension of the function  $f(t) = t^2$  for  $0 \leq t \leq \pi$ .

We want to write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt),$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} t^2 dt = \frac{2\pi^2}{3},$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} t^2 \cos(nt) dt = \frac{2}{\pi} \left[ t^2 \frac{1}{n} \sin(nt) \right]_0^{\pi} - \frac{4}{n\pi} \int_0^{\pi} t \sin(nt) dt \\ &= \frac{4}{n^2\pi} \left[ t \cos(nt) \right]_0^{\pi} + \frac{4}{n^2\pi} \int_0^{\pi} \cos(nt) dt = \frac{4(-1)^n}{n^2}. \end{aligned}$$

Note that we have detected the “continuity” of the extension since the coefficients decay as  $\frac{1}{n^2}$ . That is, the even extension of  $t^2$  has no jump discontinuities. It will have corners, since the derivative (which will be an odd function and a sine series) will have a series whose coefficients decay only as  $\frac{1}{n}$  so the derivative will have jumps.

Explicitly, the first few terms of the series are

$$\frac{\pi^2}{3} - 4 \cos(t) + \cos(2t) - \frac{4}{9} \cos(3t) + \cdots$$

**Exercise 4.4.3:** a) Compute the derivative of the even extension of  $f(t)$  above and verify it has jump discontinuities. Use the actual definition of  $f(t)$ , not its cosine series! b) Why is it that the derivative of the even extension of  $f(t)$  is the odd extension of  $f'(t)$ .

### 4.4.3 Application

We said that Fourier series ties in to the boundary value problems we studied earlier. Let us see this connection in more detail.

Suppose we have the boundary value problem for  $0 < t < L$ ,

$$x''(t) + \lambda x(t) = f(t),$$

for the *Dirichlet boundary conditions*  $x(0) = 0$ ,  $x(L) = 0$ . By using the Fredholm alternative (Theorem 4.1.2 on page 154) we note that as long as  $\lambda$  is not an eigenvalue of the underlying homogeneous problem, there will exist a unique solution. Note that the eigenfunctions of this eigenvalue problem were the functions  $\sin\left(\frac{n\pi}{L}t\right)$ . Therefore, to find the solution, we first find the Fourier sine series for  $f(t)$ . We write  $x$  also as a sine series, but with unknown coefficients. We substitute the series for  $x$  into the equation and solve for the unknown coefficients.

If we have the *Neumann boundary conditions*  $x'(0) = 0$ ,  $x'(L) = 0$ , we do the same procedure using the cosine series. These methods are best seen by examples.

**Example 4.4.3:** Take the boundary value problem for  $0 < t < 1$ ,

$$x''(t) + 2x(t) = f(t),$$

where  $f(t) = t$  on  $0 < t < 1$ , and satisfying the Dirichlet boundary conditions  $x(0) = 0$ ,  $x(1) = 0$ . We write  $f(t)$  as a sine series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t),$$

where

$$c_n = 2 \int_0^1 t \sin(n\pi t) dt = \frac{2(-1)^{n+1}}{n\pi}.$$

We write  $x(t)$  as

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t).$$

We plug in to obtain

$$\begin{aligned} x''(t) + 2x(t) &= \sum_{n=1}^{\infty} -b_n n^2 \pi^2 \sin(n\pi t) + 2 \sum_{n=1}^{\infty} b_n \sin(n\pi t) \\ &= \sum_{n=1}^{\infty} b_n (2 - n^2 \pi^2) \sin(n\pi t) \\ &= f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t). \end{aligned}$$

Therefore,

$$b_n (2 - n^2 \pi^2) = \frac{2(-1)^{n+1}}{n\pi}$$

or

$$b_n = \frac{2(-1)^{n+1}}{n\pi(2 - n^2 \pi^2)}.$$

We have thus obtained a Fourier series for the solution

$$x(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(2-n^2\pi^2)} \sin(n\pi t).$$

**Example 4.4.4:** Similarly we handle the Neumann conditions. Take the boundary value problem for  $0 < t < 1$ ,

$$x''(t) + 2x(t) = f(t),$$

where again  $f(t) = t$  on  $0 < t < 1$ , but now satisfying the Neumann boundary conditions  $x'(0) = 0$ ,  $x'(1) = 0$ . We write  $f(t)$  as a cosine series

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi t),$$

where

$$c_0 = 2 \int_0^1 t \, dt = 1,$$

and

$$c_n = 2 \int_0^1 t \cos(n\pi t) \, dt = \frac{2((-1)^n - 1)}{\pi^2 n^2} = \begin{cases} \frac{-4}{\pi^2 n^2} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

We write  $x(t)$  as a cosine series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t).$$

We plug in to obtain

$$\begin{aligned} x''(t) + 2x(t) &= \sum_{n=1}^{\infty} [-a_n n^2 \pi^2 \cos(n\pi t)] + a_0 + 2 \sum_{n=1}^{\infty} [a_n \cos(n\pi t)] \\ &= a_0 + \sum_{n=1}^{\infty} a_n (2 - n^2 \pi^2) \cos(n\pi t) \\ &= f(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{\pi^2 n^2} \cos(n\pi t). \end{aligned}$$

Therefore,  $a_0 = \frac{1}{2}$ ,  $a_n = 0$  for  $n$  even ( $n \geq 2$ ) and for  $n$  odd we have

$$a_n (2 - n^2 \pi^2) = \frac{-4}{\pi^2 n^2},$$

or

$$a_n = \frac{-4}{n^2\pi^2(2 - n^2\pi^2)}.$$

We have thus obtained a Fourier series for the solution

$$x(t) = \frac{1}{4} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2(2 - n^2\pi^2)} \cos(n\pi t).$$

#### 4.4.4 Exercises

**Exercise 4.4.4:** Take  $f(t) = (t - 1)^2$  defined on  $0 \leq t \leq 1$ . a) Sketch the plot of the even periodic extension of  $f$ . b) Sketch the plot of the odd periodic extension of  $f$ .

**Exercise 4.4.5:** Find the Fourier series of both the odd and even periodic extension of the function  $f(t) = (t - 1)^2$  for  $0 \leq t \leq 1$ . Can you tell which extension is continuous from the Fourier series coefficients?

**Exercise 4.4.6:** Find the Fourier series of both the odd and even periodic extension of the function  $f(t) = t$  for  $0 \leq t \leq \pi$ .

**Exercise 4.4.7:** Find the Fourier series of the even periodic extension of the function  $f(t) = \sin t$  for  $0 \leq t \leq \pi$ .

**Exercise 4.4.8:** Consider

$$x''(t) + 4x(t) = f(t),$$

where  $f(t) = 1$  on  $0 < t < 1$ . a) Solve for the Dirichlet conditions  $x(0) = 0, x(1) = 0$ . b) Solve for the Neumann conditions  $x'(0) = 0, x'(1) = 0$ .

**Exercise 4.4.9:** Consider

$$x''(t) + 9x(t) = f(t),$$

for  $f(t) = \sin(2\pi t)$  on  $0 < t < 1$ . a) Solve for the Dirichlet conditions  $x(0) = 0, x(1) = 0$ . b) Solve for the Neumann conditions  $x'(0) = 0, x'(1) = 0$ .

**Exercise 4.4.10:** Consider

$$x''(t) + 3x(t) = f(t), \quad x(0) = 0, \quad x(1) = 0,$$

where  $f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$ . Write the solution  $x(t)$  as a Fourier series, where the coefficients are given in terms of  $b_n$ .

**Exercise 4.4.11:** Let  $f(t) = t^2(2 - t)$  for  $0 \leq t \leq 2$ . Let  $F(t)$  be the odd periodic extension. Compute  $F(1), F(2), F(3), F(-1), F(9/2), F(101), F(103)$ . Note: Do **not** compute using the sine series.

**Exercise 4.4.101:** Let  $f(t) = t/3$  on  $0 \leq t < 3$ . a) Find the Fourier series of the even periodic extension. b) Find the Fourier series of the odd periodic extension.

**Exercise 4.4.102:** Let  $f(t) = \cos(2t)$  on  $0 \leq t < \pi$ . a) Find the Fourier series of the even periodic extension. b) Find the Fourier series of the odd periodic extension.

**Exercise 4.4.103:** Let  $f(t)$  be defined on  $0 \leq t < 1$ . Now take the average of the two extensions  $g(t) = \frac{F_{\text{odd}}(t) + F_{\text{even}}(t)}{2}$ . a) What is  $g(t)$  if  $0 \leq t < 1$  (Justify!) b) What is  $g(t)$  if  $-1 < t < 0$  (Justify!)

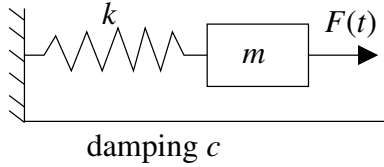
**Exercise 4.4.104:** Let  $f(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nt)$ . Solve  $x'' - x = f(t)$  for the Dirichlet conditions  $x(0) = 0$  and  $x(\pi) = 0$ .

**Exercise 4.4.105** (challenging): Let  $f(t) = t + \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(nt)$ . Solve  $x'' + \pi x = f(t)$  for the Dirichlet conditions  $x(0) = 0$  and  $x(\pi) = 1$ . Hint: Note that  $\frac{t}{\pi}$  satisfies the given Dirichlet conditions.

## 4.5 Applications of Fourier series

Note: 2 lectures, §9.4 in [EP], not in [BD]

### 4.5.1 Periodically forced oscillation



Let us return to the forced oscillations. We have a mass-spring system as before, where we have a mass  $m$  on a spring with spring constant  $k$ , with damping  $c$ , and a force  $F(t)$  applied to the mass. Suppose that the forcing function  $F(t)$  is  $2L$ -periodic for some  $L > 0$ . We have already seen this problem in chapter 2 with a simple  $F(t)$ . The equation that governs this particular setup is

$$mx''(t) + cx'(t) + kx(t) = F(t). \quad (4.9)$$

We know that the general solution will consist of  $x_c$ , which solves the associated homogeneous equation  $mx'' + cx' + kx = 0$ , and a particular solution of (4.9) we will call  $x_p$ . For  $c > 0$ , the complementary solution  $x_c$  will decay as time goes on. Therefore, we are mostly interested in particular solution  $x_p$  that does not decay and is periodic with the same period as  $F(t)$ . We call this particular solution the *steady periodic solution* and we write it as  $x_{sp}$  as before. The difference in what we will do now is that we consider an arbitrary forcing function  $F(t)$  instead of a simple cosine.

For simplicity, let us suppose that  $c = 0$ . The problem with  $c > 0$  is very similar. The equation

$$mx'' + kx = 0$$

has the general solution

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

where  $\omega_0 = \sqrt{\frac{k}{m}}$ . Any solution to  $mx''(t) + kx(t) = F(t)$  will be of the form  $A \cos(\omega_0 t) + B \sin(\omega_0 t) + x_{sp}$ . The steady periodic solution  $x_{sp}$  has the same period as  $F(t)$ .

In the spirit of the last section and the idea of undetermined coefficients we will first write

$$F(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi}{L} t\right) + d_n \sin\left(\frac{n\pi}{L} t\right).$$

Then we write a proposed steady periodic solution  $x$  as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right),$$

where  $a_n$  and  $b_n$  are unknowns. We plug  $x$  into the differential equation and solve for  $a_n$  and  $b_n$  in terms of  $c_n$  and  $d_n$ . This process is perhaps best understood by example.



**Example 4.5.1:** Suppose that  $k = 2$ , and  $m = 1$ . The units are the mks units (meters-kilograms-seconds) again. There is a jetpack strapped to the mass, which fires with a force of 1 newton for 1 second and then is off for 1 second, and so on. We want to find the steady periodic solution.

The equation is, therefore,

$$x'' + 2x = F(t),$$

where  $F(t)$  is the step function

$$F(t) = \begin{cases} 0 & \text{if } -1 < t < 0, \\ 1 & \text{if } 0 < t < 1, \end{cases}$$

extended periodically. We write

$$F(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\pi t) + d_n \sin(n\pi t).$$

We compute

$$\begin{aligned} c_n &= \int_{-1}^1 F(t) \cos(n\pi t) dt = \int_0^1 \cos(n\pi t) dt = 0 \quad \text{for } n \geq 1, \\ c_0 &= \int_{-1}^1 F(t) dt = \int_0^1 dt = 1, \\ d_n &= \int_{-1}^1 F(t) \sin(n\pi t) dt \\ &= \int_0^1 \sin(n\pi t) dt \\ &= \left[ \frac{-\cos(n\pi t)}{n\pi} \right]_{t=0}^1 \\ &= \frac{1 - (-1)^n}{\pi n} = \begin{cases} \frac{2}{\pi n} & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases} \end{aligned}$$

So

$$F(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n} \sin(n\pi t).$$

We want to try

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + b_n \sin(n\pi t).$$

Once we plug  $x$  into the differential equation  $x'' + 2x = F(t)$ , it is clear that  $a_n = 0$  for  $n \geq 1$  as there are no corresponding terms in the series for  $F(t)$ . Similarly  $b_n = 0$  for  $n$  even. Hence we try

$$x(t) = \frac{a_0}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n \sin(n\pi t).$$

We plug into the differential equation and obtain

$$\begin{aligned}
 x'' + 2x &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} [-b_n n^2 \pi^2 \sin(n\pi t)] + a_0 + 2 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} [b_n \sin(n\pi t)] \\
 &= a_0 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n (2 - n^2 \pi^2) \sin(n\pi t) \\
 &= F(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n} \sin(n\pi t).
 \end{aligned}$$

So  $a_0 = \frac{1}{2}$ ,  $b_n = 0$  for even  $n$ , and for odd  $n$  we get

$$b_n = \frac{2}{\pi n(2 - n^2 \pi^2)}.$$

The steady periodic solution has the Fourier series

$$x_{sp}(t) = \frac{1}{4} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n(2 - n^2 \pi^2)} \sin(n\pi t).$$

We know this is the steady periodic solution as it contains no terms of the complementary solution and it is periodic with the same period as  $F(t)$  itself. See Figure 4.12 for the plot of this solution.

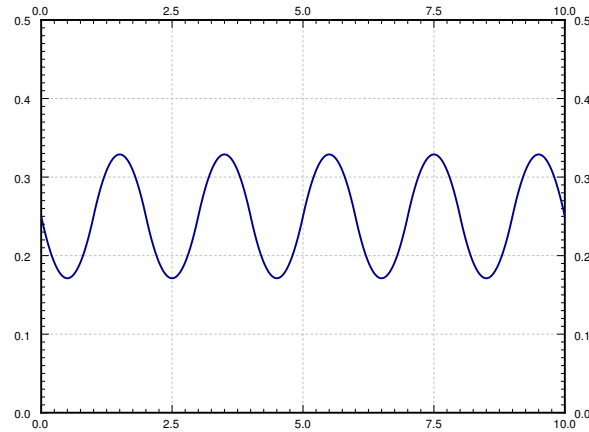


Figure 4.12: Plot of the steady periodic solution  $x_{sp}$  of Example 4.5.1.

### 4.5.2 Resonance

Just like when the forcing function was a simple cosine, resonance could still happen. Let us assume  $c = 0$  and we will discuss only pure resonance. Again, take the equation

$$mx''(t) + kx(t) = F(t).$$

When we expand  $F(t)$  and find that some of its terms coincide with the complementary solution to  $mx'' + kx = 0$ , we cannot use those terms in the guess. Just like before, they will disappear when we plug into the left hand side and we will get a contradictory equation (such as  $0 = 1$ ). That is, suppose

$$x_c = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

where  $\omega_0 = \frac{N\pi}{L}$  for some positive integer  $N$ . In this case we have to modify our guess and try

$$x(t) = \frac{a_0}{2} + t \left( a_N \cos\left(\frac{N\pi}{L} t\right) + b_N \sin\left(\frac{N\pi}{L} t\right) \right) + \sum_{\substack{n=1 \\ n \neq N}}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right).$$

In other words, we multiply the offending term by  $t$ . From then on, we proceed as before.

Of course, the solution will not be a Fourier series (it will not even be periodic) since it contains these terms multiplied by  $t$ . Further, the terms  $t \left( a_N \cos\left(\frac{N\pi}{L} t\right) + b_N \sin\left(\frac{N\pi}{L} t\right) \right)$  will eventually dominate and lead to wild oscillations. As before, this behavior is called *pure resonance* or just *resonance*.

Note that there now may be infinitely many resonance frequencies to hit. That is, as we change the frequency of  $F$  (we change  $L$ ), different terms from the Fourier series of  $F$  may interfere with the complementary solution and will cause resonance. However, we should note that since everything is an approximation and in particular  $c$  is never actually zero but something very close to zero, only the first few resonance frequencies will matter.

**Example 4.5.2:** Find the steady periodic solution to the equation

$$2x'' + 18\pi^2 x = F(t),$$

where

$$F(t) = \begin{cases} -1 & \text{if } -1 < t < 0, \\ 1 & \text{if } 0 < t < 1, \end{cases}$$

extended periodically. We note that

$$F(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t).$$

**Exercise 4.5.1:** Compute the Fourier series of  $F$  to verify the above equation.

The solution must look like

$$x(t) = c_1 \cos(3\pi t) + c_2 \sin(3\pi t) + x_p(t)$$

for some particular solution  $x_p$ .

We note that if we just tried a Fourier series with  $\sin(n\pi t)$  as usual, we would get duplication when  $n = 3$ . Therefore, we pull out that term and multiply by  $t$ . We also have to add a cosine term to get everything right. That is, we must try

$$x_p(t) = a_3 t \cos(3\pi t) + b_3 t \sin(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} b_n \sin(n\pi t).$$

Let us compute the second derivative.

$$\begin{aligned} x_p''(t) = & -6a_3\pi \sin(3\pi t) - 9\pi^2 a_3 t \cos(3\pi t) + 6b_3\pi \cos(3\pi t) - 9\pi^2 b_3 t \sin(3\pi t) + \\ & + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-n^2\pi^2 b_n) \sin(n\pi t). \end{aligned}$$

We now plug into the left hand side of the differential equation.

$$\begin{aligned} 2x_p'' + 18\pi^2 x = & -12a_3\pi \sin(3\pi t) - 18\pi^2 a_3 t \cos(3\pi t) + 12b_3\pi \cos(3\pi t) - 18\pi^2 b_3 t \sin(3\pi t) + \\ & + 18\pi^2 a_3 t \cos(3\pi t) + 18\pi^2 b_3 t \sin(3\pi t) + \\ & + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-2n^2\pi^2 b_n + 18\pi^2 b_n) \sin(n\pi t). \end{aligned}$$

If we simplify we obtain

$$2x_p'' + 18\pi^2 x = -12a_3\pi \sin(3\pi t) + 12b_3\pi \cos(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} (-2n^2\pi^2 b_n + 18\pi^2 b_n) \sin(n\pi t).$$

This series has to equal to the series for  $F(t)$ . We equate the coefficients and solve for  $a_3$  and  $b_n$ .

$$\begin{aligned} a_3 &= \frac{4/(3\pi)}{-12\pi} = \frac{-1}{9\pi^2}, \\ b_3 &= 0, \\ b_n &= \frac{4}{n\pi(18\pi^2 - 2n^2\pi^2)} = \frac{2}{\pi^3 n(9 - n^2)} \quad \text{for } n \text{ odd and } n \neq 3. \end{aligned}$$

That is,

$$x_p(t) = \frac{-1}{9\pi^2} t \cos(3\pi t) + \sum_{\substack{n=1 \\ n \text{ odd} \\ n \neq 3}}^{\infty} \frac{2}{\pi^3 n(9 - n^2)} \sin(n\pi t).$$

When  $c > 0$ , you will not have to worry about pure resonance. That is, there will never be any conflicts and you do not need to multiply any terms by  $t$ . There is a corresponding concept of practical resonance and it is very similar to the ideas we already explored in chapter 2. We will not go into details here.

### 4.5.3 Exercises

**Exercise 4.5.2:** Let  $F(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi t)$ . Find the steady periodic solution to  $x'' + 2x = F(t)$ . Express your solution as a Fourier series.

**Exercise 4.5.3:** Let  $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin(n\pi t)$ . Find the steady periodic solution to  $x'' + x' + x = F(t)$ . Express your solution as a Fourier series.

**Exercise 4.5.4:** Let  $F(t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi t)$ . Find the steady periodic solution to  $x'' + 4x = F(t)$ . Express your solution as a Fourier series.

**Exercise 4.5.5:** Let  $F(t) = t$  for  $-1 < t < 1$  and extended periodically. Find the steady periodic solution to  $x'' + x = F(t)$ . Express your solution as a series.

**Exercise 4.5.6:** Let  $F(t) = t$  for  $-1 < t < 1$  and extended periodically. Find the steady periodic solution to  $x'' + \pi^2 x = F(t)$ . Express your solution as a series.

**Exercise 4.5.101:** Let  $F(t) = \sin(2\pi t) + 0.1 \cos(10\pi t)$ . Find the steady periodic solution to  $x'' + \sqrt{2}x = F(t)$ . Express your solution as a Fourier series.

**Exercise 4.5.102:** Let  $F(t) = \sum_{n=1}^{\infty} e^{-n} \cos(2nt)$ . Find the steady periodic solution to  $x'' + 3x = F(t)$ . Express your solution as a Fourier series.

**Exercise 4.5.103:** Let  $F(t) = |t|$  for  $-1 \leq t \leq 1$  extended periodically. Find the steady periodic solution to  $x'' + \sqrt{3}x = F(t)$ . Express your solution as a series.

**Exercise 4.5.104:** Let  $F(t) = |t|$  for  $-1 \leq t \leq 1$  extended periodically. Find the steady periodic solution to  $x'' + \pi^2 x = F(t)$ . Express your solution as a series.

## 4.6 PDEs, separation of variables, and the heat equation

*Note: 2 lectures, §9.5 in [EP], §10.5 in [BD]*

Let us recall that a *partial differential equation* or *PDE* is an equation containing the partial derivatives with respect to *several* independent variables. Solving PDEs will be our main application of Fourier series.

A PDE is said to be *linear* if the dependent variable and its derivatives appear at most to the first power and in no functions. We will only talk about linear PDEs. Together with a PDE, we usually have specified some *boundary conditions*, where the value of the solution or its derivatives is specified along the boundary of a region, and/or some *initial conditions* where the value of the solution or its derivatives is specified for some initial time. Sometimes such conditions are mixed together and we will refer to them simply as *side conditions*.

We will study three specific partial differential equations, each one representing a more general class of equations. First, we will study the *heat equation*, which is an example of a *parabolic PDE*. Next, we will study the *wave equation*, which is an example of a *hyperbolic PDE*. Finally, we will study the *Laplace equation*, which is an example of an *elliptic PDE*. Each of our examples will illustrate behavior that is typical for the whole class.

### 4.6.1 Heat on an insulated wire

Let us first study the heat equation. Suppose that we have a wire (or a thin metal rod) of length  $L$  that is insulated except at the endpoints. Let  $x$  denote the position along the wire and let  $t$  denote time. See Figure 4.13.

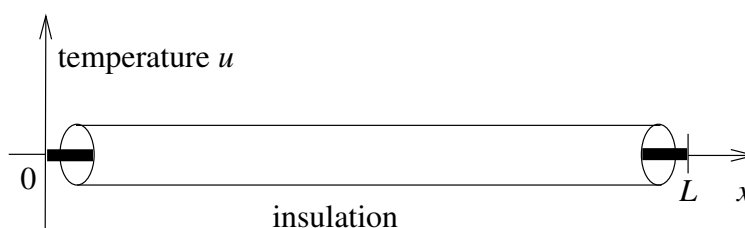


Figure 4.13: Insulated wire.

Let  $u(x, t)$  denote the temperature at point  $x$  at time  $t$ . The equation governing this setup is the so-called *one-dimensional heat equation*:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

where  $k > 0$  is a constant (the *thermal conductivity* of the material). That is, the change in heat at a specific point is proportional to the second derivative of the heat along the wire. This makes sense;

if at a fixed  $t$  the graph of the heat distribution has a maximum (the graph is concave down), then heat flows away from the maximum. And vice-versa.

We will generally use a more convenient notation for partial derivatives. We will write  $u_t$  instead of  $\frac{\partial u}{\partial t}$ , and we will write  $u_{xx}$  instead of  $\frac{\partial^2 u}{\partial x^2}$ . With this notation the heat equation becomes

$$u_t = ku_{xx}.$$

For the heat equation, we must also have some boundary conditions. We assume that the ends of the wire are either exposed and touching some body of constant heat, or the ends are insulated. For example, if the ends of the wire are kept at temperature 0, then we must have the conditions

$$u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.$$

If, on the other hand, the ends are also insulated we get the conditions

$$u_x(0, t) = 0 \quad \text{and} \quad u_x(L, t) = 0.$$

In other words, heat is not flowing in nor out of the wire at the ends. We always have two conditions along the  $x$  axis as there are two derivatives in the  $x$  direction. These side conditions are called *homogeneous* (that is,  $u$  or a derivative of  $u$  is set to zero).

Furthermore, suppose that we know the initial temperature distribution at time  $t = 0$ . That is,

$$u(x, 0) = f(x),$$

for some known function  $f(x)$ . This initial condition is not a homogeneous side condition.

## 4.6.2 Separation of variables

The heat equation is linear as  $u$  and its derivatives do not appear to any powers or in any functions. Thus the principle of superposition still applies for the heat equation (without side conditions). If  $u_1$  and  $u_2$  are solutions and  $c_1, c_2$  are constants, then  $u = c_1u_1 + c_2u_2$  is also a solution.

**Exercise 4.6.1:** Verify the principle of superposition for the heat equation.

Superposition also preserves some of the side conditions. In particular, if  $u_1$  and  $u_2$  are solutions that satisfy  $u(0, t) = 0$  and  $u(L, t) = 0$ , and  $c_1, c_2$  are constants, then  $u = c_1u_1 + c_2u_2$  is still a solution that satisfies  $u(0, t) = 0$  and  $u(L, t) = 0$ . Similarly for the side conditions  $u_x(0, t) = 0$  and  $u_x(L, t) = 0$ . In general, superposition preserves all homogeneous side conditions.

The method of *separation of variables* is to try to find solutions that are sums or products of functions of one variable. For example, for the heat equation, we try to find solutions of the form

$$u(x, t) = X(x)T(t).$$

That the desired solution we are looking for is of this form is too much to hope for. What is perfectly reasonable to ask, however, is to find enough “building-block” solutions of the form  $u(x, t) = X(x)T(t)$  using this procedure so that the desired solution to the PDE is somehow constructed from these building blocks by the use of superposition.

Let us try to solve the heat equation

$$u_t = ku_{xx} \quad \text{with} \quad u(0, t) = 0, \quad u(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x).$$

Let us guess  $u(x, t) = X(x)T(t)$ . We plug into the heat equation to obtain

$$X(x)T'(t) = kX''(x)T(t).$$

We rewrite as

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

This equation must hold for all  $x$  and all  $t$ . But the left hand side does not depend on  $x$  and the right hand side does not depend on  $t$ . Hence, each side must be a constant. Let us call this constant  $-\lambda$  (the minus sign is for convenience later). We obtain the two equations

$$\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

In other words

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T'(t) + \lambda kT(t) &= 0. \end{aligned}$$

The boundary condition  $u(0, t) = 0$  implies  $X(0)T(t) = 0$ . We are looking for a nontrivial solution and so we can assume that  $T(t)$  is not identically zero. Hence  $X(0) = 0$ . Similarly,  $u(L, t) = 0$  implies  $X(L) = 0$ . We are looking for nontrivial solutions  $X$  of the eigenvalue problem  $X'' + \lambda X = 0$ ,  $X(0) = 0$ ,  $X(L) = 0$ . We have previously found that the only eigenvalues are  $\lambda_n = \frac{n^2\pi^2}{L^2}$ , for integers  $n \geq 1$ , where eigenfunctions are  $\sin\left(\frac{n\pi}{L}x\right)$ . Hence, let us pick the solutions

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

The corresponding  $T_n$  must satisfy the equation

$$T'_n(t) + \frac{n^2\pi^2}{L^2}kT_n(t) = 0.$$

By the method of integrating factor, the solution of this problem is

$$T_n(t) = e^{\frac{-n^2\pi^2}{L^2}kt}.$$



It will be useful to note that  $T_n(0) = 1$ . Our building-block solutions are

$$u_n(x, t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2}{L^2}kt}.$$

We note that  $u_n(x, 0) = \sin\left(\frac{n\pi}{L}x\right)$ . Let us write  $f(x)$  as the sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

That is, we find the Fourier series of the odd periodic extension of  $f(x)$ . We used the sine series as it corresponds to the eigenvalue problem for  $X(x)$  above. Finally, we use superposition to write the solution as

$$u(x, t) = \sum_{n=1}^{\infty} b_n u_n(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-n^2\pi^2}{L^2}kt}.$$

Why does this solution work? First note that it is a solution to the heat equation by superposition. It satisfies  $u(0, t) = 0$  and  $u(L, t) = 0$ , because  $x = 0$  or  $x = L$  makes all the sines vanish. Finally, plugging in  $t = 0$ , we notice that  $T_n(0) = 1$  and so

$$u(x, 0) = \sum_{n=1}^{\infty} b_n u_n(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = f(x).$$

**Example 4.6.1:** Suppose that we have an insulated wire of length 1, such that the ends of the wire are embedded in ice (temperature 0). Let  $k = 0.003$ . Then suppose that initial heat distribution is  $u(x, 0) = 50x(1 - x)$ . See Figure 4.14.

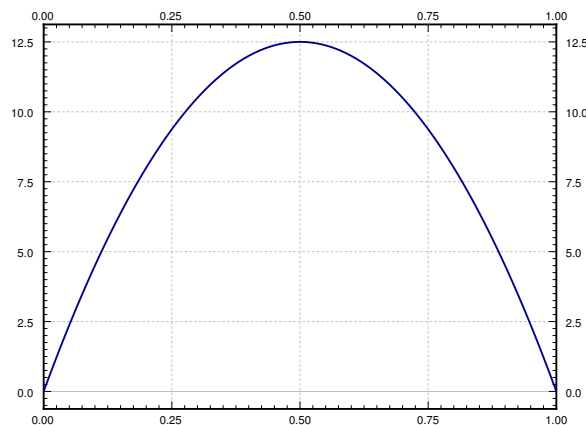


Figure 4.14: Initial distribution of temperature in the wire.

We want to find the temperature function  $u(x, t)$ . Let us suppose we also want to find when (at what  $t$ ) does the maximum temperature in the wire drop to one half of the initial maximum of 12.5.

We are solving the following PDE problem:

$$\begin{aligned} u_t &= 0.003 u_{xx}, \\ u(0, t) &= u(1, t) = 0, \\ u(x, 0) &= 50x(1-x) \quad \text{for } 0 < x < 1. \end{aligned}$$

We write  $f(x) = 50x(1-x)$  for  $0 < x < 1$  as a sine series. That is,  $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ , where

$$b_n = 2 \int_0^1 50x(1-x) \sin(n\pi x) dx = \frac{200}{\pi^3 n^3} - \frac{200(-1)^n}{\pi^3 n^3} = \begin{cases} 0 & \text{if } n \text{ even,} \\ \frac{400}{\pi^3 n^3} & \text{if } n \text{ odd.} \end{cases}$$

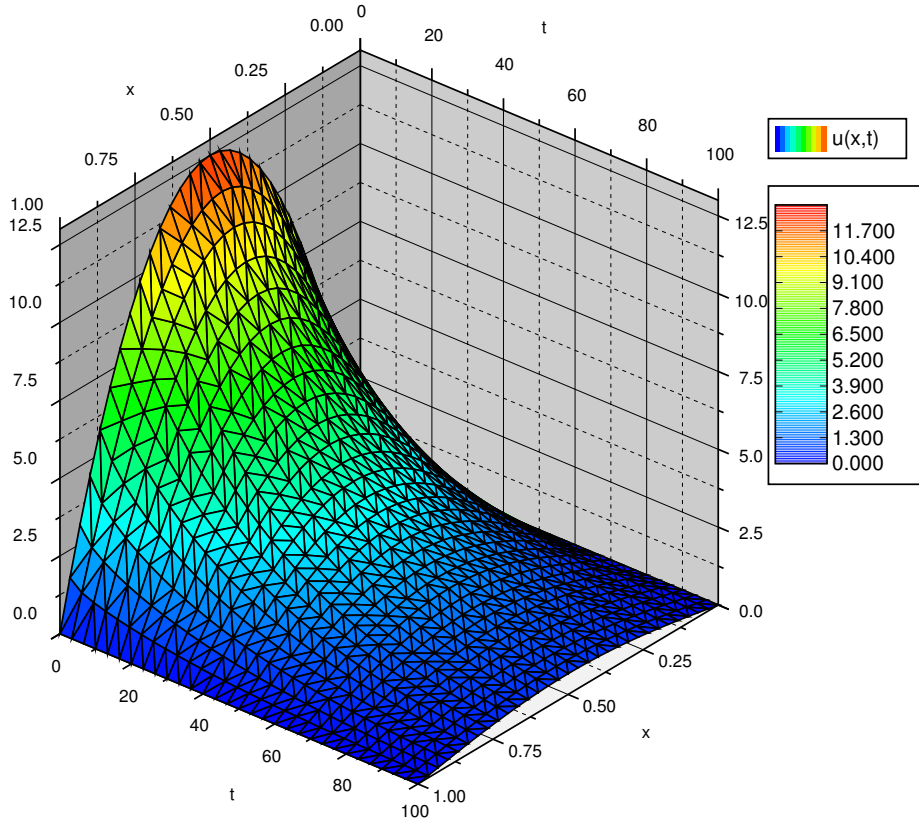


Figure 4.15: Plot of the temperature of the wire at position  $x$  at time  $t$ .

The solution  $u(x, t)$ , plotted in Figure 4.15 for  $0 \leq t \leq 100$ , is given by the series:

$$u(x, t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{\pi^3 n^3} \sin(n\pi x) e^{-n^2 \pi^2 0.003 t}.$$

Finally, let us answer the question about the maximum temperature. It is relatively easy to see that the maximum temperature will always be at  $x = 0.5$ , in the middle of the wire. The plot of  $u(x, t)$  confirms this intuition.

If we plug in  $x = 0.5$  we get

$$u(0.5, t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{400}{\pi^3 n^3} \sin(n\pi 0.5) e^{-n^2 \pi^2 0.003 t}.$$

For  $n = 3$  and higher (remember  $n$  is only odd), the terms of the series are insignificant compared to the first term. The first term in the series is already a very good approximation of the function. Hence

$$u(0.5, t) \approx \frac{400}{\pi^3} e^{-\pi^2 0.003 t}.$$

The approximation gets better and better as  $t$  gets larger as the other terms decay much faster. Let us plot the function  $u(0.5, t)$ , the temperature at the midpoint of the wire at time  $t$ , in Figure 4.16. The figure also plots the approximation by the first term.

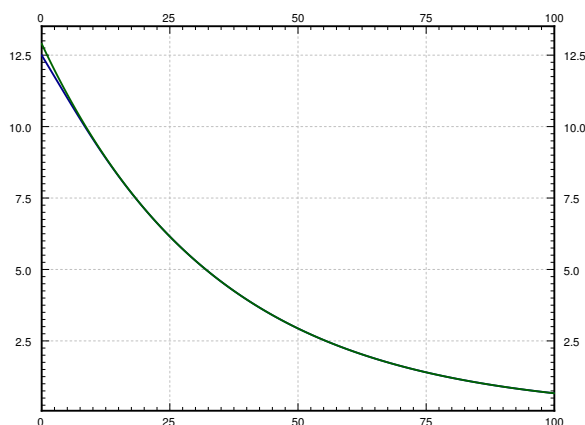


Figure 4.16: Temperature at the midpoint of the wire (the bottom curve), and the approximation of this temperature by using only the first term in the series (top curve).

After  $t = 5$  or so it would be hard to tell the difference between the first term of the series for  $u(x, t)$  and the real solution  $u(x, t)$ . This behavior is a general feature of solving the heat equation. If you are interested in behavior for large enough  $t$ , only the first one or two terms may be necessary.

Let us get back to the question of when is the maximum temperature one half of the initial maximum temperature. That is, when is the temperature at the midpoint  $12.5/2 = 6.25$ . We notice on the graph that if we use the approximation by the first term we will be close enough. We solve

$$6.25 = \frac{400}{\pi^3} e^{-\pi^2 0.003 t}.$$

That is,

$$t = \frac{\ln \frac{6.25\pi^3}{400}}{-\pi^2 0.003} \approx 24.5.$$

So the maximum temperature drops to half at about  $t = 24.5$ .

We mention an interesting behavior of the solution to the heat equation. The heat equation “smoothes” out the function  $f(x)$  as  $t$  grows. For a fixed  $t$ , the solution is a Fourier series with coefficients  $b_n e^{-\frac{n^2\pi^2}{L^2}kt}$ . If  $t > 0$ , then these coefficients go to zero faster than any  $\frac{1}{n^p}$  for any power  $p$ . In other words, the Fourier series has infinitely many derivatives everywhere. Thus even if the function  $f(x)$  has jumps and corners, the solution  $u(x, t)$  as a function of  $x$  for a fixed  $t > 0$  is as smooth as we want it.

### 4.6.3 Insulated ends

Now suppose the ends of the wire are insulated. In this case, we are solving the equation

$$u_t = ku_{xx} \quad \text{with} \quad u_x(0, t) = 0, \quad u_x(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x).$$

Yet again we try a solution of the form  $u(x, t) = X(x)T(t)$ . By the same procedure as before we plug into the heat equation and arrive at the following two equations

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T'(t) + \lambda k T(t) &= 0. \end{aligned}$$

At this point the story changes slightly. The boundary condition  $u_x(0, t) = 0$  implies  $X'(0)T(t) = 0$ . Hence  $X'(0) = 0$ . Similarly,  $u_x(L, t) = 0$  implies  $X'(L) = 0$ . We are looking for nontrivial solutions  $X$  of the eigenvalue problem  $X'' + \lambda X = 0$ ,  $X'(0) = 0$ ,  $X'(L) = 0$ . We have previously found that the only eigenvalues are  $\lambda_n = \frac{n^2\pi^2}{L^2}$ , for integers  $n \geq 0$ , where eigenfunctions are  $\cos\left(\frac{n\pi}{L}x\right)$  (we include the constant eigenfunction). Hence, let us pick solutions

$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right) \quad \text{and} \quad X_0(x) = 1.$$

The corresponding  $T_n$  must satisfy the equation

$$T'_n(t) + \frac{n^2\pi^2}{L^2}kT_n(t) = 0.$$

For  $n \geq 1$ , as before,

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt}.$$

For  $n = 0$ , we have  $T'_0(t) = 0$  and hence  $T_0(t) = 1$ . Our building-block solutions will be

$$u_n(x, t) = X_n(x)T_n(t) = \cos\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt},$$

and

$$u_0(x, t) = 1.$$

We note that  $u_n(x, 0) = \cos\left(\frac{n\pi}{L} x\right)$ . Let us write  $f$  using the cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right).$$

That is, we find the Fourier series of the even periodic extension of  $f(x)$ .

We use superposition to write the solution as

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n u_n(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) e^{\frac{-n^2 \pi^2}{L^2} kt}.$$

**Example 4.6.2:** Let us try the same equation as before, but for insulated ends. We are solving the following PDE problem

$$\begin{aligned} u_t &= 0.003 u_{xx}, \\ u_x(0, t) &= u_x(1, t) = 0, \\ u(x, 0) &= 50x(1-x) \quad \text{for } 0 < x < 1. \end{aligned}$$

For this problem, we must find the cosine series of  $u(x, 0)$ . For  $0 < x < 1$  we have

$$50x(1-x) = \frac{25}{3} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left( \frac{-200}{\pi^2 n^2} \right) \cos(n\pi x).$$

The calculation is left to the reader. Hence, the solution to the PDE problem, plotted in Figure 4.17 on the next page, is given by the series

$$u(x, t) = \frac{25}{3} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \left( \frac{-200}{\pi^2 n^2} \right) \cos(n\pi x) e^{-n^2 \pi^2 0.003 t}.$$

Note in the graph that the temperature evens out across the wire. Eventually, all the terms except the constant die out, and you will be left with a uniform temperature of  $\frac{25}{3} \approx 8.33$  along the entire length of the wire.

#### 4.6.4 Exercises

**Exercise 4.6.2:** Suppose you have a wire of length 2, with  $k = 0.001$  and an initial temperature distribution of  $u(x, 0) = 50x$ . Suppose that both the ends are embedded in ice (temperature 0). Find the solution as a series.

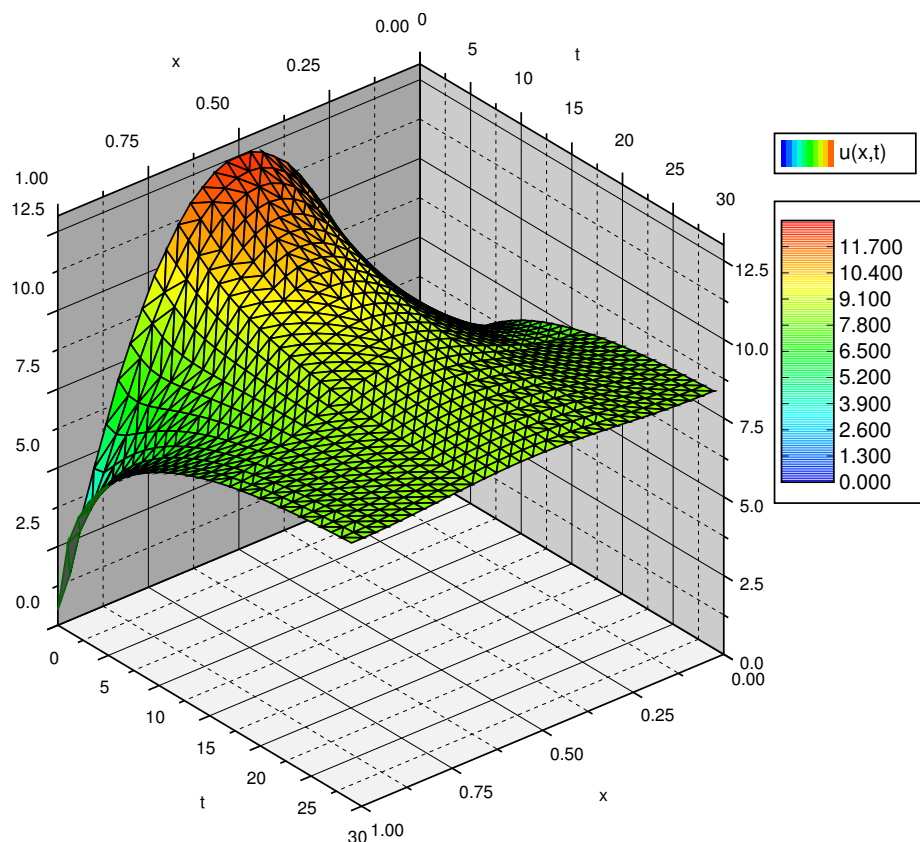


Figure 4.17: Plot of the temperature of the insulated wire at position  $x$  at time  $t$ .

**Exercise 4.6.3:** Find a series solution of

$$\begin{aligned} u_t &= u_{xx}, \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= 100 \quad \text{for } 0 < x < 1. \end{aligned}$$

**Exercise 4.6.4:** Find a series solution of

$$\begin{aligned} u_t &= u_{xx}, \\ u_x(0,t) &= u_x(\pi,t) = 0, \\ u(x,0) &= 3 \cos(x) + \cos(3x) \quad \text{for } 0 < x < \pi. \end{aligned}$$

**Exercise 4.6.5:** Find a series solution of

$$\begin{aligned}u_t &= \frac{1}{3} u_{xx}, \\u_x(0, t) &= u_x(\pi, t) = 0, \\u(x, 0) &= \frac{10x}{\pi} \quad \text{for } 0 < x < \pi.\end{aligned}$$

**Exercise 4.6.6:** Find a series solution of

$$\begin{aligned}u_t &= u_{xx}, \\u(0, t) &= 0, \quad u(1, t) = 100, \\u(x, 0) &= \sin(\pi x) \quad \text{for } 0 < x < 1.\end{aligned}$$

*Hint:* Use the fact that  $u(x, t) = 100x$  is a solution satisfying  $u_t = u_{xx}$ ,  $u(0, t) = 0$ ,  $u(1, t) = 100$ . Then use superposition.

**Exercise 4.6.7:** Find the steady state temperature solution as a function of  $x$  alone, by letting  $t \rightarrow \infty$  in the solution from exercises 4.6.5 and 4.6.6. Verify that it satisfies the equation  $u_{xx} = 0$ .

**Exercise 4.6.8:** Use separation variables to find a nontrivial solution to  $u_{xx} + u_{yy} = 0$ , where  $u(x, 0) = 0$  and  $u(0, y) = 0$ . *Hint:* Try  $u(x, y) = X(x)Y(y)$ .

**Exercise 4.6.9** (challenging): Suppose that one end of the wire is insulated (say at  $x = 0$ ) and the other end is kept at zero temperature. That is, find a series solution of

$$\begin{aligned}u_t &= ku_{xx}, \\u_x(0, t) &= u(L, t) = 0, \\u(x, 0) &= f(x) \quad \text{for } 0 < x < L.\end{aligned}$$

Express any coefficients in the series by integrals of  $f(x)$ .

**Exercise 4.6.10** (challenging): Suppose that the wire is circular and insulated, so there are no ends. You can think of this as simply connecting the two ends and making sure the solution matches up at the ends. That is, find a series solution of

$$\begin{aligned}u_t &= ku_{xx}, \\u(0, t) &= u(L, t), \quad u_x(0, t) = u_x(L, t), \\u(x, 0) &= f(x) \quad \text{for } 0 < x < L.\end{aligned}$$

Express any coefficients in the series by integrals of  $f(x)$ .

**Exercise 4.6.101:** Find a series solution of

$$\begin{aligned}u_t &= 3u_{xx}, \\u(0, t) &= u(\pi, t) = 0, \\u(x, 0) &= 5 \sin(x) + 2 \sin(5x) \quad \text{for } 0 < x < \pi.\end{aligned}$$

**Exercise 4.6.102:** Find a series solution of

$$\begin{aligned}u_t &= 0.1u_{xx}, \\u_x(0, t) &= u_x(\pi, t) = 0, \\u(x, 0) &= 1 + 2 \cos(x) \quad \text{for } 0 < x < \pi.\end{aligned}$$

**Exercise 4.6.103:** Use separation of variables to find a nontrivial solution to  $u_{xt} = u_{xx}$ .

**Exercise 4.6.104:** Use separation of variables (Hint: try  $u(x, t) = X(x) + T(t)$ ) to find a nontrivial solution to  $u_x + u_t = u$ .



## 4.7 One dimensional wave equation

*Note: 1 lecture, §9.6 in [EP], §10.7 in [BD]*

Suppose we have a string such as on a guitar of length  $L$ . Suppose we only consider vibrations in one direction. That is let  $x$  denote the position along the string, let  $t$  denote time and let  $y$  denote the displacement of the string from the rest position. See Figure 4.18.

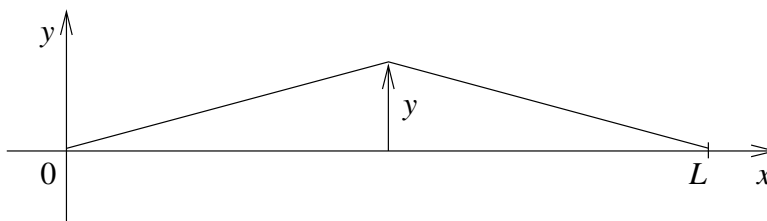


Figure 4.18: Vibrating string.

The equation that governs this setup is the so-called *one-dimensional wave equation*:

$$y_{tt} = a^2 y_{xx},$$

for some  $a > 0$ . We will assume that the ends of the string are fixed and hence we get

$$y(0, t) = 0 \quad \text{and} \quad y(L, t) = 0.$$

Note that we always have two conditions along the  $x$  axis as there are two derivatives in the  $x$  direction.

There are also two derivatives along the  $t$  direction and hence we will need two further conditions here. We will need to know the initial position and the initial velocity of the string.

$$y(x, 0) = f(x) \quad \text{and} \quad y_t(x, 0) = g(x),$$

for some known functions  $f(x)$  and  $g(x)$ .

As the equation is again linear, superposition works just as it did for the heat equation. And again we will use separation of variables to find enough building-block solutions to get the overall solution. There is one change however. It will be easier to solve two separate problems and add their solutions.

The two problems we will solve are

$$\begin{aligned} w_{tt} &= a^2 w_{xx}, \\ w(0, t) &= w(L, t) = 0, \\ w(x, 0) &= 0 & \text{for } 0 < x < L, \\ w_t(x, 0) &= g(x) & \text{for } 0 < x < L. \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
 z_{tt} &= a^2 z_{xx}, \\
 z(0, t) &= z(L, t) = 0, \\
 z(x, 0) &= f(x) && \text{for } 0 < x < L, \\
 z_t(x, 0) &= 0 && \text{for } 0 < x < L.
 \end{aligned} \tag{4.11}$$

The principle of superposition will then imply that  $y = w + z$  solves the wave equation and furthermore  $y(x, 0) = w(x, 0) + z(x, 0) = f(x)$  and  $y_t(x, 0) = w_t(x, 0) + z_t(x, 0) = g(x)$ . Hence,  $y$  is a solution to

$$\begin{aligned}
 y_{tt} &= a^2 y_{xx}, \\
 y(0, t) &= y(L, t) = 0, \\
 y(x, 0) &= f(x) && \text{for } 0 < x < L, \\
 y_t(x, 0) &= g(x) && \text{for } 0 < x < L.
 \end{aligned} \tag{4.12}$$

The reason for all this complexity is that superposition only works for homogeneous conditions such as  $y(0, t) = y(L, t) = 0$ ,  $y(x, 0) = 0$ , or  $y_t(x, 0) = 0$ . Therefore, we will be able to use the idea of separation of variables to find many building-block solutions solving all the homogeneous conditions. We can then use them to construct a solution solving the remaining nonhomogeneous condition.

Let us start with (4.10). We try a solution of the form  $w(x, t) = X(x)T(t)$  again. We plug into the wave equation to obtain

$$X(x)T''(t) = a^2 X''(x)T(t).$$

Rewriting we get

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Again, left hand side depends only on  $t$  and the right hand side depends only on  $x$ . Therefore, both equal a constant, which we will denote by  $-\lambda$ .

$$\frac{T''(t)}{a^2 T(t)} = -\lambda = \frac{X''(x)}{X(x)}.$$

We solve to get two ordinary differential equations

$$\begin{aligned}
 X''(x) + \lambda X(x) &= 0, \\
 T''(t) + \lambda a^2 T(t) &= 0.
 \end{aligned}$$

The conditions  $0 = w(0, t) = X(0)T(t)$  implies  $X(0) = 0$  and  $w(L, t) = 0$  implies that  $X(L) = 0$ . Therefore, the only nontrivial solutions for the first equation are when  $\lambda = \lambda_n = \frac{n^2 \pi^2}{L^2}$  and they are

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right).$$

The general solution for  $T$  for this particular  $\lambda_n$  is

$$T_n(t) = A \cos\left(\frac{n\pi a}{L} t\right) + B \sin\left(\frac{n\pi a}{L} t\right).$$

We also have the condition that  $w(x, 0) = 0$  or  $X(x)T(0) = 0$ . This implies that  $T(0) = 0$ , which in turn forces  $A = 0$ . It will be convenient to pick  $B = \frac{L}{n\pi a}$  (you will see why in a moment) and hence

$$T_n(t) = \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L} t\right).$$

Our building-block solution will be

$$w_n(x, t) = \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi a}{L} t\right).$$

We differentiate in  $t$ , that is

$$(w_n)_t(x, t) = \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right).$$

Hence,

$$(w_n)_t(x, 0) = \sin\left(\frac{n\pi}{L} x\right).$$

We expand  $g(x)$  in terms of these sines as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right).$$

Using superposition we can just write down the solution to (4.10) as a series

$$w(x, t) = \sum_{n=1}^{\infty} b_n w_n(x, t) = \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi a}{L} t\right).$$

**Exercise 4.7.1:** Check that  $w(x, 0) = 0$  and  $w_t(x, 0) = g(x)$ .

Similarly we proceed to solve (4.11). We again try  $z(x, y) = X(x)T(t)$ . The procedure works exactly the same at first. We obtain

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \\ T''(t) + \lambda a^2 T(t) &= 0. \end{aligned}$$

and the conditions  $X(0) = 0$ ,  $X(L) = 0$ . So again  $\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$  and

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right).$$

This time the condition on  $T$  is  $T'(0) = 0$ . Thus we get that  $B = 0$  and we take

$$T_n(t) = \cos\left(\frac{n\pi a}{L} t\right).$$

Our building-block solution will be

$$z_n(x, t) = \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right).$$

We expand  $f(x)$  in terms of these sines as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right).$$

And we write down the solution to (4.11) as a series

$$z(x, t) = \sum_{n=1}^{\infty} c_n z_n(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right).$$

**Exercise 4.7.2:** Fill in the details in the derivation of the solution of (4.11). Check that the solution satisfies all the side conditions.

Putting these two solutions together we will state the result as a theorem.

**Theorem 4.7.1.** Take the equation

$$\begin{aligned} y_{tt} &= a^2 y_{xx}, \\ y(0, t) &= y(L, t) = 0, \\ y(x, 0) &= f(x) && \text{for } 0 < x < L, \\ y_t(x, 0) &= g(x) && \text{for } 0 < x < L, \end{aligned} \tag{4.13}$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right).$$

and

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L} x\right).$$

Then the solution  $y(x, t)$  can be written as a sum of the solutions of (4.10) and (4.11). In other words,

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi a}{L} t\right) + c_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{n\pi a}{L} t\right) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \left[ b_n \frac{L}{n\pi a} \sin\left(\frac{n\pi a}{L} t\right) + c_n \cos\left(\frac{n\pi a}{L} t\right) \right]. \end{aligned}$$

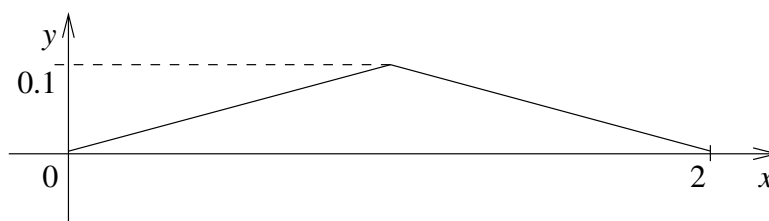


Figure 4.19: Plucked string.

**Example 4.7.1:** Let us try a simple example of a plucked string. Suppose that a string of length 2 is plucked in the middle such that it has the initial shape given in Figure 4.19. That is

$$f(x) = \begin{cases} 0.1x & \text{if } 0 \leq x \leq 1, \\ 0.1(2-x) & \text{if } 1 < x \leq 2. \end{cases}$$

The string starts at rest ( $g(x) = 0$ ). Suppose that  $a = 1$  in the wave equation for simplicity.

We leave it to the reader to compute the sine series of  $f(x)$ . The series will be

$$f(x) = \sum_{n=1}^{\infty} \frac{0.8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}x\right).$$

Note that  $\sin\left(\frac{n\pi}{2}\right)$  is the sequence  $1, 0, -1, 0, 1, 0, -1, \dots$  for  $n = 1, 2, 3, 4, \dots$ . Therefore,

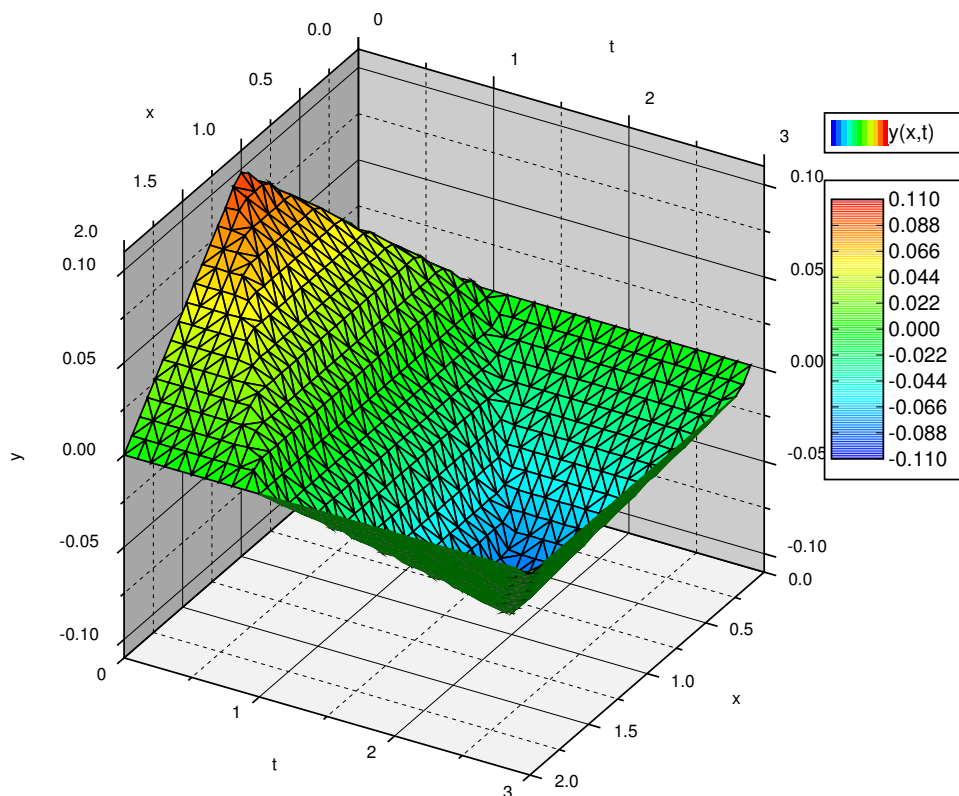
$$f(x) = \frac{0.8}{\pi^2} \sin\left(\frac{\pi}{2}x\right) - \frac{0.8}{9\pi^2} \sin\left(\frac{3\pi}{2}x\right) + \frac{0.8}{25\pi^2} \sin\left(\frac{5\pi}{2}x\right) - \dots$$

The solution  $y(x, t)$  is given by

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} \frac{0.8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}x\right) \cos\left(\frac{n\pi}{2}t\right) \\ &= \sum_{m=1}^{\infty} \frac{0.8(-1)^{m+1}}{(2m-1)^2\pi^2} \sin\left(\frac{(2m-1)\pi}{2}x\right) \cos\left(\frac{(2m-1)\pi}{2}t\right) \\ &= \frac{0.8}{\pi^2} \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}t\right) - \frac{0.8}{9\pi^2} \sin\left(\frac{3\pi}{2}x\right) \cos\left(\frac{3\pi}{2}t\right) + \frac{0.8}{25\pi^2} \sin\left(\frac{5\pi}{2}x\right) \cos\left(\frac{5\pi}{2}t\right) - \dots \end{aligned}$$

A plot for  $0 < t < 3$  is given in Figure 4.20 on the following page. Notice that unlike the heat equation, the solution does not become “smoother,” the “sharp edges” remain. We will see the reason for this behavior in the next section where we derive the solution to the wave equation in a different way.

Make sure you understand what the plot such as the one in the figure is telling you. For each fixed  $t$ , you can think of the function  $y(x, t)$  as just a function of  $x$ . This function gives you the shape of the string at time  $t$ .

Figure 4.20: Shape of the plucked string for  $0 < t < 3$ .

### 4.7.1 Exercises

**Exercise 4.7.3:** Solve

$$\begin{aligned}
 y_{tt} &= 9y_{xx}, \\
 y(0, t) &= y(1, t) = 0, \\
 y(x, 0) &= \sin(3\pi x) + \frac{1}{4} \sin(6\pi x) && \text{for } 0 < x < 1, \\
 y_t(x, 0) &= 0 && \text{for } 0 < x < 1.
 \end{aligned}$$

**Exercise 4.7.4:** Solve

$$\begin{aligned}
 y_{tt} &= 4y_{xx}, \\
 y(0, t) &= y(1, t) = 0, \\
 y(x, 0) &= \sin(3\pi x) + \frac{1}{4} \sin(6\pi x) && \text{for } 0 < x < 1, \\
 y_t(x, 0) &= \sin(9\pi x) && \text{for } 0 < x < 1.
 \end{aligned}$$

**Exercise 4.7.5:** Derive the solution for a general plucked string of length  $L$ , where we raise the string some distance  $b$  at the midpoint and let go, and for any constant  $a$  (in the equation  $y_{tt} = a^2 y_{xx}$ ).

**Exercise 4.7.6:** Suppose that a stringed musical instrument falls on the floor. Suppose that the length of the string is 1 and  $a = 1$ . When the musical instrument hits the ground the string was in rest position and hence  $y(x, 0) = 0$ . However, the string was moving at some velocity at impact ( $t = 0$ ), say  $y_t(x, 0) = -1$ . Find the solution  $y(x, t)$  for the shape of the string at time  $t$ .

**Exercise 4.7.7 (challenging):** Suppose that you have a vibrating string and that there is air resistance proportional to the velocity. That is, you have

$$\begin{aligned} y_{tt} &= a^2 y_{xx} - ky_t, \\ y(0, t) &= y(1, t) = 0, \\ y(x, 0) &= f(x) && \text{for } 0 < x < 1, \\ y_t(x, 0) &= 0 && \text{for } 0 < x < 1. \end{aligned}$$

Suppose that  $0 < k < 2\pi a$ . Derive a series solution to the problem. Any coefficients in the series should be expressed as integrals of  $f(x)$ .

**Exercise 4.7.101:** Solve

$$\begin{aligned} y_{tt} &= y_{xx}, \\ y(0, t) &= y(\pi, t) = 0, \\ y(x, 0) &= \sin(x) && \text{for } 0 < x < \pi, \\ y_t(x, 0) &= \sin(x) && \text{for } 0 < x < \pi. \end{aligned}$$

**Exercise 4.7.102:** Solve

$$\begin{aligned} y_{tt} &= 25y_{xx}, \\ y(0, t) &= y(2, t) = 0, \\ y(x, 0) &= 0 && \text{for } 0 < x < 2, \\ y_t(x, 0) &= \sin(\pi t) + 0.1 \sin(2\pi t) && \text{for } 0 < x < 2. \end{aligned}$$

**Exercise 4.7.103:** Solve

$$\begin{aligned} y_{tt} &= 2y_{xx}, \\ y(0, t) &= y(\pi, t) = 0, \\ y(x, 0) &= x && \text{for } 0 < x < \pi, \\ y_t(x, 0) &= 0 && \text{for } 0 < x < \pi. \end{aligned}$$

**Exercise 4.7.104:** let's see what happens when  $a = 0$ . find a solution to  $y_{tt} = 0$ ,  $y(0, t) = y(\pi, t) = 0$ ,  $y(x, 0) = \sin(2x)$ ,  $y_t(x, 0) = \sin(x)$ .

## 4.8 D'Alembert solution of the wave equation

*Note: 1 lecture, different from §9.6 in [EP], part of §10.7 in [BD]*

We have solved the wave equation by using Fourier series. But it is often more convenient to use the so-called *d'Alembert solution to the wave equation*<sup>‡</sup>. This solution can be derived using Fourier series as well, but it is really an awkward use of those concepts. It is much easier to derive this solution by making a correct change of variables to get an equation that can be solved by simple integration.

Suppose we have the wave equation

$$y_{tt} = a^2 y_{xx}. \quad (4.14)$$

And we wish to solve the equation (4.14) given the conditions

$$\begin{aligned} y(0, t) = y(L, t) &= 0 \quad \text{for all } t, \\ y(x, 0) &= f(x) \quad 0 < x < L, \\ y_t(x, 0) &= g(x) \quad 0 < x < L. \end{aligned} \quad (4.15)$$

### 4.8.1 Change of variables

We will transform the equation into a simpler form where it can be solved by simple integration. We change variables to  $\xi = x - at$ ,  $\eta = x + at$  and we use the chain rule:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} = -a \frac{\partial}{\partial \xi} + a \frac{\partial}{\partial \eta}. \end{aligned}$$

We compute

$$\begin{aligned} y_{xx} &= \frac{\partial^2 y}{\partial x^2} = \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \left( \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta} \right) = \frac{\partial^2 y}{\partial \xi^2} + 2 \frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2}, \\ y_{tt} &= \frac{\partial^2 y}{\partial t^2} = \left( -a \frac{\partial}{\partial \xi} + a \frac{\partial}{\partial \eta} \right) \left( -a \frac{\partial y}{\partial \xi} + a \frac{\partial y}{\partial \eta} \right) = a^2 \frac{\partial^2 y}{\partial \xi^2} - 2a^2 \frac{\partial^2 y}{\partial \xi \partial \eta} + a^2 \frac{\partial^2 y}{\partial \eta^2}. \end{aligned}$$

In the above computations, we have used the fact from calculus that  $\frac{\partial^2 y}{\partial \xi \partial \eta} = \frac{\partial^2 y}{\partial \eta \partial \xi}$ . Then we plug into the wave equation,

$$0 = a^2 y_{xx} - y_{tt} = 4a^2 \frac{\partial^2 y}{\partial \xi \partial \eta} = 4a^2 y_{\xi \eta}.$$

---

<sup>‡</sup>Named after the French mathematician Jean le Rond d'Alembert (1717 – 1783).



Therefore, the wave equation (4.14) transforms into  $y_{\xi\eta} = 0$ . It is easy to find the general solution to this equation by integrating twice. Let us integrate with respect to  $\eta$  first<sup>§</sup> and notice that the constant of integration depends on  $\xi$ . We get  $y_\xi = C(\xi)$ . Next, we integrate with respect to  $\xi$  and notice that the constant of integration must depend on  $\eta$ . Thus,  $y = \int C(\xi) d\xi + B(\eta)$ . The solution must, therefore, be of the following form for some functions  $A(\xi)$  and  $B(\eta)$ :

$$y = A(\xi) + B(\eta) = A(x - at) + B(x + at).$$

### 4.8.2 The formula

We know what any solution must look like, but we need to solve for the given side conditions. We will just give the formula and see that it works. First let  $F(x)$  denote the *odd extension* of  $f(x)$ , and let  $G(x)$  denote the *odd extension* of  $g(x)$ . Now define

$$A(x) = \frac{1}{2}F(x) - \frac{1}{2a} \int_0^x G(s) ds, \quad B(x) = \frac{1}{2}F(x) + \frac{1}{2a} \int_0^x G(s) ds.$$

We claim this  $A(x)$  and  $B(x)$  give the solution. Explicitly, the solution is  $y(x, t) = A(x - at) + B(x + at)$  or in other words:

$$\begin{aligned} y(x, t) &= \frac{1}{2}F(x - at) - \frac{1}{2a} \int_0^{x-at} G(s) ds + \frac{1}{2}F(x + at) + \frac{1}{2a} \int_0^{x+at} G(s) ds \\ &= \frac{F(x - at) + F(x + at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} G(s) ds. \end{aligned} \tag{4.16}$$

Let us check that the d'Alembert formula really works.

$$y(x, 0) = \frac{1}{2}F(x) - \frac{1}{2a} \int_0^x G(s) ds + \frac{1}{2}F(x) + \frac{1}{2a} \int_0^x G(s) ds = F(x).$$

So far so good. Assume for simplicity  $F$  is differentiable. By the fundamental theorem of calculus we have

$$y_t(x, t) = \frac{-a}{2}F'(x - at) + \frac{1}{2}G(x - at) + \frac{a}{2}F'(x + at) + \frac{1}{2}G(x + at).$$

So

$$y_t(x, 0) = \frac{-a}{2}F'(x) + \frac{1}{2}G(x) + \frac{a}{2}F'(x) + \frac{1}{2}G(x) = G(x).$$

Yay! We're smoking now. OK, now the boundary conditions. Note that  $F(x)$  and  $G(x)$  are odd. Also  $\int_0^x G(s) ds$  is an even function of  $x$  because  $G(x)$  is odd (to see this fact, do the substitution

---

<sup>§</sup>We can just as well integrate with  $\xi$  first, if we wish.

$s = -v$ ). So

$$\begin{aligned} y(0, t) &= \frac{1}{2}F(-at) - \frac{1}{2a} \int_0^{-at} G(s) ds + \frac{1}{2}F(at) + \frac{1}{2a} \int_0^{at} G(s) ds \\ &= \frac{-1}{2}F(at) - \frac{1}{2a} \int_0^{at} G(s) ds + \frac{1}{2}F(at) + \frac{1}{2a} \int_0^{at} G(s) ds = 0. \end{aligned}$$

Note that  $F(x)$  and  $G(x)$  are  $2L$  periodic. We compute

$$\begin{aligned} y(L, t) &= \frac{1}{2}F(L - at) - \frac{1}{2a} \int_0^{L-at} G(s) ds + \frac{1}{2}F(L + at) + \frac{1}{2a} \int_0^{L+at} G(s) ds \\ &= \frac{1}{2}F(-L - at) - \frac{1}{2a} \int_0^L G(s) ds - \frac{1}{2a} \int_0^{-at} G(s) ds + \\ &\quad + \frac{1}{2}F(L + at) + \frac{1}{2a} \int_0^L G(s) ds + \frac{1}{2a} \int_0^{at} G(s) ds \\ &= \frac{-1}{2}F(L + at) - \frac{1}{2a} \int_0^{at} G(s) ds + \frac{1}{2}F(L + at) + \frac{1}{2a} \int_0^{at} G(s) ds = 0. \end{aligned}$$

And voilà, it works.

**Example 4.8.1:** What the d'Alembert solution says is that the solution is a superposition of two functions (waves) moving in the opposite direction at “speed”  $a$ . To get an idea of how it works, let us do an example. Suppose that we have the simpler setup

$$\begin{aligned} y_{tt} &= y_{xx}, \\ y(0, t) &= y(1, t) = 0, \\ y(x, 0) &= f(x), \\ y_t(x, 0) &= 0. \end{aligned}$$

Here  $f(x)$  is an impulse of height 1 centered at  $x = 0.5$ :

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 0.45, \\ 20(x - 0.45) & \text{if } 0.45 \leq x < 0.55, \\ 20(0.55 - x) & \text{if } 0.55 \leq x < 0.65, \\ 0 & \text{if } 0.65 \leq x \leq 1. \end{cases}$$

The graph of this pulse is the top left plot in Figure 4.21 on the next page.

Let  $F(x)$  be the odd periodic extension of  $f(x)$ . Then from (4.16) we know that the solution is given as

$$y(x, t) = \frac{F(x - t) + F(x + t)}{2}.$$

It is not hard to compute specific values of  $y(x, t)$ . For example, to compute  $y(0.1, 0.6)$  we notice  $x - t = -0.5$  and  $x + t = 0.7$ . Now  $F(-0.5) = -f(0.5) = -20(0.55 - 0.5) = -1$  and  $F(0.7) = f(0.7) = 0$ . Hence  $y(0.1, 0.6) = \frac{-1+0}{2} = -0.5$ . As you can see the d'Alembert solution is much easier to actually compute and to plot than the Fourier series solution. See Figure 4.21 for plots of the solution  $y$  for several different  $t$ .

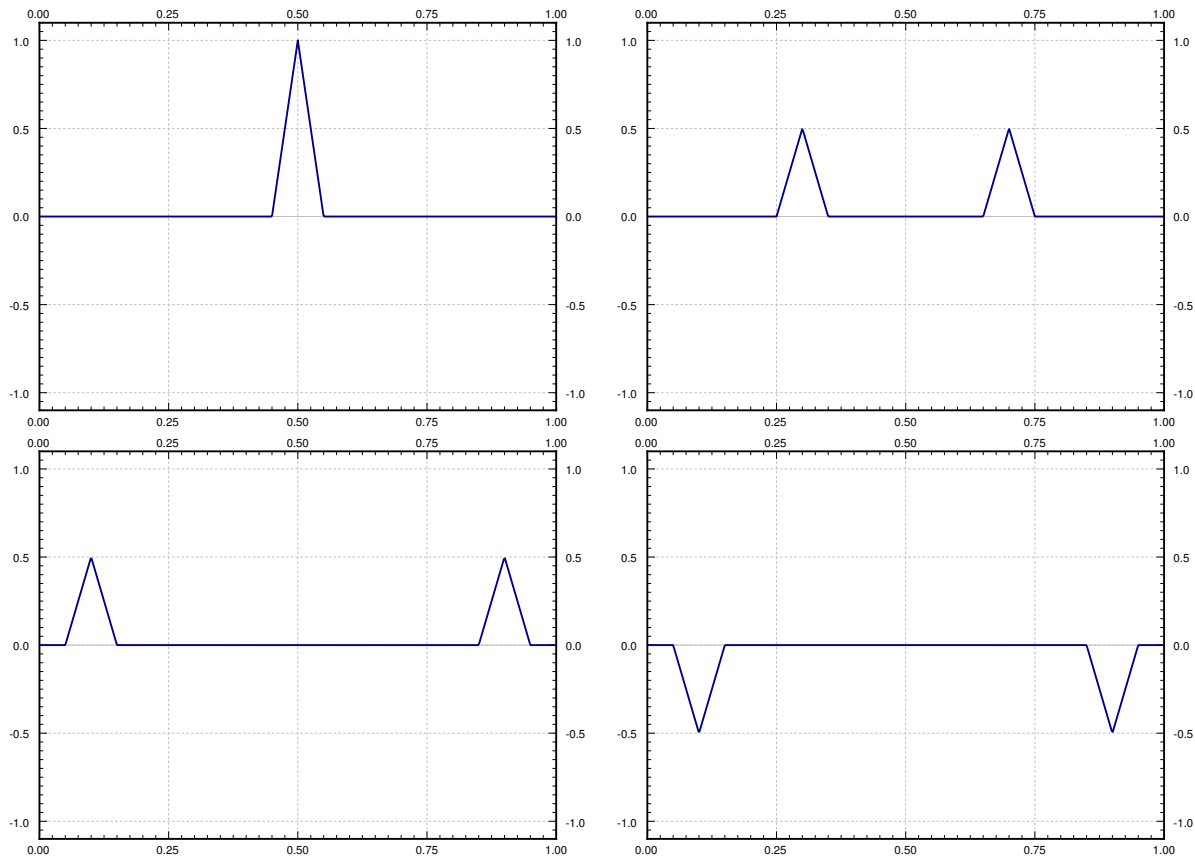


Figure 4.21: Plot of the d'Alembert solution for  $t = 0$ ,  $t = 0.2$ ,  $t = 0.4$ , and  $t = 0.6$ .

### 4.8.3 Notes

It is perhaps easier and more useful to memorize the procedure rather than the formula itself. The important thing to remember is that a solution to the wave equation is a superposition of two waves traveling in opposite directions. That is,

$$y(x, t) = A(x - at) + B(x + at).$$

If you think about it, the exact formulas for  $A$  and  $B$  are not hard to guess once you realize what kind of side conditions  $y(x, t)$  is supposed to satisfy. Let us give the formula again, but slightly differently. Best approach is to do this in stages. When  $g(x) = 0$  (and hence  $G(x) = 0$ ) we have the solution

$$\frac{F(x - at) + F(x + at)}{2}.$$

On the other hand, when  $f(x) = 0$  (and hence  $F(x) = 0$ ), we let

$$H(x) = \int_0^x G(s) ds.$$

The solution in this case is

$$\frac{1}{2a} \int_{x-at}^{x+at} G(s) ds = \frac{-H(x - at) + H(x + at)}{2a}.$$

By superposition we get a solution for the general side conditions (4.15) (when neither  $f(x)$  nor  $g(x)$  are identically zero).

$$y(x, t) = \frac{F(x - at) + F(x + at)}{2} + \frac{-H(x - at) + H(x + at)}{2a}. \quad (4.17)$$

Do note the minus sign before the  $H$ , and the  $a$  in the second denominator.

**Exercise 4.8.1:** Check that the new formula (4.17) satisfies the side conditions (4.15).

**Warning:** Make sure you use the odd extensions  $F(x)$  and  $G(x)$ , when you have formulas for  $f(x)$  and  $g(x)$ . The thing is, those formulas in general hold only for  $0 < x < L$ , and are not usually equal to  $F(x)$  and  $G(x)$  for other  $x$ .

## 4.8.4 Exercises

**Exercise 4.8.2:** Using the d'Alembert solution solve  $y_{tt} = 4y_{xx}$ ,  $0 < x < \pi$ ,  $t > 0$ ,  $y(0, t) = y(\pi, t) = 0$ ,  $y(x, 0) = \sin x$ , and  $y_t(x, 0) = \sin x$ . Hint: Note that  $\sin x$  is the odd extension of  $y(x, 0)$  and  $y_t(x, 0)$ .

**Exercise 4.8.3:** Using the d'Alembert solution solve  $y_{tt} = 2y_{xx}$ ,  $0 < x < 1$ ,  $t > 0$ ,  $y(0, t) = y(1, t) = 0$ ,  $y(x, 0) = \sin^5(\pi x)$ , and  $y_t(x, 0) = \sin^3(\pi x)$ .

**Exercise 4.8.4:** Take  $y_{tt} = 4y_{xx}$ ,  $0 < x < \pi$ ,  $t > 0$ ,  $y(0, t) = y(\pi, t) = 0$ ,  $y(x, 0) = x(\pi - x)$ , and  $y_t(x, 0) = 0$ . a) Solve using the d'Alembert formula (Hint: You can use the sine series for  $y(x, 0)$ .) b) Find the solution as a function of  $x$  for a fixed  $t = 0.5$ ,  $t = 1$ , and  $t = 2$ . Do not use the sine series here.

**Exercise 4.8.5:** Derive the d'Alembert solution for  $y_{tt} = a^2 y_{xx}$ ,  $0 < x < \pi$ ,  $t > 0$ ,  $y(0, t) = y(\pi, t) = 0$ ,  $y(x, 0) = f(x)$ , and  $y_t(x, 0) = 0$ , using the Fourier series solution of the wave equation, by applying an appropriate trigonometric identity.

**Exercise 4.8.6:** The d'Alembert solution still works if there are no boundary conditions and the initial condition is defined on the whole real line. Suppose that  $y_{tt} = y_{xx}$  (for all  $x$  on the real line and  $t \geq 0$ ),  $y(x, 0) = f(x)$ , and  $y_t(x, 0) = 0$ , where

$$f(x) = \begin{cases} 0 & \text{if } x < -1, \\ x + 1 & \text{if } -1 \leq x < 0, \\ -x + 1 & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Solve using the d'Alembert solution. That is, write down a piecewise definition for the solution. Then sketch the solution for  $t = 0$ ,  $t = 1/2$ ,  $t = 1$ , and  $t = 2$ .

**Exercise 4.8.101:** Using the d'Alembert solution solve  $y_{tt} = 9y_{xx}$ ,  $0 < x < 1$ ,  $t > 0$ ,  $y(0, t) = y(1, t) = 0$ ,  $y(x, 0) = \sin(2\pi x)$ , and  $y_t(x, 0) = \sin(3\pi x)$ .

**Exercise 4.8.102:** Take  $y_{tt} = 4y_{xx}$ ,  $0 < x < 1$ ,  $t > 0$ ,  $y(0, t) = y(1, t) = 0$ ,  $y(x, 0) = x - x^2$ , and  $y_t(x, 0) = 0$ . Using the D'Alembert solution find the solution at a)  $t = 0.1$ , b)  $t = 1/2$ , c)  $t = 1$ . (You may have to split up your answer by cases)

**Exercise 4.8.103:** Take  $y_{tt} = 100y_{xx}$ ,  $0 < x < 4$ ,  $t > 0$ ,  $y(0, t) = y(4, t) = 0$ ,  $y(x, 0) = F(x)$ , and  $y_t(x, 0) = 0$ . Suppose that  $F(0) = 0$ ,  $F(1) = 2$ ,  $F(2) = 3$ ,  $F(3) = 1$ . Using the D'Alembert solution find a)  $y(1, 1)$ , b)  $y(4, 3)$ , c)  $y(3, 9)$ .

## 4.9 Steady state temperature and the Laplacian

*Note: 1 lecture, §9.7 in [EP], §10.8 in [BD]*

Suppose we have an insulated wire, a plate, or a 3-dimensional object. We apply certain fixed temperatures on the ends of the wire, the edges of the plate or on all sides of the 3-dimensional object. We wish to find out what is the *steady state temperature* distribution. That is, we wish to know what will be the temperature after long enough period of time.

We are really looking for a solution to the heat equation that is not dependent on time. Let us first do this in one space variable. We are looking for a function  $u$  that satisfies

$$u_t = ku_{xx},$$

but such that  $u_t = 0$  for all  $x$  and  $t$ . Hence, we are looking for a function of  $x$  alone that satisfies  $u_{xx} = 0$ . It is easy to solve this equation by integration and we see that  $u = Ax + B$  for some constants  $A$  and  $B$ .

Suppose we have an insulated wire, and we apply constant temperature  $T_1$  at one end (say where  $x = 0$ ) and  $T_2$  on the other end (at  $x = L$  where  $L$  is the length of the wire). Then our steady state solution is

$$u(x) = \frac{T_2 - T_1}{L}x + T_1.$$

This solution agrees with our common sense intuition with how the heat should be distributed in the wire. So in one dimension, the steady state solutions are basically just straight lines.

Things are more complicated in two or more space dimensions. Let us restrict to two space dimensions for simplicity. The heat equation in two variables is

$$u_t = k(u_{xx} + u_{yy}), \quad (4.18)$$

or more commonly written as  $u_t = k\Delta u$  or  $u_t = k\nabla^2 u$ . Here the  $\Delta$  and  $\nabla^2$  symbols mean  $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ . We will use  $\Delta$  from now on. The reason for that notation is that you can define  $\Delta$  to be the right thing for any number of space dimensions and then the heat equation is always  $u_t = k\Delta u$ . The  $\Delta$  is called the *Laplacian*.

OK, now that we have notation out of the way, let us see what does an equation for the steady state solution look like. We are looking for a solution to (4.18) that does not depend on  $t$ . Hence we are looking for a function  $u(x, y)$  such that

$$\Delta u = u_{xx} + u_{yy} = 0.$$

This equation is called the *Laplace equation*<sup>¶</sup>. Solutions to the Laplace equation are called *harmonic functions* and have many nice properties and applications far beyond the steady state heat problem.

<sup>¶</sup>Named after the French mathematician Pierre-Simon, marquis de Laplace (1749 – 1827).

Harmonic functions in two variables are no longer just linear (plane graphs). For example, you can check that the functions  $x^2 - y^2$  and  $xy$  are harmonic. However, if you remember your multi-variable calculus we note that if  $u_{xx}$  is positive,  $u$  is concave up in the  $x$  direction, then  $u_{yy}$  must be negative and  $u$  must be concave down in the  $y$  direction. Therefore, a harmonic function can never have any “hilltop” or “valley” on the graph. This observation is consistent with our intuitive idea of steady state heat distribution.

Commonly the Laplace equation is part of a so-called *Dirichlet problem*<sup>||</sup>. That is, we have some region in the  $xy$ -plane and we specify certain values along the boundaries of the region. We then try to find a solution  $u$  defined on this region such that  $u$  agrees with the values we specified on the boundary.

For simplicity, we will consider a rectangular region. Also for simplicity we will specify boundary values to be zero at 3 of the four edges and only specify an arbitrary function at one edge. As we still have the principle of superposition, you can use this simpler solution to derive the general solution for arbitrary boundary values by solving 4 different problems, one for each edge, and adding those solutions together. This setup is left as an exercise.

We wish to solve the following problem. Let  $h$  and  $w$  be the height and width of our rectangle, with one corner at the origin and lying in the first quadrant.

$$\begin{aligned} \Delta u &= 0, & (4.19) \\ u(0, y) &= 0 \text{ for } 0 < y < h, & (4.20) \\ u(x, h) &= 0 \text{ for } 0 < x < w, & (4.21) \\ u(w, y) &= 0 \text{ for } 0 < y < h, & (4.22) \\ u(x, 0) &= f(x) \text{ for } 0 < x < w. & (4.23) \end{aligned}$$

The method we will apply is separation of variables. Again, we will come up with enough building-block solutions satisfying all the homogeneous boundary conditions (all conditions except (4.23)). We notice that superposition still works for the equation and all the homogeneous conditions. Therefore, we can use the Fourier series for  $f(x)$  to solve the problem as before.

We try  $u(x, y) = X(x)Y(y)$ . We plug  $u$  into the equation to get

$$X''Y + XY'' = 0.$$

We put the  $X$ s on one side and the  $Y$ s on the other to get

$$-\frac{X''}{X} = \frac{Y''}{Y}.$$

<sup>||</sup>Named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805 – 1859).

The left hand side only depends on  $x$  and the right hand side only depends on  $y$ . Therefore, there is some constant  $\lambda$  such that  $\lambda = \frac{-X''}{X} = \frac{Y''}{Y}$ . And we get two equations

$$\begin{aligned} X'' + \lambda X &= 0, \\ Y'' - \lambda Y &= 0. \end{aligned}$$

Furthermore, the homogeneous boundary conditions imply that  $X(0) = X(w) = 0$  and  $Y(h) = 0$ . Taking the equation for  $X$  we have already seen that we have a nontrivial solution if and only if  $\lambda = \lambda_n = \frac{n^2\pi^2}{w^2}$  and the solution is a multiple of

$$X_n(x) = \sin\left(\frac{n\pi}{w}x\right).$$

For these given  $\lambda_n$ , the general solution for  $Y$  (one for each  $n$ ) is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi}{w}y\right) + B_n \sinh\left(\frac{n\pi}{w}y\right). \quad (4.24)$$

We only have one condition on  $Y_n$  and hence we can pick one of  $A_n$  or  $B_n$  to be something convenient. It will be useful to have  $Y_n(0) = 1$ , so we let  $A_n = 1$ . Setting  $Y_n(h) = 0$  and solving for  $B_n$  we get that

$$B_n = \frac{-\cosh\left(\frac{n\pi h}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}.$$

After we plug the  $A_n$  and  $B_n$  we into (4.24) and simplify, we find

$$Y_n(y) = \frac{\sinh\left(\frac{n\pi(h-y)}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)}.$$

We define  $u_n(x, y) = X_n(x)Y_n(y)$ . And note that  $u_n$  satisfies (4.19)–(4.22).

Observe that

$$u_n(x, 0) = X_n(x)Y_n(0) = \sin\left(\frac{n\pi}{w}x\right).$$

Suppose

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{w}\right).$$

Then we get a solution of (4.19)–(4.23) of the following form.

$$u(x, y) = \sum_{n=1}^{\infty} b_n u_n(x, y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{w}x\right) \left( \frac{\sinh\left(\frac{n\pi(h-y)}{w}\right)}{\sinh\left(\frac{n\pi h}{w}\right)} \right).$$

As  $u_n$  satisfies (4.19)–(4.22) and any linear combination (finite or infinite) of  $u_n$  must also satisfy (4.19)–(4.22), we see that  $u$  must satisfy (4.19)–(4.22). By plugging in  $y = 0$  it is easy to see that  $u$  satisfies (4.23) as well.



**Example 4.9.1:** Suppose that we take  $w = h = \pi$  and we let  $f(x) = \pi$ . We compute the sine series for the function  $\pi$  (we will get the square wave). We find that for  $0 < x < \pi$  we have

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n} \sin(nx).$$

Therefore the solution  $u(x, y)$ , see Figure 4.22, to the corresponding Dirichlet problem is given as

$$u(x, y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n} \sin(nx) \left( \frac{\sinh(n(\pi - y))}{\sinh(n\pi)} \right).$$

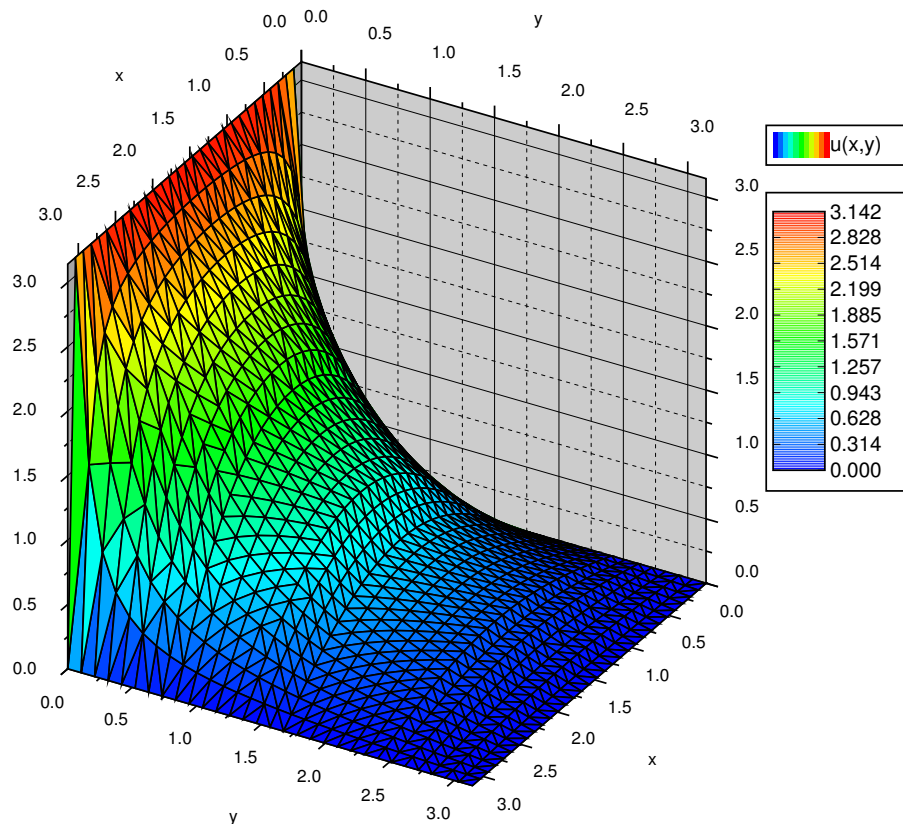


Figure 4.22: Steady state temperature of a square plate with three sides held at zero and one side held at  $\pi$ .

This scenario corresponds to the steady state temperature on a square plate of width  $\pi$  with 3 sides held at 0 degrees and one side held at  $\pi$  degrees. If we have arbitrary initial data on all sides,

then we solve four problems, each using one piece of nonhomogeneous data. Then we use the principle of superposition to add up all four solutions to have a solution to the original problem.

There is another way to visualize the solutions of the Laplace equation. Take a wire and bend it in just the right way so that it corresponds to the graph of the temperature above the boundary of your region. Then dip the wire in soapy water and let it form a soapy film stretched between the edges of the wire. It turns out that this soap film is precisely the graph of the solution to the Laplace equation. Harmonic functions come up frequently in problems when we are trying to minimize area of some surface or minimize energy in some system.

### 4.9.1 Exercises

**Exercise 4.9.1:** Let  $R$  be the region described by  $0 < x < \pi$  and  $0 < y < \pi$ . Solve the problem

$$\Delta u = 0, \quad u(x, 0) = \sin x, \quad u(x, \pi) = 0, \quad u(0, y) = 0, \quad u(\pi, y) = 0.$$

**Exercise 4.9.2:** Let  $R$  be the region described by  $0 < x < 1$  and  $0 < y < 1$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= \sin(\pi x) - \sin(2\pi x), \quad u(x, 1) = 0, \\ u(0, y) &= 0, \quad u(1, y) = 0. \end{aligned}$$

**Exercise 4.9.3:** Let  $R$  be the region described by  $0 < x < 1$  and  $0 < y < 1$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= u(x, 1) = u(0, y) = u(1, y) = C. \end{aligned}$$

for some constant  $C$ . Hint: Guess, then check your intuition.

**Exercise 4.9.4:** Let  $R$  be the region described by  $0 < x < \pi$  and  $0 < y < \pi$ . Solve

$$\Delta u = 0, \quad u(x, 0) = 0, \quad u(x, \pi) = \pi, \quad u(0, y) = y, \quad u(\pi, y) = y.$$

Hint: Try a solution of the form  $u(x, y) = X(x) + Y(y)$  (different separation of variables).

**Exercise 4.9.5:** Use the solution of Exercise 4.9.4 to solve

$$\Delta u = 0, \quad u(x, 0) = \sin x, \quad u(x, \pi) = \pi, \quad u(0, y) = y, \quad u(\pi, y) = y.$$

Hint: Use superposition.

**Exercise 4.9.6:** Let  $R$  be the region described by  $0 < x < w$  and  $0 < y < h$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= 0, \quad u(x, h) = f(x), \\ u(0, y) &= 0, \quad u(w, y) = 0. \end{aligned}$$

The solution should be in series form using the Fourier series coefficients of  $f(x)$ .

**Exercise 4.9.7:** Let  $R$  be the region described by  $0 < x < w$  and  $0 < y < h$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= 0, \quad u(x, h) = 0, \\ u(0, y) &= f(y), \quad u(w, y) = 0. \end{aligned}$$

The solution should be in series form using the Fourier series coefficients of  $f(y)$ .

**Exercise 4.9.8:** Let  $R$  be the region described by  $0 < x < w$  and  $0 < y < h$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= 0, \quad u(x, h) = 0, \\ u(0, y) &= 0, \quad u(w, y) = f(y). \end{aligned}$$

The solution should be in series form using the Fourier series coefficients of  $f(y)$ .

**Exercise 4.9.9:** Let  $R$  be the region described by  $0 < x < 1$  and  $0 < y < 1$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= \sin(9\pi x), \quad u(x, 1) = \sin(2\pi x), \\ u(0, y) &= 0, \quad u(1, y) = 0. \end{aligned}$$

*Hint: Use superposition.*

**Exercise 4.9.10:** Let  $R$  be the region described by  $0 < x < 1$  and  $0 < y < 1$ . Solve the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \\ u(x, 0) &= \sin(\pi x), \quad u(x, 1) = \sin(\pi x), \\ u(0, y) &= \sin(\pi y), \quad u(1, y) = \sin(\pi y). \end{aligned}$$

*Hint: Use superposition.*

**Exercise 4.9.101:** Let  $R$  be the region described by  $0 < x < 1$  and  $0 < y < 1$ . Solve the problem

$$\Delta u = 0, \quad u(x, 0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x), \quad u(x, 1) = 0, \quad u(0, y) = 0, \quad u(1, y) = 0.$$

**Exercise 4.9.102:** Let  $R$  be the region described by  $0 < x < 1$  and  $0 < y < 2$ . Solve the problem

$$\Delta u = 0, \quad u(x, 0) = 0.1 \sin(\pi x), \quad u(x, 2) = 0, \quad u(0, y) = 0, \quad u(1, y) = 0.$$



# Chapter 5

## Eigenvalue problems

### 5.1 Sturm-Liouville problems

*Note: 2 lectures, §10.1 in [EP], §11.2 in [BD]*

#### 5.1.1 Boundary value problems

We have encountered several different eigenvalue problems such as:

$$X''(x) + \lambda X(x) = 0$$

with different boundary conditions

$$\begin{array}{lll} X(0) = 0 & X(L) = 0 & \text{(Dirichlet) or,} \\ X'(0) = 0 & X'(L) = 0 & \text{(Neumann) or,} \\ X'(0) = 0 & X(L) = 0 & \text{(Mixed) or,} \\ X(0) = 0 & X'(L) = 0 & \text{(Mixed), \dots} \end{array}$$

For example for the insulated wire, Dirichlet conditions correspond to applying a zero temperature at the ends, Neumann means insulating the ends, etc. . . . Other types of endpoint conditions also arise naturally, such as

$$hX(0) - X'(0) = 0 \quad hX(L) + X'(L) = 0,$$

for some constant  $h$ .

These problems came up, for example, in the study of the heat equation  $u_t = ku_{xx}$  when we were trying to solve the equation by the method of separation of variables. In the computation we encountered a certain eigenvalue problem and found the eigenfunctions  $X_n(x)$ . We then found the *eigenfunction decomposition* of the initial temperature  $f(x) = u(x, 0)$  in terms of the eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x).$$

Once we had this decomposition and once we found suitable  $T_n(t)$  such that  $T_n(0) = 1$ , we noted that a solution to the original problem could be written as

$$u(x, t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x).$$

We will try to solve more general problems using this method. First, we will study second order linear equations of the form

$$\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) - q(x)y + \lambda r(x)y = 0. \quad (5.1)$$

Essentially any second order linear equation of the form  $a(x)y'' + b(x)y' + c(x)y + \lambda d(x)y = 0$  can be written as (5.1) after multiplying by a proper factor.

**Example 5.1.1** (Bessel): Put the following equation into the form (5.1):

$$x^2 y'' + xy' + (\lambda x^2 - n^2)y = 0.$$

Multiply both sides by  $\frac{1}{x}$  to obtain

$$\frac{1}{x} (x^2 y'' + xy' + (\lambda x^2 - n^2)y) = xy'' + y' + \left( \lambda x - \frac{n^2}{x} \right) y = \frac{d}{dx} \left( x \frac{dy}{dx} \right) - \frac{n^2}{x} y + \lambda xy = 0.$$

We can state the general *Sturm-Liouville problem*\*. We seek nontrivial solutions to

$$\boxed{\begin{aligned} \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) - q(x)y + \lambda r(x)y &= 0, & a < x < b, \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0. \end{aligned}} \quad (5.2)$$

In particular, we seek  $\lambda$ s that allow for nontrivial solutions. The  $\lambda$ s for which there are nontrivial solutions are called the *eigenvalues* and the corresponding nontrivial solutions are called *eigenfunctions*. The constants  $\alpha_1$  and  $\alpha_2$  should not be both zero, same for  $\beta_1$  and  $\beta_2$ .

**Theorem 5.1.1.** Suppose  $p(x)$ ,  $p'(x)$ ,  $q(x)$  and  $r(x)$  are continuous on  $[a, b]$  and suppose  $p(x) > 0$  and  $r(x) > 0$  for all  $x$  in  $[a, b]$ . Then the Sturm-Liouville problem (5.2) has an increasing sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

such that

$$\lim_{n \rightarrow \infty} \lambda_n = +\infty$$

and such that to each  $\lambda_n$  there is (up to a constant multiple) a single eigenfunction  $y_n(x)$ .

Moreover, if  $q(x) \geq 0$  and  $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ , then  $\lambda_n \geq 0$  for all  $n$ .

---

\*Named after the French mathematicians Jacques Charles François Sturm (1803 – 1855) and Joseph Liouville (1809 – 1882).

Note: Be careful about the signs. Also be careful about the inequalities for  $r$  and  $p$ , they must be strict for all  $x$ ! Problems satisfying the hypothesis of the theorem are called *regular Sturm-Liouville problems* and we will only consider such problems here. That is, a regular problem is one where  $p(x)$ ,  $p'(x)$ ,  $q(x)$  and  $r(x)$  are continuous,  $p(x) > 0$ ,  $r(x) > 0$ ,  $q(x) \geq 0$ , and  $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ .

When zero is an eigenvalue, we will usually start labeling the eigenvalues at 0 rather than 1 for convenience.

**Example 5.1.2:** The problem  $y'' + \lambda y = 0$ ,  $0 < x < L$ ,  $y(0) = 0$ , and  $y(L) = 0$  is a regular Sturm-Liouville problem.  $p(x) = 1$ ,  $q(x) = 0$ ,  $r(x) = 1$ , and we have  $p(x) = 1 > 0$  and  $r(x) = 1 > 0$ . The eigenvalues are  $\lambda_n = \frac{n^2\pi^2}{L^2}$  and eigenfunctions are  $y_n(x) = \sin(\frac{n\pi}{L}x)$ . All eigenvalues are nonnegative as predicted by the theorem.

**Exercise 5.1.1:** Find eigenvalues and eigenfunctions for

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0.$$

Identify the  $p, q, r, \alpha_j, \beta_j$ . Can you use the theorem to make the search for eigenvalues easier? (Hint: Consider the condition  $-y'(0) = 0$ )

**Example 5.1.3:** Find eigenvalues and eigenfunctions of the problem

$$\begin{aligned} y'' + \lambda y &= 0, & 0 < x < 1, \\ hy(0) - y'(0) &= 0, & y'(1) = 0, & h > 0. \end{aligned}$$

These equations give a regular Sturm-Liouville problem.

**Exercise 5.1.2:** Identify  $p, q, r, \alpha_j, \beta_j$  in the example above.

First note that  $\lambda \geq 0$  by Theorem 5.1.1. Therefore, the general solution (without boundary conditions) is

$$\begin{aligned} y(x) &= A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x) & \text{if } \lambda > 0, \\ y(x) &= Ax + B & \text{if } \lambda = 0. \end{aligned}$$

Let us see if  $\lambda = 0$  is an eigenvalue: We must satisfy  $0 = hB - A$  and  $A = 0$ , hence  $B = 0$  (as  $h > 0$ ), therefore, 0 is not an eigenvalue (no eigenfunction).

Now let us try  $\lambda > 0$ . We plug in the boundary conditions.

$$\begin{aligned} 0 &= hA - \sqrt{\lambda} B, \\ 0 &= -A \sqrt{\lambda} \sin(\sqrt{\lambda}) + B \sqrt{\lambda} \cos(\sqrt{\lambda}). \end{aligned}$$

Note that if  $A = 0$ , then  $B = 0$  and vice-versa, hence both are nonzero. So  $B = \frac{hA}{\sqrt{\lambda}}$ , and  $0 = -A \sqrt{\lambda} \sin(\sqrt{\lambda}) + \frac{hA}{\sqrt{\lambda}} \sqrt{\lambda} \cos(\sqrt{\lambda})$ . As  $A \neq 0$  we get

$$0 = -\sqrt{\lambda} \sin(\sqrt{\lambda}) + h \cos(\sqrt{\lambda}),$$

or

$$\frac{h}{\sqrt{\lambda}} = \tan \sqrt{\lambda}.$$

Now use a computer to find  $\lambda_n$ . There are tables available, though using a computer or a graphing calculator will probably be far more convenient nowadays. Easiest method is to plot the functions  $h/x$  and  $\tan x$  and see for which  $x$  they intersect. There will be an infinite number of intersections. So denote by  $\sqrt{\lambda_1}$  the first intersection, by  $\sqrt{\lambda_2}$  the second intersection, etc. . . . For example, when  $h = 1$ , we get that  $\lambda_1 \approx 0.86$ , and  $\lambda_2 \approx 3.43$ . A plot for  $h = 1$  is given in Figure 5.1. The appropriate eigenfunction (let  $A = 1$  for convenience, then  $B = h/\sqrt{\lambda}$ ) is

$$y_n(x) = \cos(\sqrt{\lambda_n} x) + \frac{h}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n} x).$$

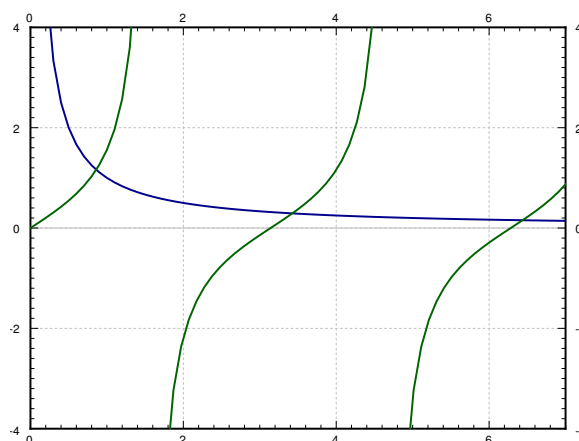


Figure 5.1: Plot of  $\frac{1}{x}$  and  $\tan x$ .

### 5.1.2 Orthogonality

We have seen the notion of orthogonality before. For example, we have shown that  $\sin(nx)$  are orthogonal for distinct  $n$  on  $[0, \pi]$ . For general Sturm-Liouville problems we will need a more general setup. Let  $r(x)$  be a *weight function* (any function, though generally we will assume it is positive) on  $[a, b]$ . Then two functions  $f(x)$ ,  $g(x)$  are said to be *orthogonal* with respect to the weight function  $r(x)$  when

$$\int_a^b f(x) g(x) r(x) dx = 0.$$



In this setting, we define the *inner product* as

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_a^b f(x) g(x) r(x) dx,$$

and then say  $f$  and  $g$  are orthogonal whenever  $\langle f, g \rangle = 0$ . The results and concepts are again analogous to finite dimensional linear algebra.

The idea of the given inner product is that those  $x$  where  $r(x)$  is greater have more weight. Nontrivial (nonconstant)  $r(x)$  arise naturally, for example from a change of variables. Hence, you could think of a change of variables such that  $d\xi = r(x) dx$ .

We have the following orthogonality property of eigenfunctions of a regular Sturm-Liouville problem.

**Theorem 5.1.2.** *Suppose we have a regular Sturm-Liouville problem*

$$\begin{aligned} \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) - q(x)y + \lambda r(x)y &= 0, \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0. \end{aligned}$$

*Let  $y_j$  and  $y_k$  be two distinct eigenfunctions for two distinct eigenvalues  $\lambda_j$  and  $\lambda_k$ . Then*

$$\int_a^b y_j(x) y_k(x) r(x) dx = 0,$$

*that is,  $y_j$  and  $y_k$  are orthogonal with respect to the weight function  $r$ .*

Proof is very similar to the analogous theorem from § 4.1. It can also be found in many books including, for example, Edwards and Penney [EP].

### 5.1.3 Fredholm alternative

We also have the *Fredholm alternative* theorem we talked about before for all regular Sturm-Liouville problems. We state it here for completeness.

**Theorem 5.1.3** (Fredholm alternative). *Suppose that we have a regular Sturm-Liouville problem. Then either*

$$\begin{aligned} \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) - q(x)y + \lambda r(x)y &= 0, \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0, \end{aligned}$$

has a nonzero solution, or

$$\begin{aligned}\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) - q(x)y + \lambda r(x)y &= f(x), \\ \alpha_1 y(a) - \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0,\end{aligned}$$

has a unique solution for any  $f(x)$  continuous on  $[a, b]$ .

This theorem is used in much the same way as we did before in § 4.4. It is used when solving more general nonhomogeneous boundary value problems. The theorem does not help us solve the problem, but it tells us when a solution exists and when it exists if it is unique, so that we know when to spend time looking for a solution. To solve the problem we decompose  $f(x)$  and  $y(x)$  in terms of the eigenfunctions of the homogeneous problem, and then solve for the coefficients of the series for  $y(x)$ .

### 5.1.4 Eigenfunction series

What we want to do with the eigenfunctions once we have them is to compute the *eigenfunction decomposition* of an arbitrary function  $f(x)$ . That is, we wish to write

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad (5.3)$$

where  $y_n(x)$  the eigenfunctions. We wish to find out if we can represent any function  $f(x)$  in this way, and if so, we wish to calculate  $c_n$  (and of course we would want to know if the sum converges). OK, so imagine we could write  $f(x)$  as (5.3). We will assume convergence and the ability to integrate the series term by term. Because of orthogonality we have

$$\begin{aligned}\langle f, y_m \rangle &= \int_a^b f(x) y_m(x) r(x) dx \\ &= \sum_{n=1}^{\infty} c_n \int_a^b y_n(x) y_m(x) r(x) dx \\ &= c_m \int_a^b y_m(x) y_m(x) r(x) dx = c_m \langle y_m, y_m \rangle.\end{aligned}$$

Hence,

$$c_m = \frac{\langle f, y_m \rangle}{\langle y_m, y_m \rangle} = \frac{\int_a^b f(x) y_m(x) r(x) dx}{\int_a^b (y_m(x))^2 r(x) dx}. \quad (5.4)$$

Note that  $y_m$  are known up to a constant multiple, so we could have picked a scalar multiple of an eigenfunction such that  $\langle y_m, y_m \rangle = 1$  (if we had an arbitrary eigenfunction  $\tilde{y}_m$ , divide it by  $\sqrt{\langle \tilde{y}_m, \tilde{y}_m \rangle}$ ). In the case that  $\langle y_m, y_m \rangle = 1$  we would have the simpler form  $c_m = \langle f, y_m \rangle$  as we essentially did for the Fourier series. The following theorem holds more generally, but the statement given is enough for our purposes.

**Theorem 5.1.4.** *Suppose  $f$  is a piecewise smooth continuous function on  $[a, b]$ . If  $y_1, y_2, \dots$  are the eigenfunctions of a regular Sturm-Liouville problem, then there exist real constants  $c_1, c_2, \dots$  given by (5.4) such that (5.3) converges and holds for  $a < x < b$ .*

**Example 5.1.4:** Take the simple Sturm-Liouville problem

$$\begin{aligned} y'' + \lambda y &= 0, \quad 0 < x < \frac{\pi}{2}, \\ y(0) &= 0, \quad y'\left(\frac{\pi}{2}\right) = 0. \end{aligned}$$

The above is a regular problem and furthermore we actually know by Theorem 5.1.1 on page 222 that  $\lambda \geq 0$ .

Suppose  $\lambda = 0$ , then the general solution is  $y(x) = Ax + B$ , we plug in the initial conditions to get  $0 = y(0) = B$ , and  $0 = y'(\frac{\pi}{2}) = A$ , hence  $\lambda = 0$  is not an eigenvalue.

The general solution, therefore, is

$$y(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x).$$

Plugging in the boundary conditions we get  $0 = y(0) = A$  and  $0 = y'(\frac{\pi}{2}) = \sqrt{\lambda} B \cos(\sqrt{\lambda} \frac{\pi}{2})$ .  $B$  cannot be zero and hence  $\cos(\sqrt{\lambda} \frac{\pi}{2}) = 0$ . This means that  $\sqrt{\lambda} \frac{\pi}{2}$  must be an odd integral multiple of  $\frac{\pi}{2}$ , i.e.  $(2n-1)\frac{\pi}{2} = \sqrt{\lambda_n} \frac{\pi}{2}$ . Hence

$$\lambda_n = (2n-1)^2.$$

We can take  $B = 1$ . And hence our eigenfunctions are

$$y_n(x) = \sin((2n-1)x).$$

We finally compute

$$\int_0^{\frac{\pi}{2}} (\sin((2n-1)x))^2 dx = \frac{\pi}{4}.$$

So any piecewise smooth function on  $[0, \frac{\pi}{2}]$  can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin((2n-1)x),$$

where

$$c_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\int_0^{\frac{\pi}{2}} f(x) \sin((2n-1)x) dx}{\int_0^{\frac{\pi}{2}} (\sin((2n-1)x))^2 dx} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} f(x) \sin((2n-1)x) dx.$$

Note that the series converges to an odd  $2\pi$ -periodic (not  $\pi$ -periodic!) extension of  $f(x)$ .

**Exercise 5.1.3** (challenging): In the above example, the function is defined on  $0 < x < \frac{\pi}{2}$ , yet the series converges to an odd  $2\pi$ -periodic extension of  $f(x)$ . Find out how is the extension defined for  $\frac{\pi}{2} < x < \pi$ .

### 5.1.5 Exercises

**Exercise 5.1.4:** Find eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) - y'(0) = 0, \quad y(1) = 0.$$

**Exercise 5.1.5:** Expand the function  $f(x) = x$  on  $0 \leq x \leq 1$  using the eigenfunctions of the system

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0.$$

**Exercise 5.1.6:** Suppose that you had a Sturm-Liouville problem on the interval  $[0, 1]$  and came up with  $y_n(x) = \sin(\gamma_n x)$ , where  $\gamma > 0$  is some constant. Decompose  $f(x) = x$ ,  $0 < x < 1$  in terms of these eigenfunctions.

**Exercise 5.1.7:** Find eigenvalues and eigenfunctions of

$$y^{(4)} + \lambda y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y(1) = 0, \quad y'(1) = 0.$$

This problem is not a Sturm-Liouville problem, but the idea is the same.

**Exercise 5.1.8** (more challenging): Find eigenvalues and eigenfunctions for

$$\frac{d}{dx}(e^x y') + \lambda e^x y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

*Hint:* First write the system as a constant coefficient system to find general solutions. Do note that Theorem 5.1.1 on page 222 guarantees  $\lambda \geq 0$ .

**Exercise 5.1.101:** Find eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(-1) = 0, \quad y(1) = 0.$$

**Exercise 5.1.102:** Put the following problems into the standard form for Sturm-Liouville problems, that is, find  $p(x)$ ,  $q(x)$ ,  $r(x)$ ,  $\alpha_1$ ,  $\alpha_2$ ,  $\beta_1$ , and  $\beta_2$ , and decide if the problems are regular or not.

a)  $xy'' + \lambda y = 0$  for  $0 < x < 1$ ,  $y(0) = 0$ ,  $y(1) = 0$ ,

b)  $(1 + x^2)y'' - 2xy' + (\lambda - x^2)y = 0$  for  $-1 < x < 1$ ,  $y(-1) = 0$ ,  $y(1) + y'(1) = 0$ .

## 5.2 Application of eigenfunction series

*Note: 1 lecture, §10.2 in [EP], exercises in §11.2 in [BD]*

The eigenfunction series can arise even from higher order equations. Suppose we have an elastic beam (say made of steel). We will study the transversal vibrations of the beam. That is, assume the beam lies along the  $x$  axis and let  $y(x, t)$  measure the displacement of the point  $x$  on the beam at time  $t$ . See Figure 5.2.

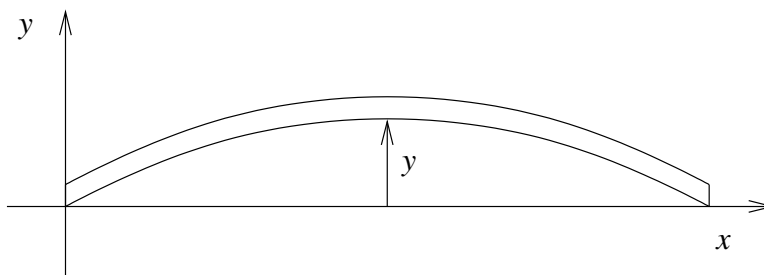


Figure 5.2: Transversal vibrations of a beam.

The equation that governs this setup is

$$a^4 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

for some constant  $a > 0$ .

Suppose the beam is of length 1 simply supported (hinged) at the ends. Suppose the beam is displaced by some function  $f(x)$  at time  $t = 0$  and then let go (initial velocity is 0). Then  $y$  satisfies:

$$\begin{aligned} a^4 y_{xxxx} + y_{tt} &= 0 \quad (0 < x < 1, t > 0), \\ y(0, t) &= y_{xx}(0, t) = 0, \\ y(1, t) &= y_{xx}(1, t) = 0, \\ y(x, 0) &= f(x), \quad y_t(x, 0) = 0. \end{aligned} \tag{5.5}$$

Again we try  $y(x, t) = X(x)T(t)$  and plug in to get  $a^4 X^{(4)}T + XT'' = 0$  or

$$\frac{X^{(4)}}{X} = \frac{-T''}{a^4 T} = \lambda.$$

We note that we want  $T'' + \lambda a^4 T = 0$ . Let us assume that  $\lambda > 0$ . We can argue that we expect vibration and not exponential growth nor decay in the  $t$  direction (there is no friction in our model for instance). Similarly  $\lambda = 0$  will not occur.

**Exercise 5.2.1:** Try to justify  $\lambda > 0$  just from the equations.

Write  $\omega^4 = \lambda$ , so that we do not need to write the fourth root all the time. For  $X$  we get the equation  $X^{(4)} - \omega^4 X = 0$ . The general solution is

$$X(x) = Ae^{\omega x} + Be^{-\omega x} + C \sin(\omega x) + D \cos(\omega x).$$

Now  $0 = X(0) = A + B + D$ ,  $0 = X''(0) = \omega^2(A + B - D)$ . Hence,  $D = 0$  and  $A + B = 0$ , or  $B = -A$ . So we have

$$X(x) = Ae^{\omega x} - Ae^{-\omega x} + C \sin(\omega x).$$

Also  $0 = X(1) = A(e^\omega - e^{-\omega}) + C \sin \omega$ , and  $0 = X''(1) = A\omega^2(e^\omega - e^{-\omega}) - C\omega^2 \sin \omega$ . This means that  $C \sin \omega = 0$  and  $A(e^\omega - e^{-\omega}) = 2A \sinh \omega = 0$ . If  $\omega > 0$ , then  $\sinh \omega \neq 0$  and so  $A = 0$ . This means that  $C \neq 0$  otherwise  $\lambda$  is not an eigenvalue. Also  $\omega$  must be an integer multiple of  $\pi$ . Hence  $\omega = n\pi$  and  $n \geq 1$  (as  $\omega > 0$ ). We can take  $C = 1$ . So the eigenvalues are  $\lambda_n = n^4\pi^4$  and the eigenfunctions are  $\sin(n\pi x)$ .

Now  $T'' + n^4\pi^4 a^4 T = 0$ . The general solution is  $T(t) = A \sin(n^2\pi^2 a^2 t) + B \cos(n^2\pi^2 a^2 t)$ . But  $T'(0) = 0$  and hence we must have  $A = 0$  and we can take  $B = 1$  to make  $T(0) = 1$  for convenience. So our solutions are  $T_n(t) = \cos(n^2\pi^2 a^2 t)$ .

As the eigenfunctions are just sines again, we can decompose the function  $f(x)$  on  $0 < x < 1$  using the sine series. We find numbers  $b_n$  such that for  $0 < x < 1$  we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Then the solution to (5.5) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \cos(n^2\pi^2 a^2 t).$$

The point is that  $X_n T_n$  is a solution that satisfies all the homogeneous conditions (that is, all conditions except the initial position). And since  $T_n(0) = 1$ , we have

$$y(x, 0) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0) = \sum_{n=1}^{\infty} b_n X_n(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = f(x).$$

So  $y(x, t)$  solves (5.5).

Note that the natural (circular) frequency of the system is  $n^2\pi^2 a^2$ . These frequencies are all integer multiples of the fundamental frequency  $\pi^2 a^2$ , so we will get a nice musical note. The exact frequencies and their amplitude are what we call the timbre of the note.

The timbre of a beam is different than for a vibrating string where we will get “more” of the smaller frequencies since we will get all integer multiples,  $1, 2, 3, 4, 5, \dots$ . For a steel beam we will get only the square multiples  $1, 4, 9, 16, 25, \dots$ . That is why when you hit a steel beam you hear a very pure sound. The sound of a xylophone or vibraphone is, therefore, very different from a guitar or piano.

**Example 5.2.1:** Let us assume that  $f(x) = \frac{x(x-1)}{10}$ . On  $0 < x < 1$  we have (you know how to do this by now)

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{5\pi^3 n^3} \sin(n\pi x).$$

Hence, the solution to (5.5) with the given initial position  $f(x)$  is

$$y(x, t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{5\pi^3 n^3} \sin(n\pi x) \cos(n^2 \pi^2 a^2 t).$$

### 5.2.1 Exercises

**Exercise 5.2.2:** Suppose you have a beam of length 5 with free ends. Let  $y$  be the transverse deviation of the beam at position  $x$  on the beam ( $0 < x < 5$ ). You know that the constants are such that this satisfies the equation  $y_{tt} + 4y_{xxxx} = 0$ . Suppose you know that the initial shape of the beam is the graph of  $x(5 - x)$ , and the initial velocity is uniformly equal to 2 (same for each  $x$ ) in the positive  $y$  direction. Set up the equation together with the boundary and initial conditions. Just set up, do not solve.

**Exercise 5.2.3:** Suppose you have a beam of length 5 with one end free and one end fixed (the fixed end is at  $x = 5$ ). Let  $u$  be the longitudinal deviation of the beam at position  $x$  on the beam ( $0 < x < 5$ ). You know that the constants are such that this satisfies the equation  $u_{tt} = 4u_{xx}$ . Suppose you know that the initial displacement of the beam is  $\frac{x-5}{50}$ , and the initial velocity is  $\frac{-(x-5)}{100}$  in the positive  $u$  direction. Set up the equation together with the boundary and initial conditions. Just set up, do not solve.

**Exercise 5.2.4:** Suppose the beam is  $L$  units long, everything else kept the same as in (5.5). What is the equation and the series solution.

**Exercise 5.2.5:** Suppose you have

$$\begin{aligned} a^4 y_{xxxx} + y_{tt} &= 0 \quad (0 < x < 1, t > 0), \\ y(0, t) &= y_{xx}(0, t) = 0, \\ y(1, t) &= y_{xx}(1, t) = 0, \\ y(x, 0) &= f(x), \quad y_t(x, 0) = g(x). \end{aligned}$$

That is, you have also an initial velocity. Find a series solution. Hint: Use the same idea as we did for the wave equation.

**Exercise 5.2.101:** Suppose you have a beam of length 1 with hinged ends. Let  $y$  be the transverse deviation of the beam at position  $x$  on the beam ( $0 < x < 1$ ). You know that the constants are such that this satisfies the equation  $y_{tt} + 4y_{xxxx} = 0$ . Suppose you know that the initial shape of the beam is the graph of  $\sin(\pi x)$ , and the initial velocity is 0. Solve for  $y$ .

**Exercise 5.2.102:** Suppose you have a beam of length 10 with two fixed ends. Let  $y$  be the transverse deviation of the beam at position  $x$  on the beam ( $0 < x < 10$ ). You know that the constants are such that this satisfies the equation  $y_{tt} + 9y_{xxxx} = 0$ . Suppose you know that the initial shape of the beam is the graph of  $\sin(\pi x)$ , and the initial velocity is uniformly equal to  $x(10 - x)$ . Set up the equation together with the boundary and initial conditions. Just set up, do not solve.



## 5.3 Steady periodic solutions

*Note: 1–2 lectures, §10.3 in [EP], not in [BD]*

### 5.3.1 Forced vibrating string.

Suppose that we have a guitar string of length  $L$ . We have studied the wave equation problem in this case, where  $x$  was the position on the string,  $t$  was time and  $y$  was the displacement of the string. See Figure 5.3.

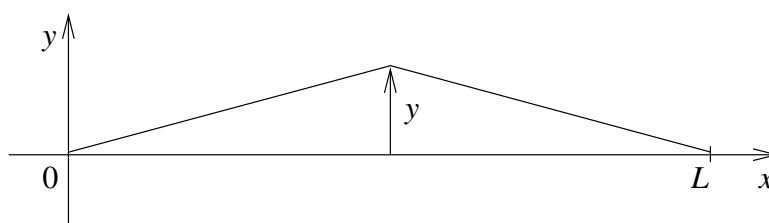


Figure 5.3: Vibrating string.

The problem is governed by the equations

$$\begin{aligned} y_{tt} &= a^2 y_{xx}, \\ y(0, t) &= 0, \quad y(L, t) = 0, \\ y(x, 0) &= f(x), \quad y_t(x, 0) = g(x). \end{aligned} \tag{5.6}$$

We saw previously that the solution is of the form

$$y = \sum_{n=1}^{\infty} \left( A_n \cos\left(\frac{n\pi a}{L} t\right) + B_n \sin\left(\frac{n\pi a}{L} t\right) \right) \sin\left(\frac{n\pi}{L} x\right)$$

where  $A_n$  and  $B_n$  were determined by the initial conditions. The natural frequencies of the system are the (circular) frequencies  $\frac{n\pi a}{L}$  for integers  $n \geq 1$ .

But these are free vibrations. What if there is an external force acting on the string. Let us assume say air vibrations (noise), for example a second string. Or perhaps a jet engine. For simplicity, assume nice pure sound and assume the force is uniform at every position on the string. Let us say  $F(t) = F_0 \cos(\omega t)$  as force per unit mass. Then our wave equation becomes (remember force is mass times acceleration)

$$y_{tt} = a^2 y_{xx} + F_0 \cos(\omega t), \tag{5.7}$$

with the same boundary conditions of course.

We will want to find the solution here that satisfies the above equation and

$$y(0, t) = 0, \quad y(L, t) = 0, \quad y(x, 0) = 0, \quad y_t(x, 0) = 0. \quad (5.8)$$

That is, the string is initially at rest. First we find a particular solution  $y_p$  of (5.7) that satisfies  $y(0, t) = y(L, t) = 0$ . We define the functions  $f$  and  $g$  as

$$f(x) = -y_p(x, 0), \quad g(x) = -\frac{\partial y_p}{\partial t}(x, 0).$$

We then find solution  $y_c$  of (5.6). If we add the two solutions, we find that  $y = y_c + y_p$  solves (5.7) with the initial conditions.

**Exercise 5.3.1:** Check that  $y = y_c + y_p$  solves (5.7) and the side conditions (5.8).

So the big issue here is to find the particular solution  $y_p$ . We look at the equation and we make an educated guess

$$y_p(x, t) = X(x) \cos(\omega t).$$

We plug in to get

$$-\omega^2 X \cos(\omega t) = a^2 X'' \cos(\omega t) + F_0 \cos(\omega t),$$

or  $-\omega^2 X = a^2 X'' + F_0$  after canceling the cosine. We know how to find a general solution to this equation (it is an nonhomogeneous constant coefficient equation) and we get that the general solution is

$$X(x) = A \cos\left(\frac{\omega}{a} x\right) + B \sin\left(\frac{\omega}{a} x\right) - \frac{F_0}{\omega^2}.$$

The endpoint conditions imply  $X(0) = X(L) = 0$ . So

$$0 = X(0) = A - \frac{F_0}{\omega^2},$$

or  $A = \frac{F_0}{\omega^2}$ , and also

$$0 = X(L) = \frac{F_0}{\omega^2} \cos\left(\frac{\omega L}{a}\right) + B \sin\left(\frac{\omega L}{a}\right) - \frac{F_0}{\omega^2}.$$

Assuming that  $\sin(\frac{\omega L}{a})$  is not zero we can solve for  $B$  to get

$$B = \frac{-F_0 \left( \cos\left(\frac{\omega L}{a}\right) - 1 \right)}{\omega^2 \sin\left(\frac{\omega L}{a}\right)}. \quad (5.9)$$

Therefore,

$$X(x) = \frac{F_0}{\omega^2} \left( \cos\left(\frac{\omega}{a} x\right) - \frac{\cos\left(\frac{\omega L}{a}\right) - 1}{\sin\left(\frac{\omega L}{a}\right)} \sin\left(\frac{\omega}{a} x\right) - 1 \right).$$

The particular solution  $y_p$  we are looking for is

$$y_p(x, t) = \frac{F_0}{\omega^2} \left( \cos\left(\frac{\omega}{a}x\right) - \frac{\cos\left(\frac{\omega L}{a}\right) - 1}{\sin\left(\frac{\omega L}{a}\right)} \sin\left(\frac{\omega}{a}x\right) - 1 \right) \cos(\omega t).$$

**Exercise 5.3.2:** Check that  $y_p$  works.

Now we get to the point that we skipped. Suppose that  $\sin(\frac{\omega L}{a}) = 0$ . What this means is that  $\omega$  is equal to one of the natural frequencies of the system, i.e. a multiple of  $\frac{\pi a}{L}$ . We notice that if  $\omega$  is not equal to a multiple of the base frequency, but is very close, then the coefficient  $B$  in (5.9) seems to become very large. But let us not jump to conclusions just yet. When  $\omega = \frac{n\pi a}{L}$  for  $n$  even, then  $\cos(\frac{\omega L}{a}) = 1$  and hence we really get that  $B = 0$ . So resonance occurs only when both  $\cos(\frac{\omega L}{a}) = -1$  and  $\sin(\frac{\omega L}{a}) = 0$ . That is when  $\omega = \frac{n\pi a}{L}$  for odd  $n$ .

We could again solve for the resonance solution if we wanted to, but it is, in the right sense, the limit of the solutions as  $\omega$  gets close to a resonance frequency. In real life, pure resonance never occurs anyway.

The above calculation explains why a string will begin to vibrate if the identical string is plucked close by. In the absence of friction this vibration would get louder and louder as time goes on. On the other hand, you are unlikely to get large vibration if the forcing frequency is not close to a resonance frequency even if you have a jet engine running close to the string. That is, the amplitude will not keep increasing unless you tune to just the right frequency.

Similar resonance phenomena occur when you break a wine glass using human voice (yes this is possible, but not easy<sup>†</sup>) if you happen to hit just the right frequency. Remember a glass has much purer sound, i.e. it is more like a vibraphone, so there are far fewer resonance frequencies to hit.

When the forcing function is more complicated, you decompose it in terms of the Fourier series and apply the above result. You may also need to solve the above problem if the forcing function is a sine rather than a cosine, but if you think about it, the solution is almost the same.

**Example 5.3.1:** Let us do the computation for specific values. Suppose  $F_0 = 1$  and  $\omega = 1$  and  $L = 1$  and  $a = 1$ . Then

$$y_p(x, t) = \left( \cos(x) - \frac{\cos(1) - 1}{\sin(1)} \sin(x) - 1 \right) \cos(t).$$

Write  $B = \frac{\cos(1)-1}{\sin(1)}$  for simplicity.

Then plug in  $t = 0$  to get

$$f(x) = -y_p(x, 0) = -\cos x + B \sin x + 1,$$

and after differentiating in  $t$  we see that  $g(x) = -\frac{\partial y_p}{\partial t}(x, 0) = 0$ .

<sup>†</sup>Mythbusters, episode 31, Discovery Channel, originally aired may 18th 2005.

Hence to find  $y_c$  we need to solve the problem

$$\begin{aligned} y_{tt} &= y_{xx}, \\ y(0, t) &= 0, \quad y(1, t) = 0, \\ y(x, 0) &= -\cos x + B \sin x + 1, \\ y_t(x, 0) &= 0. \end{aligned}$$

Note that the formula that we use to define  $y(x, 0)$  is not odd, hence it is not a simple matter of plugging in to apply the D'Alembert formula directly! You must define  $F$  to be the odd, 2-periodic extension of  $y(x, 0)$ . Then our solution would look like

$$y(x, t) = \frac{F(x+t) + F(x-t)}{2} + \left( \cos(x) - \frac{\cos(1)-1}{\sin(1)} \sin(x) - 1 \right) \cos(t). \quad (5.10)$$

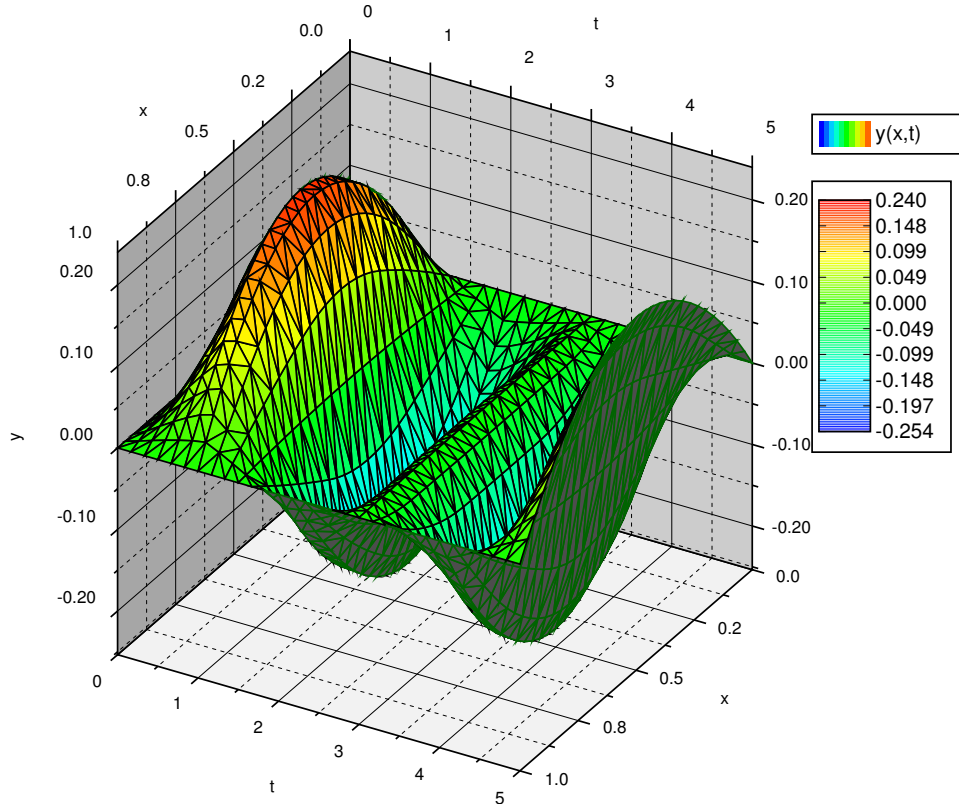


Figure 5.4: Plot of  $y(x, t) = \frac{F(x+t)+F(x-t)}{2} + \left( \cos(x) - \frac{\cos(1)-1}{\sin(1)} \sin(x) - 1 \right) \cos(t)$ .

It is not hard to compute specific values for an odd extension of a function and hence (5.10) is a wonderful solution to the problem. For example it is very easy to have a computer do it, unlike a series solution. A plot is given in Figure 5.4.

### 5.3.2 Underground temperature oscillations

Let  $u(x, t)$  be the temperature at a certain location at depth  $x$  underground at time  $t$ . See Figure 5.5.

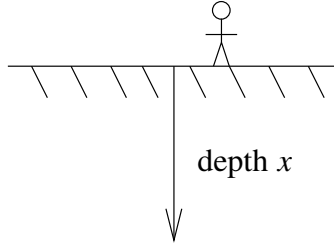


Figure 5.5: Underground temperature.

The temperature  $u$  satisfies the heat equation  $u_t = ku_{xx}$ , where  $k$  is the diffusivity of the soil. We know the temperature at the surface  $u(0, t)$  from weather records. Let us assume for simplicity that

$$u(0, t) = T_0 + A_0 \cos(\omega t).$$

For some base temperature  $T_0$ , then  $t = 0$  is midsummer (could put negative sign above to make it midwinter).  $A_0$  is picked properly to make this the typical variation for the year. That is, the hottest temperature is  $T_0 + A_0$  and the coldest is  $T_0 - A_0$ . For simplicity, we will assume that  $T_0 = 0$ .  $\omega$  is picked depending on the units of  $t$ , such that when  $t = 1$  year, then  $\omega t = 2\pi$ .

It seems reasonable that the temperature at depth  $x$  will also oscillate with the same frequency. And this in fact will be the steady periodic solution, independent of the initial conditions. So we are looking for a solution of the form

$$u(x, t) = V(x) \cos(\omega t) + W(x) \sin(\omega t).$$

for the problem

$$u_t = ku_{xx}, \quad u(0, t) = A_0 \cos(\omega t). \quad (5.11)$$

We will employ the complex exponential here to make calculations simpler. Suppose we have a complex valued function

$$h(x, t) = X(x) e^{i\omega t}.$$

We will look for an  $h$  such that  $\text{Re } h = u$ . To find an  $h$ , whose real part satisfies (5.11), we look for an  $h$  such that

$$h_t = kh_{xx}, \quad h(0, t) = A_0 e^{i\omega t}. \quad (5.12)$$

**Exercise 5.3.3:** Suppose  $h$  satisfies (5.12). Use Euler's formula for the complex exponential to check that  $u = \text{Re } h$  satisfies (5.11).

Substitute  $h$  into (5.12).

$$i\omega X e^{i\omega t} = k X'' e^{i\omega t}.$$

Hence,

$$k X'' - i\omega X = 0,$$

or

$$X'' - \alpha^2 X = 0,$$

where  $\alpha = \pm \sqrt{\frac{i\omega}{k}}$ . Note that  $\pm \sqrt{i} = \pm \frac{1+i}{\sqrt{2}}$  so you could simplify to  $\alpha = \pm(1+i)\sqrt{\frac{\omega}{2k}}$ . Hence the general solution is

$$X(x) = A e^{-(1+i)\sqrt{\frac{\omega}{2k}}x} + B e^{(1+i)\sqrt{\frac{\omega}{2k}}x}.$$

We assume that an  $X(x)$  that solves the problem must be bounded as  $x \rightarrow \infty$  since  $u(x, t)$  should be bounded (we are not worrying about the earth core!). If you use Euler's formula to expand the complex exponentials, you will note that the second term will be unbounded (if  $B \neq 0$ ), while the first term is always bounded. Hence  $B = 0$ .

**Exercise 5.3.4:** Use Euler's formula to show that  $e^{(1+i)\sqrt{\frac{\omega}{2k}}x}$  will be unbounded as  $x \rightarrow \infty$ , while  $e^{-(1+i)\sqrt{\frac{\omega}{2k}}x}$  will be bounded as  $x \rightarrow \infty$ .

Furthermore,  $X(0) = A_0$  since  $h(0, t) = A_0 e^{i\omega t}$ . Thus  $A = A_0$ . This means that

$$h(x, t) = A_0 e^{-(1+i)\sqrt{\frac{\omega}{2k}}x} e^{i\omega t} = A_0 e^{-(1+i)\sqrt{\frac{\omega}{2k}}x+i\omega t} = A_0 e^{-\sqrt{\frac{\omega}{2k}}x} e^{i(\omega t - \sqrt{\frac{\omega}{2k}}x)}.$$

We will need to get the real part of  $h$ , so we apply Euler's formula to get

$$h(x, t) = A_0 e^{-\sqrt{\frac{\omega}{2k}}x} \left( \cos\left(\omega t - \sqrt{\frac{\omega}{2k}}x\right) + i \sin\left(\omega t - \sqrt{\frac{\omega}{2k}}x\right) \right).$$

Then finally

$$u(x, t) = \operatorname{Re} h(x, t) = A_0 e^{-\sqrt{\frac{\omega}{2k}}x} \cos\left(\omega t - \sqrt{\frac{\omega}{2k}}x\right),$$

Yay!

Notice the phase is different at different depths. At depth  $x$  the phase is delayed by  $x\sqrt{\frac{\omega}{2k}}$ . For example in cgs units (centimeters-grams-seconds) we have  $k = 0.005$  (typical value for soil),  $\omega = \frac{2\pi}{\text{seconds in a year}} = \frac{2\pi}{31,557,341} \approx 1.99 \times 10^{-7}$ . Then if we compute where the phase shift  $x\sqrt{\frac{\omega}{2k}} = \pi$  we find the depth in centimeters where the seasons are reversed. That is, we get the depth at which summer is the coldest and winter is the warmest. We get approximately 700 centimeters, which is approximately 23 feet below ground.

Be careful not to jump to conclusions. The temperature swings decay rapidly as you dig deeper. The amplitude of the temperature swings is  $A_0 e^{-\sqrt{\frac{\omega}{2k}}x}$ . This function decays *very* quickly as  $x$  (the depth) grows. Let us again take typical parameters as above. We will also assume that our surface

temperature swing is  $\pm 15^\circ$  Celsius, that is,  $A_0 = 15$ . Then the maximum temperature variation at 700 centimeters is only  $\pm 0.66^\circ$  Celsius.

You need not dig very deep to get an effective “refrigerator,” with nearly constant temperature. That is why wines are kept in a cellar; you need consistent temperature. The temperature differential could also be used for energy. A home could be heated or cooled by taking advantage of the above fact. Even without the earth core you could heat a home in the winter and cool it in the summer. The earth core makes the temperature higher the deeper you dig, although you need to dig somewhat deep to feel a difference. We did not take that into account above.

### 5.3.3 Exercises

**Exercise 5.3.5:** Suppose that the forcing function for the vibrating string is  $F_0 \sin(\omega t)$ . Derive the particular solution  $y_p$ .

**Exercise 5.3.6:** Take the forced vibrating string. Suppose that  $L = 1$ ,  $a = 1$ . Suppose that the forcing function is the square wave that is 1 on the interval  $0 < x < 1$  and  $-1$  on the interval  $-1 < x < 0$ . Find the particular solution. Hint: You may want to use result of Exercise 5.3.5.

**Exercise 5.3.7:** The units are cgs (centimeters-grams-seconds). For  $k = 0.005$ ,  $\omega = 1.991 \times 10^{-7}$ ,  $A_0 = 20$ . Find the depth at which the temperature variation is half ( $\pm 10$  degrees) of what it is on the surface.

**Exercise 5.3.8:** Derive the solution for underground temperature oscillation without assuming that  $T_0 = 0$ .

**Exercise 5.3.101:** Take the forced vibrating string. Suppose that  $L = 1$ ,  $a = 1$ . Suppose that the forcing function is a sawtooth, that is  $|x| - \frac{1}{2}$  on  $-1 < x < 1$  extended periodically. Find the particular solution.

**Exercise 5.3.102:** The units are cgs (centimeters-grams-seconds). For  $k = 0.01$ ,  $\omega = 1.991 \times 10^{-7}$ ,  $A_0 = 25$ . Find the depth at which the summer is again the hottest point.





# Chapter 6

## The Laplace transform

### 6.1 The Laplace transform

*Note: 1.5 – 2 lectures, §10.1 in [EP], §6.1 and parts of §6.2 in [BD]*

#### 6.1.1 The transform

In this chapter we will discuss the Laplace transform\*. The Laplace transform turns out to be a very efficient method to solve certain ODE problems. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The Laplace transform also has applications in the analysis of electrical circuits, NMR spectroscopy, signal processing, and elsewhere. Finally, understanding the Laplace transform will also help with understanding the related Fourier transform, which, however, requires more understanding of complex numbers. We will not cover the Fourier transform.

The Laplace transform also gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. For example, take the standard equation

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

We can think of  $t$  as time and  $f(t)$  as incoming signal. The Laplace transform will convert the equation from a differential equation in time to an algebraic (no derivatives) equation, where the new independent variable  $s$  is the frequency.

We can think of the *Laplace transform* as a black box. It eats functions and spits out functions in a new variable. We write  $\mathcal{L}\{f(t)\} = F(s)$  for the Laplace transform of  $f(t)$ . It is common to write lower case letters for functions in the time domain and upper case letters for functions in the

---

\*Just like the Laplace equation and the Laplacian, the Laplace transform is also named after Pierre-Simon, marquis de Laplace (1749 – 1827).

frequency domain. We use the same letter to denote that one function is the Laplace transform of the other. For example  $F(s)$  is the Laplace transform of  $f(t)$ . Let us define the transform.

$$\mathcal{L}\{f(t)\} = F(s) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-st} f(t) dt.$$

We note that we are only considering  $t \geq 0$  in the transform. Of course, if we think of  $t$  as time there is no problem, we are generally interested in finding out what will happen in the future (Laplace transform is one place where it is safe to ignore the past). Let us compute some simple transforms.

**Example 6.1.1:** Suppose  $f(t) = 1$ , then

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} dt = \left[ \frac{e^{-st}}{-s} \right]_{t=0}^{\infty} = \lim_{h \rightarrow \infty} \left[ \frac{e^{-st}}{-s} \right]_{t=0}^h = \lim_{h \rightarrow \infty} \left( \frac{e^{-sh}}{-s} - \frac{1}{-s} \right) = \frac{1}{s}.$$

The limit (the improper integral) only exists if  $s > 0$ . So  $\mathcal{L}\{1\}$  is only defined for  $s > 0$ .

**Example 6.1.2:** Suppose  $f(t) = e^{-at}$ , then

$$\mathcal{L}\{e^{-at}\} = \int_0^{\infty} e^{-st} e^{-at} dt = \int_0^{\infty} e^{-(s+a)t} dt = \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_{t=0}^{\infty} = \frac{1}{s+a}.$$

The limit only exists if  $s + a > 0$ . So  $\mathcal{L}\{e^{-at}\}$  is only defined for  $s + a > 0$ .

**Example 6.1.3:** Suppose  $f(t) = t$ , then using integration by parts

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^{\infty} e^{-st} t dt \\ &= \left[ \frac{-te^{-st}}{s} \right]_{t=0}^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \\ &= 0 + \frac{1}{s} \left[ \frac{e^{-st}}{-s} \right]_{t=0}^{\infty} \\ &= \frac{1}{s^2}. \end{aligned}$$

Again, the limit only exists if  $s > 0$ .

**Example 6.1.4:** A common function is the *unit step function*, which is sometimes called the *Heaviside function*<sup>†</sup>. This function is generally given as

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

---

<sup>†</sup>The function is named after the English mathematician, engineer, and physicist Oliver Heaviside (1850–1925). Only by coincidence is the function “heavy” on “one side.”

Let us find the Laplace transform of  $u(t - a)$ , where  $a \geq 0$  is some constant. That is, the function that is 0 for  $t < a$  and 1 for  $t \geq a$ .

$$\mathcal{L}\{u(t - a)\} = \int_0^{\infty} e^{-st} u(t - a) dt = \int_a^{\infty} e^{-st} dt = \left[ \frac{e^{-st}}{-s} \right]_{t=a}^{\infty} = \frac{e^{-as}}{s},$$

where of course  $s > 0$  (and  $a \geq 0$  as we said before).

By applying similar procedures we can compute the transforms of many elementary functions. Many basic transforms are listed in Table 6.1.

$f(t)$	$\mathcal{L}\{f(t)\}$
$C$	$\frac{C}{s}$
$t$	$\frac{1}{s^2}$
$t^2$	$\frac{2}{s^3}$
$t^3$	$\frac{6}{s^4}$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{-at}$	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$u(t - a)$	$\frac{e^{-as}}{s}$

Table 6.1: Some Laplace transforms ( $C$ ,  $\omega$ , and  $a$  are constants).

**Exercise 6.1.1:** Verify Table 6.1.

Since the transform is defined by an integral. We can use the linearity properties of the integral. For example, suppose  $C$  is a constant, then

$$\mathcal{L}\{Cf(t)\} = \int_0^{\infty} e^{-st} Cf(t) dt = C \int_0^{\infty} e^{-st} f(t) dt = C\mathcal{L}\{f(t)\}.$$

So we can “pull out” a constant out of the transform. Similarly we have linearity. Since linearity is very important we state it as a theorem.

**Theorem 6.1.1** (Linearity of the Laplace transform). *Suppose that  $A$ ,  $B$ , and  $C$  are constants, then*

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\},$$

and in particular

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}.$$

**Exercise 6.1.2:** Verify the theorem. That is, show that  $\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$ .

These rules together with Table 6.1 on the previous page make it easy to find the Laplace transform of a whole lot of functions already. But be careful. It is a common mistake to think that the Laplace transform of a product is the product of the transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

It must also be noted that not all functions have a Laplace transform. For example, the function  $\frac{1}{t}$  does not have a Laplace transform as the integral diverges for all  $s$ . Similarly,  $\tan t$  or  $e^{t^2}$  do not have Laplace transforms.

### 6.1.2 Existence and uniqueness

Let us consider when does the Laplace transform exist in more detail. First let us consider functions of exponential order. The function  $f(t)$  is of *exponential order* as  $t$  goes to infinity if

$$|f(t)| \leq Me^{ct},$$

for some constants  $M$  and  $c$ , for sufficiently large  $t$  (say for all  $t > t_0$  for some  $t_0$ ). The simplest way to check this condition is to try and compute

$$\lim_{t \rightarrow \infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then  $f(t)$  is of exponential order.

**Exercise 6.1.3:** Use L'Hopital's rule from calculus to show that a polynomial is of exponential order. Hint: Note that a sum of two exponential order functions is also of exponential order. Then show that  $t^n$  is of exponential order for any  $n$ .

For an exponential order function we have existence and uniqueness of the Laplace transform.

**Theorem 6.1.2** (Existence). *Let  $f(t)$  be continuous and of exponential order for a certain constant  $c$ . Then  $F(s) = \mathcal{L}\{f(t)\}$  is defined for all  $s > c$ .*

The transform also exists for some other functions that are not of exponential order, but that will not be relevant to us. Before dealing with uniqueness, let us note that for exponential order functions we obtain that their Laplace transform decays at infinity:

$$\lim_{s \rightarrow \infty} F(s) = 0.$$

**Theorem 6.1.3** (Uniqueness). *Let  $f(t)$  and  $g(t)$  be continuous and of exponential order. Suppose that there exists a constant  $C$ , such that  $F(s) = G(s)$  for all  $s > C$ . Then  $f(t) = g(t)$  for all  $t \geq 0$ .*

Both theorems hold for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function. Uniqueness however does not “see” values at the discontinuities. So we can only conclude that  $f(t) = g(t)$  outside of discontinuities. For example, the unit step function is sometimes defined using  $u(0) = 1/2$ . This new step function, however, has the exact same Laplace transform as the one we defined earlier where  $u(0) = 1$ .

### 6.1.3 The inverse transform

As we said, the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the frequency domain we will want to get back to the time domain, as that is what we are interested in. If we have a function  $F(s)$ , to be able to find  $f(t)$  such that  $\mathcal{L}\{f(t)\} = F(s)$ , we need to first know if such a function is unique. It turns out we are in luck by Theorem 6.1.3. So we can without fear make the following definition.

If  $F(s) = \mathcal{L}\{f(t)\}$  for some function  $f(t)$ . We define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \stackrel{\text{def}}{=} f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself (requires complex numbers). For us it will suffice to compute the inverse by using Table 6.1 on page 243.

**Example 6.1.5:** Take  $F(s) = \frac{1}{s+1}$ . Find the inverse Laplace transform.

We look at the table and we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

As the Laplace transform is linear, the inverse Laplace transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, we also have  $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$ . Let us demonstrate how linearity can be used.

**Example 6.1.6:** Take  $F(s) = \frac{s^2+s+1}{s^3+s}$ . Find the inverse Laplace transform.

First we use the *method of partial fractions* to write  $F$  in a form where we can use Table 6.1 on page 243. We factor the denominator as  $s(s^2 + 1)$  and write

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Putting the right hand side over a common denominator and equating the numerators we get  $A(s^2 + 1) + s(Bs + C) = s^2 + s + 1$ . Expanding and equating coefficients we obtain  $A + B = 1$ ,  $C = 1$ ,  $A = 1$ , and thus  $B = 0$ . In other words,

$$F(s) = \frac{s^2 + s + 1}{s^3 + s} = \frac{1}{s} + \frac{1}{s^2 + 1}.$$

By linearity of the inverse Laplace transform we get

$$\mathcal{L}^{-1} \left\{ \frac{s^2 + s + 1}{s^3 + s} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} + \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} = 1 + \sin t.$$

Another useful property is the so-called *shifting property* or the *first shifting property*

$$\boxed{\mathcal{L}\{e^{-at}f(t)\} = F(s + a),}$$

where  $F(s)$  is the Laplace transform of  $f(t)$ .

**Exercise 6.1.4:** Derive the first shifting property from the definition of the Laplace transform.

The shifting property can be used, for example, when the denominator is a more complicated quadratic that may come up in the method of partial fractions. We will write such quadratics as  $(s + a)^2 + b$  by completing the square and then use the shifting property.

**Example 6.1.7:** Find  $\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4s + 8} \right\}$ .

First we complete the square to make the denominator  $(s + 2)^2 + 4$ . Next we find

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} = \frac{1}{2} \sin(2t).$$

Putting it all together with the shifting property we find

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4s + 8} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s + 2)^2 + 4} \right\} = \frac{1}{2} e^{-2t} \sin(2t).$$

In general, we will want to be able to apply the Laplace transform to rational functions, that is functions of the form

$$\frac{F(s)}{G(s)}$$

where  $F(s)$  and  $G(s)$  are polynomials. Since normally (for functions that we are considering) the Laplace transform goes to zero as  $s \rightarrow \infty$ , it is not hard to see that the degree of  $F(s)$  will be smaller than that of  $G(s)$ . Such rational functions are called *proper rational functions* and we will always be able to apply the method of partial fractions. Of course this means we will need to be able to factor the denominator into linear and quadratic terms, which involves finding the roots of the denominator.

## 6.1.4 Exercises

**Exercise 6.1.5:** Find the Laplace transform of  $3 + t^5 + \sin(\pi t)$ .

**Exercise 6.1.6:** Find the Laplace transform of  $a + bt + ct^2$  for some constants  $a$ ,  $b$ , and  $c$ .

**Exercise 6.1.7:** Find the Laplace transform of  $A \cos(\omega t) + B \sin(\omega t)$ .

**Exercise 6.1.8:** Find the Laplace transform of  $\cos^2(\omega t)$ .

**Exercise 6.1.9:** Find the inverse Laplace transform of  $\frac{4}{s^2-9}$ .

**Exercise 6.1.10:** Find the inverse Laplace transform of  $\frac{2s}{s^2-1}$ .

**Exercise 6.1.11:** Find the inverse Laplace transform of  $\frac{1}{(s-1)^2(s+1)}$ .

**Exercise 6.1.12:** Find the Laplace transform of  $f(t) = \begin{cases} t & \text{if } t \geq 1, \\ 0 & \text{if } t < 1. \end{cases}$

**Exercise 6.1.13:** Find the inverse Laplace transform of  $\frac{s}{(s^2+s+2)(s+4)}$ .

**Exercise 6.1.14:** Find the Laplace transform of  $\sin(\omega(t-a))$ .

**Exercise 6.1.15:** Find the Laplace transform of  $t \sin(\omega t)$ . Hint: several integrations by parts.

**Exercise 6.1.101:** Find the Laplace transform of  $4(t+1)^2$ .

**Exercise 6.1.102:** Find the inverse Laplace transform of  $\frac{8}{s^3(s+2)}$ .

**Exercise 6.1.103:** Find the Laplace transform of  $te^{-t}$  (Hint: integrate by parts).

**Exercise 6.1.104:** Find the Laplace transform of  $\sin(t)e^{-t}$  (Hint: integrate by parts).

## 6.2 Transforms of derivatives and ODEs

*Note: 2 lectures, §7.2–7.3 in [EP], §6.2 and §6.3 in [BD]*

### 6.2.1 Transforms of derivatives

Let us see how the Laplace transform is used for differential equations. First let us try to find the Laplace transform of a function that is a derivative. That is, suppose  $g(t)$  is a continuous differentiable function of exponential order. Then

$$\mathcal{L}\{g'(t)\} = \int_0^{\infty} e^{-st} g'(t) dt = \left[ e^{-st} g(t) \right]_{t=0}^{\infty} - \int_0^{\infty} (-s) e^{-st} g(t) dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

We repeat this procedure for higher derivatives. The results are listed in Table 6.2. The procedure also works for piecewise smooth functions, that is functions that are piecewise continuous with a piecewise continuous derivative. The fact that the function is of exponential order is used to show that the limits appearing above exist. We will not worry much about this fact.

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Table 6.2: Laplace transforms of derivatives ( $G(s) = \mathcal{L}\{g(t)\}$  as usual).

**Exercise 6.2.1:** Verify Table 6.2.

### 6.2.2 Solving ODEs with the Laplace transform

Notice that the Laplace transform turns differentiation into multiplication by  $s$ . Let us see how to apply this fact to differential equations.

**Example 6.2.1:** Take the equation

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

We will take the Laplace transform of both sides. By  $X(s)$  we will, as usual, denote the Laplace transform of  $x(t)$ .

$$\begin{aligned} \mathcal{L}\{x''(t) + x(t)\} &= \mathcal{L}\{\cos(2t)\}, \\ s^2X(s) - sx(0) - x'(0) + X(s) &= \frac{s}{s^2 + 4}. \end{aligned}$$



We can plug in the initial conditions now (this will make computations more streamlined) to obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2 + 4}.$$

We now solve for  $X(s)$ ,

$$X(s) = \frac{s}{(s^2 + 1)(s^2 + 4)} + \frac{1}{s^2 + 1}.$$

We use partial fractions (exercise) to write

$$X(s) = \frac{1}{3} \frac{s}{s^2 + 1} - \frac{1}{3} \frac{s}{s^2 + 4} + \frac{1}{s^2 + 1}.$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3} \cos(t) - \frac{1}{3} \cos(2t) + \sin(t).$$

The procedure for linear constant coefficient equations is as follows. We take an ordinary differential equation in the time variable  $t$ . We apply the Laplace transform to transform the equation into an algebraic (non differential) equation in the frequency domain. All the  $x(t)$ ,  $x'(t)$ ,  $x''(t)$ , and so on, will be converted to  $X(s)$ ,  $sX(s) - x(0)$ ,  $s^2X(s) - sx(0) - x'(0)$ , and so on. We solve the equation for  $X(s)$ . Then taking the inverse transform, if possible, we find  $x(t)$ .

It should be noted that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

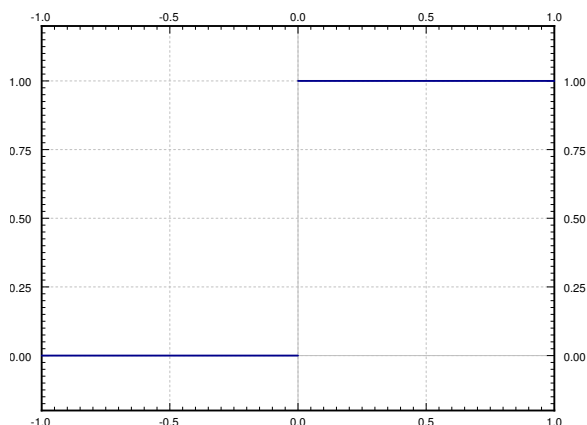
### 6.2.3 Using the Heaviside function

Before we move on to more general equations than those we could solve before, we want to consider the Heaviside function. See Figure 6.1 on the following page for the graph.

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

This function is useful for putting together functions, or cutting functions off. Most commonly it is used as  $u(t - a)$  for some constant  $a$ . This just shifts the graph to the right by  $a$ . That is, it is a function that is 0 when  $t < a$  and 1 when  $t \geq a$ . Suppose for example that  $f(t)$  is a “signal” and you started receiving the signal  $\sin t$  at time  $t = \pi$ . The function  $f(t)$  should then be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Figure 6.1: Plot of the Heaviside (unit step) function  $u(t)$ .

Using the Heaviside function,  $f(t)$  can be written as

$$f(t) = u(t - \pi) \sin t.$$

Similarly the step function that is 1 on the interval  $[1, 2)$  and zero everywhere else can be written as

$$u(t - 1) - u(t - 2).$$

The Heaviside function is useful to define functions defined piecewise. If you want the function  $t$  on when  $t$  is in  $[0, 1]$  and the function  $-t + 2$  when  $t$  is in  $[1, 2]$  and zero otherwise, you can use the expression

$$t(u(t) - u(t - 1)) + (-t + 2)(u(t - 1) - u(t - 2)).$$

Hence it is useful to know how the Heaviside function interacts with the Laplace transform. We have already seen that

$$\mathcal{L}\{u(t - a)\} = \frac{e^{-as}}{s}.$$

This can be generalized into a *shifting property* or *second shifting property*.

$$\boxed{\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}\mathcal{L}\{f(t)\}.} \quad (6.1)$$

**Example 6.2.2:** Suppose that the forcing function is not periodic. For example, suppose that we had a mass-spring system

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where  $f(t) = 1$  if  $1 \leq t < 5$  and zero otherwise. We could imagine a mass-spring system, where a rocket is fired for 4 seconds starting at  $t = 1$ . Or perhaps an RLC circuit, where the voltage is raised at a constant rate for 4 seconds starting at  $t = 1$ , and then held steady again starting at  $t = 5$ .

We can write  $f(t) = u(t - 1) - u(t - 5)$ . We transform the equation and we plug in the initial conditions as before to obtain

$$s^2 X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for  $X(s)$  to obtain

$$X(s) = \frac{e^{-s}}{s(s^2 + 1)} - \frac{e^{-5s}}{s(s^2 + 1)}.$$

We leave it as an exercise to the reader to show that

$$\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + 1)} \right\} = 1 - \cos t.$$

In other words  $\mathcal{L}\{1 - \cos t\} = \frac{1}{s(s^2 + 1)}$ . So using (6.1) we find

$$\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s^2 + 1)} \right\} = \mathcal{L}^{-1} \{e^{-s} \mathcal{L}\{1 - \cos t\}\} = (1 - \cos(t - 1)) u(t - 1).$$

Similarly

$$\mathcal{L}^{-1} \left\{ \frac{e^{-5s}}{s(s^2 + 1)} \right\} = \mathcal{L}^{-1} \{e^{-5s} \mathcal{L}\{1 - \cos t\}\} = (1 - \cos(t - 5)) u(t - 5).$$

Hence, the solution is

$$x(t) = (1 - \cos(t - 1)) u(t - 1) - (1 - \cos(t - 5)) u(t - 5).$$

The plot of this solution is given in Figure 6.2.

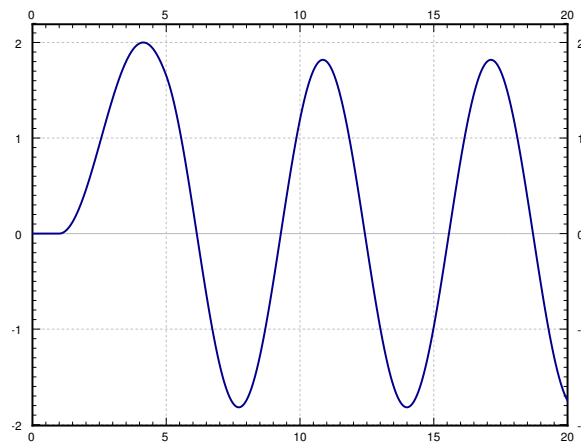


Figure 6.2: Plot of  $x(t)$ .

### 6.2.4 Transfer functions

Laplace transform leads to the following useful concept for studying the steady state behavior of a linear system. Suppose we have an equation of the form

$$Lx = f(t),$$

where  $L$  is a linear constant coefficient differential operator. Then  $f(t)$  is usually thought of as input of the system and  $x(t)$  is thought of as the output of the system. For example, for a mass-spring system the input is the forcing function and output is the behaviour of the mass. We would like to have an convenient way to study the behaviour of the system for different inputs.

Let us suppose that all the initial conditions are zero and take the Laplace transform of the equation, we obtain the equation

$$A(s)X(s) = F(s).$$

Solving for the ratio  $X(s)/F(s)$  we obtain the so called *transfer function*  $H(s) = 1/A(s)$ .

$$H(s) = \frac{X(s)}{F(s)}.$$

In other words,  $X(s) = H(s)F(s)$ . We obtain an algebraic dependence of the output of the system based on the input. We can now easily study the steady state behaviour of the system given different inputs by simply multiplying by the transfer function.

**Example 6.2.3:** Given  $x'' + \omega_0^2 x = f(t)$ , let us find the transfer function (assuming the initial conditions are zero).

First, we take the Laplace transform of the equation.

$$s^2 X(s) + \omega_0^2 X(s) = F(s).$$

Now we solve for the transfer function  $X(s)/F(s)$ .

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + \omega_0^2}.$$

Let us see how to use the transfer function. Suppose we have the constant input  $f(t) = 1$ . Hence  $F(s) = 1/s$ , and

$$X(s) = H(s)F(s) = \frac{1}{s^2 + \omega_0^2} \frac{1}{s}.$$

Taking the inverse Laplace transform of  $X(s)$  we obtain

$$x(t) = \frac{1 - \cos(\omega_0 t)}{\omega_0^2}.$$

### 6.2.5 Transforms of integrals

A feature of Laplace transforms is that it is also able to easily deal with integral equations. That is, equations in which integrals rather than derivatives of functions appear. The basic property, which can be proved by applying the definition and doing integration by parts, is

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s).$$

It is sometimes useful (e.g. for computing the inverse transform) to write this as

$$\int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\}.$$

**Example 6.2.4:** To compute  $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$  we could proceed by applying this integration rule.

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \frac{1}{s^2+1}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} d\tau = \int_0^t \sin \tau d\tau = 1 - \cos t.$$

**Example 6.2.5:** An equation containing an integral of the unknown function is called an *integral equation*. For example, take

$$t^2 = \int_0^t e^\tau x(\tau) d\tau,$$

where we wish to solve for  $x(t)$ . We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s} \mathcal{L}\{e^t x(t)\} = \frac{1}{s} X(s-1),$$

where  $X(s) = \mathcal{L}\{x(t)\}$ . Thus

$$X(s-1) = \frac{2}{s^2} \quad \text{or} \quad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again

$$x(t) = 2e^{-t}t.$$

### 6.2.6 Exercises

**Exercise 6.2.2:** Using the Heaviside function write down the piecewise function that is 0 for  $t < 0$ ,  $t^2$  for  $t$  in  $[0, 1]$  and  $t$  for  $t > 1$ .

**Exercise 6.2.3:** Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where  $m > 0$ ,  $c > 0$ ,  $k > 0$ , and  $c^2 - 4km > 0$  (system is overdamped).

**Exercise 6.2.4:** Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where  $m > 0$ ,  $c > 0$ ,  $k > 0$ , and  $c^2 - 4km < 0$  (system is underdamped).

**Exercise 6.2.5:** Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where  $m > 0$ ,  $c > 0$ ,  $k > 0$ , and  $c^2 = 4km$  (system is critically damped).

**Exercise 6.2.6:** Solve  $x'' + x = u(t - 1)$  for initial conditions  $x(0) = 0$  and  $x'(0) = 0$ .

**Exercise 6.2.7:** Show the differentiation of the transform property. Suppose  $\mathcal{L}\{f(t)\} = F(s)$ , then show

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

*Hint: Differentiate under the integral sign.*

**Exercise 6.2.8:** Solve  $x''' + x = t^3 u(t - 1)$  for initial conditions  $x(0) = 1$  and  $x'(0) = 0$ ,  $x''(0) = 0$ .

**Exercise 6.2.9:** Show the second shifting property:  $\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}\mathcal{L}\{f(t)\}$ .

**Exercise 6.2.10:** Let us think of the mass-spring system with a rocket from Example 6.2.2. We noticed that the solution kept oscillating after the rocket stopped running. The amplitude of the oscillation depends on the time that the rocket was fired (for 4 seconds in the example). a) Find a formula for the amplitude of the resulting oscillation in terms of the amount of time the rocket is fired. b) Is there a nonzero time (if so what is it?) for which the rocket fires and the resulting oscillation has amplitude 0 (the mass is not moving)?

**Exercise 6.2.11:** Define

$$f(t) = \begin{cases} (t - 1)^2 & \text{if } 1 \leq t < 2, \\ 3 - t & \text{if } 2 \leq t < 3, \\ 0 & \text{otherwise.} \end{cases}$$

a) Sketch the graph of  $f(t)$ . b) Write down  $f(t)$  using the Heaviside function. c) Solve  $x'' + x = f(t)$ ,  $x(0) = 0$ ,  $x'(0) = 0$  using Laplace transform.

**Exercise 6.2.12:** Find the transfer function for  $mx'' + cx' + kx = f(t)$  (assuming the initial conditions are zero).

**Exercise 6.2.101:** Using the Heaviside function  $u(t)$ , write down the function

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ t - 1 & \text{if } 1 \leq x < 2. \\ 1 & \text{if } x \geq 2 \end{cases}$$

**Exercise 6.2.102:** Solve  $x'' - x = (t^2 - 1)u(t - 1)$  for initial conditions  $x(0) = 1$ ,  $x'(0) = 2$  using the Laplace transform.

**Exercise 6.2.103:** Find the transfer function for  $x' + x = f(t)$  (assuming the initial conditions are zero).

## 6.3 Convolution

*Note: 1 or 1.5 lectures, §7.2 in [EP], §6.6 in [BD]*

### 6.3.1 The convolution

We said that the Laplace transformation of a product is not the product of the transforms. All hope is not lost however. We simply have to use a different type of a “product.” Take two functions  $f(t)$  and  $g(t)$  defined for  $t \geq 0$ . Define the *convolution*<sup>‡</sup> of  $f(t)$  and  $g(t)$  as

$$(f * g)(t) \stackrel{\text{def}}{=} \int_0^t f(\tau)g(t - \tau) d\tau. \quad (6.2)$$

As you can see, the convolution of two functions of  $t$  is another function of  $t$ .

**Example 6.3.1:** Take  $f(t) = e^t$  and  $g(t) = t$  for  $t \geq 0$ . Then

$$(f * g)(t) = \int_0^t e^\tau(t - \tau) d\tau = e^t - t - 1.$$

To solve the integral we did one integration by parts.

**Example 6.3.2:** Take  $f(t) = \sin(\omega t)$  and  $g(t) = \cos(\omega t)$  for  $t \geq 0$ . Then

$$(f * g)(t) = \int_0^t \sin(\omega\tau) \cos(\omega(t - \tau)) d\tau.$$

We will apply the identity

$$\cos(\theta) \sin(\psi) = \frac{1}{2} (\sin(\theta + \psi) - \sin(\theta - \psi)).$$

Hence,

$$\begin{aligned} (f * g)(t) &= \int_0^t \frac{1}{2} (\sin(\omega t) - \sin(\omega t - 2\omega\tau)) d\tau \\ &= \left[ \frac{1}{2} \tau \sin(\omega t) + \frac{1}{4\omega} \cos(2\omega\tau - \omega t) \right]_{\tau=0}^t \\ &= \frac{1}{2} t \sin(\omega t). \end{aligned}$$

The formula holds only for  $t \geq 0$ . We assumed that  $f$  and  $g$  are zero (or simply not defined) for negative  $t$ .

---

<sup>‡</sup> For those that have seen convolution defined before, you may have seen it defined as  $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau$ . This definition agrees with (6.2) if you define  $f(t)$  and  $g(t)$  to be zero for  $t < 0$ . When discussing the Laplace transform the definition we gave is sufficient. Convolution does occur in many other applications, however, where you may have to use the more general definition with infinities.



The convolution has many properties that make it behave like a product. Let  $c$  be a constant and  $f$ ,  $g$ , and  $h$  be functions then

$$\begin{aligned} f * g &= g * f, \\ (cf) * g &= f * (cg) = c(f * g), \\ (f * g) * h &= f * (g * h). \end{aligned}$$

The most interesting property for us, and the main result of this section is the following theorem.

**Theorem 6.3.1.** *Let  $f(t)$  and  $g(t)$  be of exponential type, then*

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t - \tau) d\tau\right\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse.

**Example 6.3.3:** Suppose we have the function of  $s$  defined by

$$\frac{1}{(s + 1)s^2} = \frac{1}{s + 1} \frac{1}{s^2}.$$

We recognize the two entries of Table 6.2. That is

$$\mathcal{L}^{-1}\left\{\frac{1}{s + 1}\right\} = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s + 1} \frac{1}{s^2}\right\} = \int_0^t \tau e^{-(t-\tau)} d\tau = e^{-t} + t - 1.$$

The calculation of the integral involved an integration by parts.

### 6.3.2 Solving ODEs

The next example will demonstrate the full power of the convolution and Laplace transform. We will be able to give a solution to the forced oscillation problem for any forcing function as a definite integral.

**Example 6.3.4:** Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function  $f(t)$ .

We first apply the Laplace transform to the equation. Denote the transform of  $x(t)$  by  $X(s)$  and the transform of  $f(t)$  by  $F(s)$  as usual.

$$s^2 X(s) + \omega_0^2 X(s) = F(s),$$

or in other words

$$X(s) = F(s) \frac{1}{s^2 + \omega_0^2}.$$

We know

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + \omega_0^2} \right\} = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore,

$$x(t) = \int_0^t f(\tau) \frac{\sin(\omega_0(t - \tau))}{\omega_0} d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0 t)}{\omega_0} f(t - \tau) d\tau.$$

Let us notice one more thing with this example. We can now see how Laplace transform handles resonance. Suppose that  $f(t) = \cos(\omega_0 t)$ . Then

$$x(t) = \int_0^t \frac{\sin(\omega_0 \tau)}{\omega_0} \cos(\omega_0(t - \tau)) d\tau = \frac{1}{\omega_0} \int_0^t \cos(\omega_0 \tau) \sin(\omega_0(t - \tau)) d\tau.$$

We have already computed the convolution of sine and cosine in Example 6.3.2. Hence

$$x(t) = \left( \frac{1}{\omega_0} \right) \left( \frac{1}{2} t \sin(\omega_0 t) \right) = \frac{1}{2\omega_0} t \sin(\omega_0 t).$$

Note the  $t$  in front of the sine. This solution will, therefore, grow without bound as  $t$  gets large, meaning we get resonance.

Similarly, we can solve any constant coefficient equation with an arbitrary forcing function  $f(t)$  as a definite integral using convolution. A definite integral is usually enough for most practical purposes. It is generally not hard to numerically evaluate a definite integral.

### 6.3.3 Volterra integral equation

A common integral equation is the *Volterra integral equation*<sup>§</sup>

$$x(t) = f(t) + \int_0^t g(t - \tau)x(\tau) d\tau,$$

---

<sup>§</sup>Named for the Italian mathematician Vito Volterra (1860 – 1940).

where  $f(t)$  and  $g(t)$  are known functions and  $x(t)$  is an unknown we wish to solve for. To find  $x(t)$ , we apply the Laplace transform to the equation to obtain

$$X(s) = F(s) + G(s)X(s),$$

where  $X(s)$ ,  $F(s)$ , and  $G(s)$  are the Laplace transforms of  $x(t)$ ,  $f(t)$ , and  $g(t)$  respectively. We find

$$X(s) = \frac{F(s)}{1 - G(s)}.$$

To find  $x(t)$  we now need to find the inverse Laplace transform of  $X(s)$ .

**Example 6.3.5:** Solve

$$x(t) = e^{-t} + \int_0^t \sinh(t - \tau)x(\tau) d\tau.$$

We apply Laplace transform to obtain

$$X(s) = \frac{1}{s+1} + \frac{1}{s^2-1}X(s),$$

or

$$X(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

It is not hard to apply Table 6.1 on page 243 to find

$$x(t) = \cosh(\sqrt{2}t) - \frac{1}{\sqrt{2}} \sinh(\sqrt{2}t).$$

### 6.3.4 Exercises

**Exercise 6.3.1:** Let  $f(t) = t^2$  for  $t \geq 0$ , and  $g(t) = u(t-1)$ . Compute  $f * g$ .

**Exercise 6.3.2:** Let  $f(t) = t$  for  $t \geq 0$ , and  $g(t) = \sin t$  for  $t \geq 0$ . Compute  $f * g$ .

**Exercise 6.3.3:** Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function  $f(t)$ , where  $m > 0$ ,  $c > 0$ ,  $k > 0$ , and  $c^2 - 4km > 0$  (system is overdamped). Write the solution as a definite integral.

**Exercise 6.3.4:** Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function  $f(t)$ , where  $m > 0$ ,  $c > 0$ ,  $k > 0$ , and  $c^2 - 4km < 0$  (system is underdamped). Write the solution as a definite integral.

**Exercise 6.3.5:** Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function  $f(t)$ , where  $m > 0$ ,  $c > 0$ ,  $k > 0$ , and  $c^2 = 4km$  (system is critically damped). Write the solution as a definite integral.

**Exercise 6.3.6:** Solve

$$x(t) = e^{-t} + \int_0^t \cos(t - \tau)x(\tau) d\tau.$$

**Exercise 6.3.7:** Solve

$$x(t) = \cos t + \int_0^t \cos(t - \tau)x(\tau) d\tau.$$

**Exercise 6.3.8:** Compute  $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)^2} \right\}$  using convolution.

**Exercise 6.3.9:** Write down the solution to  $x'' - 2x = e^{-t^2}$ ,  $x(0) = 0$ ,  $x'(0) = 0$  as a definite integral. Hint: Do not try to compute the Laplace transform of  $e^{-t^2}$ .

**Exercise 6.3.101:** Let  $f(t) = \cos t$  for  $t \geq 0$ , and  $g(t) = e^{-t}$ . Compute  $f * g$ .

**Exercise 6.3.102:** Compute  $\mathcal{L}^{-1} \left\{ \frac{5}{s^4+s^2} \right\}$  using convolution.

**Exercise 6.3.103:** Solve  $x'' + x = \sin t$ ,  $x(0) = 0$ ,  $x'(0) = 0$  using convolution.

**Exercise 6.3.104:** Solve  $x'' + x = \sin t$ ,  $x(0) = 0$ ,  $x'(0) = 0$  using convolution.

# Chapter 7

## Power series methods

### 7.1 Power series

*Note: 1 or 1.5 lecture , §3.1 in [EP], §5.1 in [BD]*

Many functions can be written in terms of a power series

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k.$$

If we assume that a solution of a differential equation is written as a power series, then perhaps we can use a method reminiscent of undetermined coefficients. That is, we will try to solve for the numbers  $a_k$ . Before we can carry out this process, let us review some results and concepts about power series.

#### 7.1.1 Definition

As we said, a *power series* is an expression such as

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots, \quad (7.1)$$

where  $a_0, a_1, a_2, \dots, a_k, \dots$  and  $x_0$  are constants. Let

$$S_n(x) = \sum_{k=0}^n a_k(x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n,$$

denote the so-called *partial sum*. If for some  $x$ , the limit

$$\lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k(x - x_0)^k$$

exists, then we say that the series (7.1) *converges* at  $x$ . Note that for  $x = x_0$ , the series always converges to  $a_0$ . When (7.1) converges at any other point  $x \neq x_0$ , we say that (7.1) is a *convergent power series*. In this case we write

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k(x - x_0)^k.$$

If the series does not converge for any point  $x \neq x_0$ , we say that the series is *divergent*.

**Example 7.1.1:** The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

is convergent for any  $x$ . Recall that  $k! = 1 \cdot 2 \cdot 3 \cdots k$  is the factorial. By convention we define  $0! = 1$ . In fact, you may recall that this series converges to  $e^x$ .

We say that (7.1) *converges absolutely* at  $x$  whenever the limit

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n |a_k| |x - x_0|^k$$

exists. That is, if the series  $\sum_{k=0}^{\infty} |a_k| |x - x_0|^k$  is convergent. Note that if (7.1) converges absolutely at  $x$ , then it converges at  $x$ . However, the opposite is not true.

**Example 7.1.2:** The series

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k$$

converges absolutely at any  $x \in (-1, 1)$ . It converges at  $x = -1$ , as  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$  converges (conditionally) by the alternating series test. But the power series does not converge absolutely at  $x = -1$ , because  $\sum_{k=1}^{\infty} \frac{1}{k}$  does not converge. The series diverges at  $x = 1$ .

## 7.1.2 Radius of convergence

If a series converges absolutely at some  $x_1$ , then for all  $x$  such that  $|x - x_0| \leq |x_0 - x_1|$  we have that  $|a_k(x - x_0)^k| \leq |a_k(x_1 - x_0)^k|$  for all  $k$ . As the numbers  $|a_k(x_1 - x_0)^k|$  sum to some finite limit, summing smaller positive numbers  $|a_k(x - x_0)^k|$  must also have a finite limit. Therefore, the series must converge absolutely at  $x$ . We have the following result.

**Theorem 7.1.1.** *For a power series (7.1), there exists a number  $\rho$  (we allow  $\rho = \infty$ ) called the radius of convergence such that the series converges absolutely on the interval  $(x_0 - \rho, x_0 + \rho)$  and diverges for  $x < x_0 - \rho$  and  $x > x_0 + \rho$ . We write  $\rho = \infty$  if the series converges for all  $x$ .*

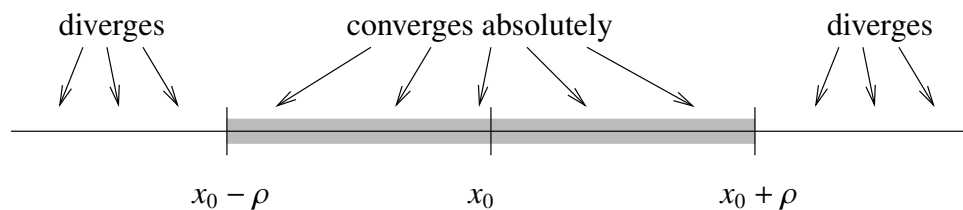


Figure 7.1: Convergence of a power series.

See Figure 7.1. In Example 7.1.1 the radius of convergence is  $\rho = \infty$  as the series converges everywhere. In Example 7.1.2 the radius of convergence is  $\rho = 1$ . We note that  $\rho = 0$  is another way of saying that the series is divergent.

A useful test for convergence of a series is the *ratio test*. Suppose that

$$\sum_{k=0}^{\infty} c_k$$

is a series such that the limit

$$L = \lim_{n \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exists. Then the series converges absolutely if  $L < 1$  and diverges if  $L > 1$ .

Let us apply this test to the series (7.1). That is we let  $c_k = a_k(x - x_0)^k$  in the test. We let

$$L = \lim_{n \rightarrow \infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{k+1}(x - x_0)^{k+1}}{a_k(x - x_0)^k} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| |x - x_0|.$$

Define  $A$  by

$$A = \lim_{n \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

Then if  $1 > L = A|x - x_0|$  the series (7.1) converges absolutely. If  $A = 0$ , then the series always converges. If  $A > 0$ , then the series converges absolutely if  $|x - x_0| < 1/A$ , and diverges if  $|x - x_0| > 1/A$ . That is, the radius of convergence is  $1/A$ . Let us summarize.

**Theorem 7.1.2.** *Let*

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

*be a power series such that*

$$A = \lim_{n \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$$

*exists. If  $A = 0$ , then the radius of convergence of the series is  $\infty$ . Otherwise the radius of convergence is  $1/A$ .*

**Example 7.1.3:** Suppose we have the series

$$\sum_{k=0}^{\infty} 2^{-k}(x-1)^k.$$

First we compute,

$$A = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{2^{-k-1}}{2^{-k}} \right| = 2^{-1} = 1/2.$$

Therefore the radius of convergence is 2, and the series converges absolutely on the interval  $(-1, 3)$ .

The ratio test does not always apply. That is the limit of  $\left| \frac{a_{k+1}}{a_k} \right|$  might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter.

### 7.1.3 Analytic functions

Functions represented by series are called *analytic functions*. Not every function is analytic, although the majority of the functions you have seen in calculus are.

An analytic function  $f(x)$  is equal to its *Taylor series*\* near a point  $x_0$ . That is, for  $x$  near  $x_0$  we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, \quad (7.2)$$

where  $f^{(k)}(x_0)$  denotes the  $k^{\text{th}}$  derivative of  $f(x)$  at the point  $x_0$ .

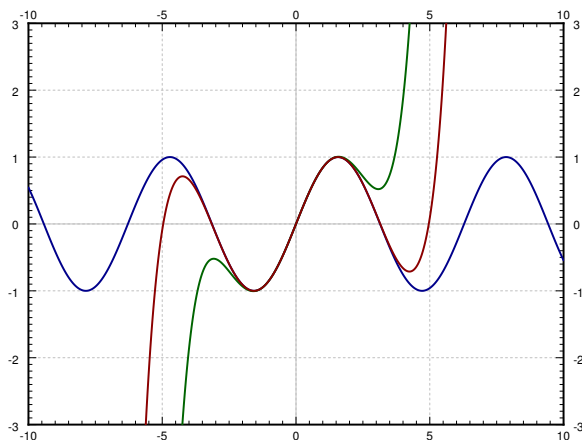


Figure 7.2: The sine function and its Taylor approximations around  $x_0 = 0$  of 5<sup>th</sup> and 9<sup>th</sup> degree.

---

\*Named after the English mathematician Sir Brook Taylor (1685 – 1731).



For example, sine is an analytic function and its Taylor series around  $x_0 = 0$  is given by

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

In Figure 7.2 on the facing page we plot  $\sin(x)$  and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for  $x$  near 0, but gets worse for larger  $x$ . This is what will happen in general. To get good approximation far away from  $x_0$  you will need to take more and more terms of the Taylor series.

### 7.1.4 Manipulating power series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is Suppose that  $\sum a_k(x - x_0)^k$  is a convergent power series. Then for  $x$  in the radius of convergence we have

$$\frac{d}{dx} \left[ \sum_{k=0}^{\infty} a_k(x - x_0)^k \right] = \sum_{k=1}^{\infty} k a_k(x - x_0)^{k-1}.$$

Notice that the term corresponding to  $k = 0$  disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

**Example 7.1.4:** Let us show that the exponential  $y = e^x$  solves  $y' = y$ . First write

$$y = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Now differentiate

$$y' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}.$$

For convenience we *reindex* the series by simply replacing  $k$  with  $k + 1$ . The series does not change, what changes is simply how we write it. After reindexing the series starts at  $k = 0$  again.

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

That was precisely the power series for  $e^x$  that we started with, so we showed that  $\frac{d}{dx} e^x = e^x$ .

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. Firstly, we can add series by adding term by term,

$$\left( \sum_{k=0}^{\infty} a_k(x - x_0)^k \right) + \left( \sum_{k=0}^{\infty} b_k(x - x_0)^k \right) = \sum_{k=0}^{\infty} (a_k + b_k)(x - x_0)^k.$$

We can multiply by constants,

$$\alpha \left( \sum_{k=0}^{\infty} a_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} \alpha a_k (x - x_0)^k.$$

We can also multiply series together,

$$\left( \sum_{k=0}^{\infty} a_k (x - x_0)^k \right) \left( \sum_{k=0}^{\infty} b_k (x - x_0)^k \right) = \sum_{k=0}^{\infty} c_k (x - x_0)^k,$$

where  $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$ . The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

### 7.1.5 Power series for rational functions

Polynomials are simply finite power series. That is, a polynomial is a power series where the  $a_k$  beyond a certain point are all zero. We can always expand a polynomial as a power series about any point  $x_0$  by writing the polynomial as a polynomial of  $(x - x_0)$ . For example, let us write  $2x^2 - 3x + 4$  as a power series around  $x_0 = 1$ :

$$2x^2 - 3x + 4 = 3 + (x - 1) + 2(x - 1)^2.$$

In other words  $a_0 = 3$ ,  $a_1 = 1$ ,  $a_2 = 2$ , and all other  $a_k = 0$ . To do this, we know that  $a_k = 0$  for all  $k \geq 3$ . So we write  $a_0 + a_1(x - 1) + a_2(x - 1)^2$ , we expand, and we solve for  $a_0$ ,  $a_1$ , and  $a_2$ . We could have also differentiated at  $x = 1$  and used the Taylor series formula (7.2).

Now let us look at rational functions. Notice that a series for a function only defines the function on an interval. For example, for  $-1 < x < 1$  we have

$$\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for  $x \leq -1$  and  $x \geq 1$ , even though  $\frac{1}{1-x}$  is defined for all  $x \neq 1$ .

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at  $x_0$ . Note that as for polynomials, we could equivalently use the Taylor series expansion (7.2).

**Example 7.1.5:** Expand  $\frac{x}{1+2x+x^2}$  as a power series around the origin and find the radius of convergence.

First, write  $1 + 2x + x^2 = (1 + x)^2 = (1 - (-x))^2$ . Now we compute

$$\begin{aligned}\frac{x}{1 + 2x + x^2} &= x \left( \frac{1}{1 - (-x)} \right)^2 \\ &= x \left( \sum_{k=0}^{\infty} (-1)^k x^k \right) \\ &= x \left( \sum_{k=0}^{\infty} c_k x^k \right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1},\end{aligned}$$

where using the formula for product of product of series we obtain,  $c_0 = 1$ ,  $c_1 = -1 - 1 = -2$ ,  $c_2 = 1 + 1 + 1 = 3$ , etc.... Therefore

$$\frac{x}{1 + 2x + x^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k = x - 2x^2 + 3x^3 - 4x^4 + \dots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k} \right| = \lim_{k \rightarrow \infty} \frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1.

### 7.1.6 Exercises

**Exercise 7.1.1:** Is the power series  $\sum_{k=0}^{\infty} e^k x^k$  convergent? If so, what is the radius of convergence?

**Exercise 7.1.2:** Is the power series  $\sum_{k=0}^{\infty} kx^k$  convergent? If so, what is the radius of convergence?

**Exercise 7.1.3:** Is the power series  $\sum_{k=0}^{\infty} k!x^k$  convergent? If so, what is the radius of convergence?

**Exercise 7.1.4:** Is the power series  $\sum_{k=0}^{\infty} \frac{1}{(2k)!} (x - 10)^k$  convergent? If so, what is the radius of convergence?

**Exercise 7.1.5:** Determine the Taylor series for  $\sin x$  around the point  $x_0 = \pi$ .

**Exercise 7.1.6:** Determine the Taylor series for  $\ln x$  around the point  $x_0 = 1$ , and find the radius of convergence.

**Exercise 7.1.7:** Determine the Taylor series and its radius of convergence of  $\frac{1}{1+x}$  around  $x_0 = 0$ .

**Exercise 7.1.8:** Determine the Taylor series and its radius of convergence of  $\frac{x}{4-x^2}$  around  $x_0 = 0$ .  
Hint: you will not be able to use the ratio test.

**Exercise 7.1.9:** Expand  $x^5 + 5x + 1$  as a power series around  $x_0 = 5$ .

**Exercise 7.1.10:** Suppose that the ratio test applies to a series  $\sum_{k=0}^{\infty} a_k x^k$ . Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.

**Exercise 7.1.101:** Is the power series  $\sum_{n=1}^{\infty} (0.1)^n x^n$  convergent? If so, what is the radius of convergence?

**Exercise 7.1.102** (challenging): Is the power series  $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$  convergent? If so, what is the radius of convergence?

**Exercise 7.1.103:** Using the geometric series, expand  $\frac{1}{1-x}$  around  $x_0 = 2$ . For what  $x$  does the series converge?

**Exercise 7.1.104** (challenging): Find the Taylor series for  $x^7 e^x$  around  $x_0 = 0$ .

## 7.2 Series solutions of linear second order ODEs

Note: 1 or 1.5 lecture, §3.1 in [EP], §5.2 and §5.3 in [BD]

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0. \quad (7.3)$$

Suppose that  $p(x)$ ,  $q(x)$ , and  $r(x)$  are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k \quad (7.4)$$

and solve for the  $a_k$  to try to obtain a solution defined in some interval around  $x_0$ .

The point  $x_0$  is called an *ordinary point* if  $p(x_0) \neq 0$ . That is, the functions

$$\frac{q(x)}{p(x)} \quad \text{and} \quad \frac{r(x)}{p(x)} \quad (7.5)$$

are defined for  $x$  near  $x_0$ . If  $p(x_0) = 0$ , then we say  $x_0$  is a *singular point*. Handling singular points is harder than ordinary points and so we will focus only on ordinary points.

**Example 7.2.1:** Let us start with a very simple example

$$y'' - y = 0.$$

Let us try a power series solution near  $x_0 = 0$ , which is an ordinary point. Every point is an ordinary point in fact, as the equation is constant coefficient. We already know we should obtain exponentials or the hyperbolic sine and cosine, but let us pretend we do not know this.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

If we differentiate, the  $k = 0$  term is a constant and hence disappears. We therefore get

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

We differentiate yet again to obtain (now the  $k = 1$  term disappears)

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We reindex the series (replace  $k$  with  $k + 2$ ) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

Now we plug  $y$  and  $y''$  into the differential equation

$$\begin{aligned} 0 = y'' - y &= \left( \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \right) - \left( \sum_{k=0}^{\infty} a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \left( (k+2)(k+1) a_{k+2} - a_k \right) x^k \\ &= \sum_{k=0}^{\infty} ((k+2)(k+1) a_{k+2} - a_k) x^k. \end{aligned}$$

As  $y'' - y$  is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)(k+1) a_{k+2} - a_k = 0, \quad \text{or} \quad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The above equation is called a *recurrence relation* for the coefficients of the power series. It did not matter what  $a_0$  or  $a_1$  was. They can be arbitrary. But once we pick  $a_0$  and  $a_1$ , then all other coefficients are determined by the recurrence relation.

So let us see what the coefficients must be. First,  $a_0$  and  $a_1$  are arbitrary

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}, \quad \dots$$

So we note that for even  $k$ , that is  $k = 2n$  we get

$$a_k = a_{2n} = \frac{a_0}{(2n)!}, \tag{7.6}$$

and for odd  $k$ , that is  $k = 2n + 1$  we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}. \tag{7.7}$$

Let us write down the series

$$y = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \left( \frac{a_0}{(2n)!} x^{2n} + \frac{a_1}{(2n+1)!} x^{2n+1} \right) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

Now we recognize the two series as the hyperbolic sine and cosine. Therefore,

$$y = a_0 \cosh x + a_1 \sinh x.$$

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

**Example 7.2.2:** Let us do a more complex example. Suppose we wish to solve *Airy's equation*<sup>†</sup>, that is

$$y'' - xy = 0,$$

near the point  $x_0 = 0$ . Note that  $x_0 = 0$  is an ordinary point.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate twice (as above) to obtain

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Now we plug into the equation

$$\begin{aligned} 0 = y'' - xy &= \left( \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - x \left( \sum_{k=0}^{\infty} a_k x^k \right) \\ &= \left( \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - \left( \sum_{k=0}^{\infty} a_k x^{k+1} \right). \end{aligned}$$

Now we reindex to make things easier to sum

$$\begin{aligned} 0 = y'' - xy &= \left( 2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k \right) - \left( \sum_{k=1}^{\infty} a_{k-1} x^k \right) \\ &= 2a_2 + \sum_{k=1}^{\infty} ((k+2)(k+1) a_{k+2} - a_{k-1}) x^k. \end{aligned}$$

Again  $y'' - xy$  is supposed to be 0 so first we notice that  $a_2 = 0$  and also

$$(k+2)(k+1) a_{k+2} - a_{k-1} = 0, \quad \text{or} \quad a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}.$$

Now we jump in steps of three. First we notice that since  $a_2 = 0$  we must have that,  $a_5 = 0$ ,  $a_8 = 0$ ,  $a_{11} = 0$ , etc. . . . In general  $a_{3n+2} = 0$ .

The constants  $a_0$  and  $a_1$  are arbitrary and we obtain

$$a_3 = \frac{a_0}{(3)(2)}, \quad a_4 = \frac{a_1}{(4)(3)}, \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)}, \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)}, \quad \dots$$

---

<sup>†</sup>Named after the English mathematician Sir George Biddell Airy (1801 – 1892).

For  $a_k$  where  $k$  is a multiple of 3, that is  $k = 3n$  we notice that

$$a_{3n} = \frac{a_0}{(2)(3)(5)(6) \cdots (3n-1)(3n)}.$$

For  $a_k$  where  $k = 3n + 1$ , we notice

$$a_{3n+1} = \frac{a_1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}.$$

In other words, if we write down the series for  $y$  we notice that it has two parts

$$\begin{aligned} y &= \left( a_0 + \frac{a_0}{6}x^3 + \frac{a_0}{180}x^6 + \cdots + \frac{a_0}{(2)(3)(5)(6) \cdots (3n-1)(3n)}x^{3n} + \cdots \right) \\ &\quad + \left( a_1x + \frac{a_1}{12}x^4 + \frac{a_1}{504}x^7 + \cdots + \frac{a_1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}x^{3n+1} + \cdots \right) \\ &= a_0 \left( 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6) \cdots (3n-1)(3n)}x^{3n} + \cdots \right) \\ &\quad + a_1 \left( x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}x^{3n+1} + \cdots \right). \end{aligned}$$

We define

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6) \cdots (3n-1)(3n)}x^{3n} + \cdots, \\ y_2(x) &= x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7) \cdots (3n)(3n+1)}x^{3n+1} + \cdots, \end{aligned}$$

and write the general solution to the equation as  $y(x) = a_0y_1(x) + a_1y_2(x)$ . Notice from the power series that  $y_1(0) = 1$  and  $y_2(0) = 0$ . Also,  $y_1'(0) = 0$  and  $y_2'(0) = 1$ . If we obtained a solution that satisfies the initial conditions  $y(0) = a_0$  and  $y'(0) = a_1$ .

The functions  $y_1$  and  $y_2$  cannot be written in terms of the elementary functions that you know. See Figure 7.3 for the plot of the solutions  $y_1$  and  $y_2$ . These functions have many interesting properties. For example, they are oscillatory for negative  $x$  (like solutions to  $y'' + y = 0$ ) and for positive  $x$  they grow without bound (like solutions to  $y'' - y = 0$ ).

Sometimes a solution may turn out to be a polynomial.

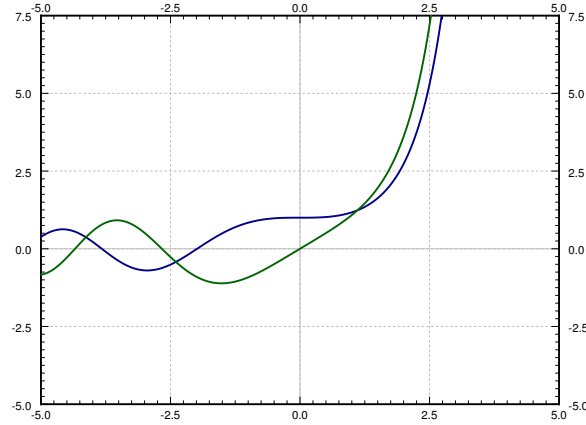
**Example 7.2.3:** Let us find a solution to the so-called *Hermite's equation of order  $n^{\ddagger}$*  is the equation

$$y'' - 2xy' + 2ny = 0.$$

---

<sup>‡</sup>Named after the French mathematician Charles Hermite (1822–1901).



Figure 7.3: The two solutions  $y_1$  and  $y_2$  to Airy's equation.

Let us find a solution around the point  $x_0 = 0$ . We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate (as above) to obtain

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1},$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Now we plug into the equation

$$\begin{aligned} 0 = y'' - 2xy' + 2ny &= \left( \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - 2x \left( \sum_{k=1}^{\infty} k a_k x^{k-1} \right) + 2n \left( \sum_{k=0}^{\infty} a_k x^k \right) \\ &= \left( \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} \right) - \left( \sum_{k=1}^{\infty} 2k a_k x^k \right) + \left( \sum_{k=0}^{\infty} 2n a_k x^k \right) \\ &= \left( 2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1) a_{k+2} x^k \right) - \left( \sum_{k=1}^{\infty} 2k a_k x^k \right) + \left( 2n a_0 + \sum_{k=1}^{\infty} 2n a_k x^k \right) \\ &= 2a_2 + 2n a_0 + \sum_{k=1}^{\infty} ((k+2)(k+1) a_{k+2} - 2k a_k + 2n a_k) x^k. \end{aligned}$$

As  $y'' - 2xy' + 2ny = 0$  we have

$$(k+2)(k+1) a_{k+2} + (-2k + 2n) a_k = 0, \quad \text{or} \quad a_{k+2} = \frac{(2k - 2n)}{(k+2)(k+1)} a_k.$$

This recurrence relation actually includes  $a_2 = -na_0$  (which comes about from  $2a_2 + 2na_0 = 0$ ). Again  $a_0$  and  $a_1$  are arbitrary.

$$\begin{aligned} a_2 &= \frac{-2n}{(2)(1)}a_0, & a_3 &= \frac{2(1-n)}{(3)(2)}a_1, \\ a_4 &= \frac{2(2-n)}{(4)(3)}a_2 = \frac{2^2(2-n)(-n)}{(4)(3)(2)(1)}a_0, \\ a_5 &= \frac{2(3-n)}{(5)(4)}a_3 = \frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)}a_1, \quad \dots \end{aligned}$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1 + \frac{2(-n)}{2!}x^2 + \frac{2^2(-n)(2-n)}{4!}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6!}x^6 + \cdots, \\ y_2(x) &= x + \frac{2(1-n)}{3!}x^3 + \frac{2^2(1-n)(3-n)}{5!}x^5 + \frac{2^3(1-n)(3-n)(5-n)}{7!}x^7 + \cdots. \end{aligned}$$

We then write

$$y(x) = a_0y_1(x) + a_1y_2(x). \quad (7.8)$$

We also notice that if  $n$  is a positive even integer, then  $y_1(x)$  is a polynomial as all the coefficients in the series beyond a certain degree are zero. If  $n$  is a positive odd integer, then  $y_2(x)$  is a polynomial. For example if  $n = 4$ , then

$$y_1(x) = 1 + \frac{2(-4)}{2!}x^2 + \frac{2^2(-4)(2-4)}{4!}x^4 = 1 - 4x^2 + \frac{4}{3}x^4. \quad (7.9)$$

### 7.2.1 Exercises

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the  $k^{\text{th}}$  coefficient.

**Exercise 7.2.1:** Use power series methods to solve  $y'' + y = 0$  at the point  $x_0 = 1$ .

**Exercise 7.2.2:** Use power series methods to solve  $y'' + 4xy = 0$  at the point  $x_0 = 0$ .

**Exercise 7.2.3:** Use power series methods to solve  $y'' - xy = 0$  at the point  $x_0 = 1$ .

**Exercise 7.2.4:** Use power series methods to solve  $y'' + x^2y = 0$  at the point  $x_0 = 0$ .

**Exercise 7.2.5:** The methods work for other orders than second order. Try the methods of this section to solve the first order system  $y' - xy = 0$  at the point  $x_0 = 0$ .

**Exercise 7.2.6** (Chebyshev's equation of order  $p$ ): a) Solve  $(1 - x^2)y'' - xy' + p^2y = 0$  using power series methods at  $x_0 = 0$ . b) For what  $p$  is there a polynomial solution?

**Exercise 7.2.7:** Find a polynomial solution to  $(x^2 + 1)y'' - 2xy' + 2y = 0$  using power series methods.

**Exercise 7.2.8:** a) Use power series methods to solve  $(1 - x)y'' + y = 0$  at the point  $x_0 = 0$ . b) Use the solution to part a) to find a solution for  $xy'' + y = 0$  around the point  $x_0 = 1$ .

**Exercise 7.2.101:** Use power series methods to solve  $y'' + 2x^3y = 0$  at the point  $x_0 = 0$ .

**Exercise 7.2.102** (challenging): We can also use power series methods in nonhomogeneous equations. a) Use power series methods to solve  $y'' - xy = \frac{1}{1-x}$  at the point  $x_0 = 0$ . Hint: recall the geometric series. b) Now solve for the initial condition  $y(0) = 0$ ,  $y'(0) = 0$ .

**Exercise 7.2.103:** Attempt to solve  $x^2y'' - y = 0$  at  $x_0 = 0$  using the power series method of this section ( $x_0$  is a singular point). Can you find at least one solution? Can you find more than one solution?



## Further Reading

- [BM] Paul W. Berg and James L. McGregor, *Elementary Partial Differential Equations*, Holden-Day, San Francisco, CA, 1966.
- [BD] William E. Boyce, Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 9th edition, John Wiley & Sons Inc., New York, NY, 2008.
- [EP] C.H. Edwards and D.E. Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling*, 4th edition, Prentice Hall, 2008.
- [F] Stanley J. Farlow, *An Introduction to Differential Equations and Their Applications*, McGraw-Hill, Inc., Princeton, NJ, 1994.
- [I] E.L. Ince, *Ordinary Differential Equations*, Dover Publications, Inc., New York, NY, 1956.



## Solutions to Selected Exercises

**0.2.101:** Compute  $x' = -2e^{-2t}$  and  $x'' = 4e^{-2t}$ . Then  $(4e^{-2t}) + 4(-2e^{-2t}) + 4(e^{-2t}) = 0$ .

**0.2.102:** Yes.

**0.2.103:**  $y = x^r$  is a solution for  $r = 0$  and  $r = 2$ .

**0.2.104:**  $C_1 = 100, C_2 = -90$

**0.2.105:**  $\varphi = -9e^{8s}$

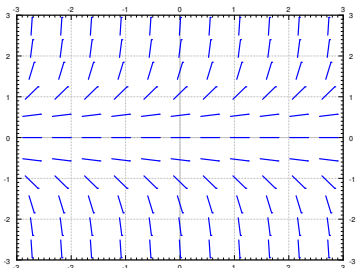
**1.1.101:**  $y = e^x + \frac{x^2}{2} + 9$

**1.1.102:**  $x = (3t - 2)^{1/3}$

**1.1.103:**  $x = \sin^{-1}(t + 1)$

**1.1.104:** 170

**1.2.101:**



$y = 0$  is a solution such that  $y(0) = 0$ .

**1.2.102:** Yes a solution exists.  $y' = f(x, y)$  where  $f(x, y) = xy$ . The function  $f(x, y)$  is continuous and  $\frac{\partial f}{\partial y} = x$ , which is also continuous near  $(0, 0)$ . So a solution exists and is unique. (In fact  $y = 0$  is the solution).

**1.2.103:** No, the equation is not defined at  $(x, y) = (1, 0)$ .

**1.3.101:**  $y = Ce^{x^2}$

**1.3.102:**  $y = e^{t^3} + 1$

**1.3.103:**  $x^3 + x = t + 2$

**1.3.104:**  $y = \frac{1}{1 - \ln x}$

**1.4.101:**  $y = Ce^{-x^3} + 1/3$

**1.4.102:**  $y = 2e^{\cos(2x)} + 1$

**1.4.103:** 250 grams

**1.4.104:**  $P(10) = 1000e^{2 \times 10 - 0.05 \times 10^2} = 1000e^{15} \approx 3.27 \times 10^9$

**1.5.101:**  $y = \frac{2}{3x-2}$

**1.5.102:**  $y = \frac{3-x^2}{2x}$

**1.5.103:**  $y = (7e^{3x} + 3x + 1)^{1/3}$

**1.5.104:**  $y = \sqrt{x^2 - \ln(C - x)}$

**1.6.101:**

a) 0, 1, 2 are critical points.

b)  $x = 0$  is unstable (semistable),  $x = 1$  is stable, and  $x = 2$  is unstable.

c) 1

**1.6.102:** a) There are no critical points. b)  $\infty$

**1.6.103:** a)  $\frac{dx}{dt} = kx(M - x) + A$  b)  $\frac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$

**1.7.101:** Approximately: 1.0000, 1.2397, 1.3829

**1.7.102:**

a) 0, 8, 12

b)  $x(4) = 16$ , so errors are: 16, 8, 4.

c) Factors are 0.5, 0.5, 0.5.

**1.7.103:** a) 0, 0, 0 b)  $x = 0$  is a solution so errors are: 0, 0, 0.

**2.1.101:** Yes. To justify try to find a constant  $A$  such that  $\sin(x) = Ae^x$  for all  $x$ .

**2.1.102:** No.  $e^{x+2} = e^2 e^x$ .

**2.1.103:**  $y = 5$

**2.1.104:**  $y = C_1 \ln(x) + C_2$

**2.2.101:**  $y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$

**2.2.102:**  $y = C_1 e^{3x} + C_2 x e^{3x}$

**2.2.103:**  $y = e^{-x/4} \cos((\sqrt{7}/4)x) - \sqrt{7} e^{-x/4} \sin((\sqrt{7}/4)x)$

**2.2.104:**  $y(x) = \frac{2(a-b)}{5} e^{-3x/2} + \frac{3a+2b}{5} e^x$

**2.2.105:**  $z(t) = 2e^{-t} \cos(t)$

**2.3.101:**  $y = C_1 e^x + C_2 x^3 + C_3 x^2 + C_4 x + C_5$

**2.3.102:** a)  $r^3 - 3r^2 + 4r - 12 = 0$ , b)  $y''' - 3y'' + 4y' - 12y = 0$ , c)  $y = C_1 e^{3x} + C_2 \sin(2x) + C_3 \cos(2x)$

**2.3.103:**  $y = 0$ .

**2.3.104:** No.  $e^1 e^x - e^{x+1} = 0$ .

**2.3.105:** Yes. (Hint: First note that  $\sin(x)$  is bounded. Then note that  $x$  and  $x \sin(x)$  cannot be multiples of each other.)



**2.4.101:**  $k = 8/9$  (and larger)

**2.4.102:**

a)  $0.05I'' + 0.1I' + (1/5)I = 0$

b)  $x(t) = Ce^{-t} \cos(\sqrt{3}t - \gamma)$

c)  $x(t) = 10e^{-t} \cos(\sqrt{3}t) + \frac{10}{\sqrt{3}}e^{-t} \sin(\sqrt{3}t)$

**2.4.103:** a)  $k = 500000$  b)  $\frac{1}{5\sqrt{2}} \approx 0.141$  c) 45000 kg d) 11250 kg

**2.5.101:**  $y = \frac{-16 \sin(3x) - 6 \cos(3x)}{73}$

**2.5.102:** a)  $y = \frac{2e^x + 3x^3 - 9x}{6}$ . b)  $y = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) + \frac{2e^x + 3x^3 - 9x}{6}$ .

**2.5.103:**  $y(x) = x^2 - 4x + 6 + e^{-x}(x - 5)$ .

**2.5.104:**  $y = \frac{2xe^x - (e^x + e^{-x}) \log(e^{2x} + 1)}{4}$

**2.6.101:**  $\omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984$   $C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$

**2.6.102:**  $x_{sp} = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2} \sin(\omega t) + \frac{A}{k}$ , where  $p = \frac{c}{2m}$  and  $\omega_0 = \sqrt{\frac{k}{m}}$ .

**2.6.103:** a)  $\omega = 2$  b) 25

**3.1.101:**  $y_1 = C_1 e^{3x}$ ,  $y_2 = y(x) = C_2 e^x + \frac{C_1}{2} e^{3x}$ ,  $y_3 = y(x) = C_3 e^x + \frac{C_1}{2} e^{3x}$

**3.1.102:**  $x = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$ ,  $y = \frac{5}{3}e^{2t} + \frac{4}{3}e^{-t}$

**3.1.103:**  $x'_1 = x_2$ ,  $x'_2 = x_3$ ,  $x'_3 = x_1 + t$

**3.1.104:**  $y'_3 + y_1 + y_2 = t$ ,  $y'_4 + y_1 - y_2 = t^2$ ,  $y'_1 = y_3$ ,  $y'_2 = y_4$

**3.2.101:** -15

**3.2.102:** -2

**3.2.103:**  $\vec{x} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

**3.2.104:** a)  $\begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$  b)  $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

**3.3.101:** Yes.

**3.3.102:** No.  $2 \begin{bmatrix} \cosh(t) \\ 1 \end{bmatrix} - \begin{bmatrix} e^t \\ 1 \end{bmatrix} - \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} = \vec{0}$

**3.3.103:**  $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

**3.3.104:** a)  $\vec{x}' = \begin{bmatrix} 0 & 2t \\ 0 & 2t \end{bmatrix} \vec{x}$  b)  $\vec{x} = \begin{bmatrix} C_2 e^{t^2} + C_1 \\ C_2 e^{t^2} \end{bmatrix}$

**3.4.101:**

a) Eigenvalues: 4, 0, -1 Eigenvectors:  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$

b)  $\vec{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} e^{-t}$

**3.4.102:**

a) Eigenvalues:  $\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$ , Eigenvectors:  $\begin{bmatrix} -2 \\ 1-\sqrt{3}i \end{bmatrix}, \begin{bmatrix} -2 \\ 1+\sqrt{3}i \end{bmatrix}$

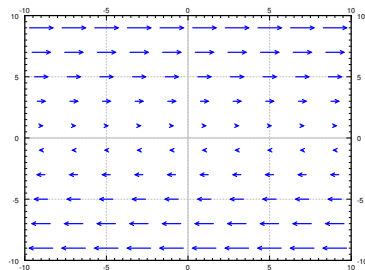
b)  $\vec{x} = C_1 e^{t/2} \begin{bmatrix} -2 \cos(\frac{\sqrt{3}t}{2}) \\ \cos(\frac{\sqrt{3}t}{2}) + \sqrt{3} \sin(\frac{\sqrt{3}t}{2}) \end{bmatrix} + C_2 e^{t/2} \begin{bmatrix} -2 \sin(\frac{\sqrt{3}t}{2}) \\ \sin(\frac{\sqrt{3}t}{2}) - \sqrt{3} \cos(\frac{\sqrt{3}t}{2}) \end{bmatrix}$

**3.4.103:**  $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

**3.4.104:**  $\vec{x} = C_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

**3.5.101:** a) Two eigenvalues:  $\pm \sqrt{2}$  so the behavior is a saddle. b) Two eigenvalues: 1 and 2, so the behavior is a source. c) Two eigenvalues:  $\pm 2i$ , so the behavior is a center (ellipses). d) Two eigenvalues: -1 and -2, so the behavior is a sink. e) Two eigenvalues:  $1 \pm 4i$ , so the behavior is a spiral source.

**3.5.102:** Spiral source.

**3.5.103:**

The solution will not move anywhere if  $y = 0$ . When  $y$  is positive, then the solution moves (with constant speed) in the positive  $x$  direction. When  $y$  is negative, then the solution moves (with constant speed) in the negative  $x$  direction. It is not one of the behaviors we have seen.

Note that the matrix has a double eigenvalue 0 and the general solution is  $x = C_1 t + C_2$  and  $y = C_1$ , which agrees with the above description.

**3.6.101:**  $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_1 \cos(\sqrt{3}t) + b_1 \sin(\sqrt{3}t)) + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} (a_2 \cos(\sqrt{2}t) + b_2 \sin(\sqrt{2}t)) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (a_3 \cos(t) + b_3 \sin(t)) + \begin{bmatrix} -1 \\ 1/2 \\ 2/3 \end{bmatrix} \cos(2t)$

**3.6.102:**  $\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \vec{x}'' = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \vec{x}$ . Solution:  $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} (a_1 \cos(\sqrt{3k/m}t) + b_1 \sin(\sqrt{3k/m}t)) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} (a_2 \cos(\sqrt{k/m}t) + b_2 \sin(\sqrt{k/m}t)) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (a_3 t + b_3)$ .

**3.6.103:**  $x_2 = (2/5) \cos(\sqrt{1/6}t) - (2/5) \cos(t)$

**3.7.101:** a) 3, 0, 0 b) No defects. c)  $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

**3.7.102:**

a) 1, 1, 2

b) Eigenvalue 1 has a defect of 1

c)  $\vec{x} = C_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^t + C_2 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) e^t + C_3 \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} e^{2t}$

**3.7.103:**

a) 2, 2, 2

b) Eigenvalue 2 has a defect of 2

$$c) \vec{x} = C_1 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} e^{2t} + C_2 \left( \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t} + C_3 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t}$$

$$\mathbf{3.7.104:} \quad A = \begin{bmatrix} 5 & 5 \\ 0 & 5 \end{bmatrix}$$

$$\mathbf{3.8.101:} \quad e^{tA} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{-t}-e^{3t}}{2} \\ \frac{e^{-t}-e^{3t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}$$

$$\mathbf{3.8.102:} \quad e^{tA} = \begin{bmatrix} 2e^{3t}-4e^{2t}+3e^t & \frac{3e^t}{2}-\frac{3e^{3t}}{2} & -e^{3t}+4e^{2t}-3e^t \\ 2e^t-2e^{2t} & e^t & 2e^{2t}-2e^t \\ 2e^{3t}-5e^{2t}+3e^t & \frac{3e^t}{2}-\frac{3e^{3t}}{2} & -e^{3t}+5e^{2t}-3e^t \end{bmatrix}$$

$$\mathbf{3.8.103:} \quad a) e^{tA} = \begin{bmatrix} (t+1)e^{2t} & -te^{2t} \\ te^{2t} & (1-t)e^{2t} \end{bmatrix} \quad b) \vec{x} = \begin{bmatrix} (1-t)e^{2t} \\ (2-t)e^{2t} \end{bmatrix}$$

$$\mathbf{3.8.104:} \quad \begin{bmatrix} 1+2t+5t^2 & 3t+6t^2 \\ 2t+4t^2 & 1+2t+5t^2 \end{bmatrix} \quad e^{0.1A} \approx \begin{bmatrix} 1.25 & 0.36 \\ 0.24 & 1.25 \end{bmatrix}$$

**3.9.101:** The general solution is (particular solutions should agree with one of these):

$$x(t) = \frac{1}{5}C_1(e^{9t} + e^{4t}) + \frac{4}{5}C_2(e^{9t} - e^{4t}) - \frac{18t+5}{54} \quad y(t) = \frac{1}{5}C_1(e^{9t} - e^{4t}) + \frac{1}{5}C_2(4e^{5t} + e^{4t}) + \frac{t}{6} + \frac{7}{216}$$

**3.9.102:** The general solution is (particular solutions should agree with one of these):

$$x(t) = \frac{1}{2}C_1(e^t + e^{-t}) + \frac{1}{2}C_2(e^t - e^{-t}) + te^t \quad y(t) = \frac{1}{2}C_1(e^t - e^{-t}) + \frac{1}{2}C_2(e^t + e^{-t}) + te^t$$

$$\mathbf{3.9.103:} \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left( \frac{5}{2}e^t - t - 1 \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{-1}{2}e^{-t}$$

$$\mathbf{3.9.104:} \quad \vec{x} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \left( \left( \frac{1}{140} + \frac{1}{120\sqrt{6}} \right) e^{\sqrt{6}t} + \left( \frac{1}{140} + \frac{1}{120\sqrt{6}} \right) e^{-\sqrt{6}t} - \frac{t}{60} - \frac{\cos(t)}{70} \right) \\ + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left( \frac{-9}{80} \sin(2t) + \frac{1}{30} \cos(2t) + \frac{9t}{40} - \frac{\cos(t)}{30} \right)$$

$$\mathbf{4.1.101:} \quad \omega = \pi \sqrt{\frac{15}{2}}$$

$$\mathbf{4.1.102:} \quad \lambda_k = 4k^2\pi^2 \text{ for } k = 1, 2, 3, \dots \quad x_k = \cos(2k\pi t) + B \sin(2k\pi t) \quad (\text{for any } B)$$

$$\mathbf{4.1.103:} \quad x(t) = -\sin(t)$$

**4.1.104:**  $x = Ce^{-\lambda t}$  and if  $x(0) = 0$  then  $C = 0$  and so  $x(t) = 0$ , so the solution is always identically zero. Note that one condition is always enough to guarantee a unique solution for a first order equation.

$$\mathbf{4.1.105:} \quad \frac{\sqrt{3}}{3}e^{\frac{-3}{2}\sqrt[3]{\lambda}} - \frac{\sqrt{3}}{3}\cos\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\right) + \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}\right) = 0$$

$$\mathbf{4.2.101:} \quad \sin(t)$$

$$\mathbf{4.2.102:} \quad \sum_{n=1}^{\infty} \frac{(\pi-n)\sin(\pi n+\pi^2)+(\pi+n)\sin(\pi n-\pi^2)}{\pi n^2-\pi^3} \sin(nt)$$

$$\mathbf{4.2.103:} \quad \frac{1}{2} - \frac{1}{2}\cos(2t)$$

$$\mathbf{4.2.104:} \quad \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{(-1)^n(8\pi^2 n^2 - 48)}{n^4} \cos(nt)$$

$$4.3.101: \quad \text{a) } \frac{8}{6} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{\pi^2 n^2} \cos\left(\frac{n\pi}{2}t\right) \quad \text{b) } \frac{8}{6} - \frac{16}{\pi^2} \cos\left(\frac{\pi}{2}t\right) + \frac{4}{\pi^2} \cos(\pi t) - \frac{16}{9\pi^2} \cos\left(\frac{3\pi}{2}t\right) + \dots$$

$$4.3.102: \quad \text{a) } \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\lambda}{n\pi} \sin\left(\frac{n\pi}{\lambda}t\right) \quad \text{b) } \frac{2\lambda}{\pi} \sin\left(\frac{\pi}{\lambda}t\right) - \frac{\lambda}{\pi} \sin\left(\frac{2\pi}{\lambda}t\right) + \frac{2\lambda}{3\pi} \sin\left(\frac{3\pi}{\lambda}t\right) - \dots$$

$$4.3.103: \quad f'(t) = \sum_{n=1}^{\infty} \frac{\pi}{n+1} \cos(n\pi t)$$

$$4.3.104: \quad \text{a) } F(t) = \frac{t}{2} + C + \sum_{n=1}^{\infty} \frac{1}{n^4} \sin(nt) \quad \text{b) no.}$$

$$4.4.101: \quad \text{a) } 1/2 + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{\pi^2 n^2} \cos\left(\frac{n\pi}{3}t\right) \quad \text{b) } \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \sin\left(\frac{n\pi}{3}t\right)$$

$$4.4.102: \quad \text{a) } \cos(2t) \quad \text{b) } \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4n}{\pi n^2 - 4\pi} \sin(nt)$$

$$4.4.103: \quad \text{a) } f(t) \quad \text{b) } 0$$

$$4.4.104: \quad \sum_{n=1}^{\infty} \frac{-1}{n^2(1+n^2)} \sin(nt)$$

$$4.4.105: \quad \frac{t}{\pi} + \sum_{n=1}^{\infty} \frac{1}{2^n(\pi - n^2)} \sin(nt)$$

$$4.5.101: \quad x = \frac{1}{\sqrt{2-4\pi^2}} \sin(2\pi t) + \frac{0.1}{\sqrt{2-100\pi^2}} \cos(10\pi t)$$

$$4.5.102: \quad x = \sum_{n=1}^{\infty} \frac{e^{-n}}{3-2n} \cos(2nt)$$

$$4.5.103: \quad x = \frac{1}{2\sqrt{3}} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^2(\sqrt{3}-n^2\pi^2)} \cos(n\pi t)$$

$$4.5.104: \quad x = \frac{1}{2\sqrt{3}} - \frac{2}{\pi^3} t \sin(\pi t) + \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi^4(1-n^2)} \cos(n\pi t)$$

$$4.6.101: \quad u(x, t) = 5 \sin(x) e^{-3t} + 2 \sin(5x) e^{-75t}$$

$$4.6.102: \quad u(x, t) = 1 + 2 \cos(x) e^{-0.1t}$$

$$4.6.103: \quad u(x, t) = e^{\lambda t} e^{\lambda x} \text{ for some } \lambda$$

$$4.6.104: \quad u(x, t) = A e^x + B e^t$$

$$4.7.101: \quad y(x, t) = \sin(x) (\sin(t) + \cos(t))$$

$$4.7.102: \quad y(x, t) = \frac{1}{5\pi} \sin(\pi x) \sin(5\pi t) + \frac{1}{100\pi} \sin(2\pi x) \sin(10\pi t)$$

$$4.7.103: \quad y(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \cos(n\sqrt{2}t)$$

$$4.7.104: \quad y(x, t) = \sin(2x) + t \sin(x)$$

$$4.8.101: \quad y(x, t) = \frac{\sin(2\pi(x-3t)) + \sin(2\pi(3t+x))}{2} + \frac{\cos(3\pi(x-3t)) - \cos(3\pi(3t+x))}{18\pi}$$

$$4.8.102: \quad a) y(x, 0.1) = \begin{cases} x - x^2 - 0.04 & \text{if } 0.2 \leq x \leq 0.8 \\ 0.6x & \text{if } x \leq 0.2 \\ 0.6 - 0.6x & \text{if } x \geq 0.8 \end{cases}$$

$$b) y(x, 1/2) = -x + x^2 \quad c) y(x, 1) = x - x^2$$

$$4.8.103: \quad a) y(1, 1) = -1/2 \quad b) y(4, 3) = 0 \quad c) y(3, 9) = 1/2$$

$$4.9.101: \quad u(x, y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) \left( \frac{\sinh(n\pi(1-y))}{\sinh(n\pi)} \right)$$

$$4.9.102: \quad u(x, y) = 0.1 \sin(\pi x) \left( \frac{\sinh(\pi(2-y))}{\sinh(2\pi)} \right)$$

$$5.1.101: \quad \lambda_n = \frac{(2n-1)\pi}{2}, n = 1, 2, 3, \dots, y_n = \cos\left(\frac{(2n-1)\pi}{2}x\right)$$

5.1.102: a)  $p(x) = 1$ ,  $q(x) = 0$ ,  $r(x) = \frac{1}{x}$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_1 = 1$ ,  $\beta_2 = 0$ . The problem is not regular. b)  $p(x) = 1 + x^2$ ,  $q(x) = x^2$ ,  $r(x) = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_1 = 1$ ,  $\beta_2 = 1$ . The problem is regular.

$$5.2.101: \quad y(x, t) = \sin(\pi x) \cos(4\pi^2 t)$$

$$5.2.102: \quad 9y_{xxxx} + y_{tt} = 0 \quad (0 < x < 10, t > 0), \quad y(0, t) = y_x(0, t) = 0, \quad y(10, t) = y_x(10, t) = 0, \\ y(x, 0) = \sin(\pi x), \quad y_t(x, 0) = x(10 - x).$$

$$5.3.101: \quad y_p(x, t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^4 \pi^4} \left( \cos(n\pi x) - \frac{\cos(n\pi) - 1}{\sin(n\pi)} \sin(n\pi x) - 1 \right) \cos(n\pi t).$$

5.3.102: Approximately 1991 centimeters

$$6.1.101: \quad \frac{8}{s^3} + \frac{8}{s^2} + \frac{4}{s}$$

$$6.1.102: \quad 2t^2 - 2t + 1 - e^{-2t}$$

$$6.1.103: \quad \frac{1}{(s+1)^2}$$

$$6.1.104: \quad \frac{1}{s^2 + 2s + 2}$$

$$6.2.101: \quad f(t) = (t-1)(u(t-1) - u(t-2)) + u(t-2)$$

$$6.2.102: \quad x(t) = (2e^{t-1} - t^2 - 1)u(t-1) - \frac{e^{-t}}{2} + \frac{3e^t}{2}$$

$$6.2.103: \quad H(s) = \frac{1}{s+1}$$

$$6.3.101: \quad \frac{1}{2}(\cos t + \sin t - e^{-t})$$

$$6.3.102: \quad 5t - 5 \sin t$$

$$6.3.103: \quad \frac{1}{2}(\sin t - t \cos t)$$

$$6.3.104: \quad \frac{1}{2}(\sin t - t \cos t)$$

7.1.101: Yes. Radius of convergence is 10.

7.1.102: Yes. Radius of convergence is  $e$ .

$$7.1.103: \quad \frac{1}{1-x} = -\frac{1}{1-(2-x)} \text{ so } \frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n, \text{ which converges for } 1 < x < 3.$$

**7.1.104:**  $\sum_{n=7}^{\infty} \frac{1}{(n-7)!} x^n$

**7.2.101:**  $a_2 = 0$ ,  $a_3 = 0$ ,  $a_4 = 0$ , recurrence relation (for  $k \geq 5$ ):  $a_k = -2a_{k-5}$ , so:

$$y(x) = a_0 + a_1x - 2a_0x^5 - 2a_1x^6 + 4a_0x^{10} + 4a_1x^{11} - 8a_0x^{15} - 8a_1x^{16} + \dots$$

**7.2.102:** a)  $a_2 = \frac{1}{2}$ , and for  $k \geq 1$  we have  $a_k = a_{k-3} + 1$ , so

$$y(x) = a_0 + a_1x + \frac{1}{2}x^2 + (a_0+1)x^3 + (a_1+1)x^4 + \frac{3}{2}x^5 + (a_0+2)x^6 + (a_1+2)x^7 + \frac{5}{2}x^8 + (a_0+3)x^9 + (a_1+3)x^{10} + \dots$$

b)  $y(x) = \frac{1}{2}x^2 + x^3 + x^4 + \frac{3}{2}x^5 + 2x^6 + 2x^7 + \frac{5}{2}x^8 + 3x^9 + 3x^{10} + \dots$

**7.2.103:** Applying the method of this section directly we obtain  $a_k = 0$  for all  $k$  and so  $y(x) = 0$  is the only solution we find.

# Index

- absolute convergence, 262
- acceleration, 16
- addition of matrices, 90
- Airy's equation, 271
- algebraic multiplicity, 124
- amplitude, 65
- analytic functions, 264
- angular frequency, 65
- antiderivative, 14
- antidifferentiate, 14
- associated homogeneous equation, 70, 101
- atan2, 66
- augmented matrix, 95
- autonomous equation, 37
- autonomous system, 87
  
- beating, 78
- Bernoulli equation, 33
- boundary conditions for a PDE, 190
- boundary value problem, 149
  
- catenary, 11
- Cauchy-Euler equation, 50
- center, 112
- cgs units, 238, 239
- characteristic equation, 52
- Chebyshev's equation of order  $p$ , 275
- Chebyshev's equation of order 1, 50
- cofactor, 94
- cofactor expansion, 94
- column vector, 90
- commute, 92
- complementary solution, 70
- complete eigenvalue, 124
- complex conjugate, 106
- complex number, 53
- complex roots, 54
- constant coefficient, 51, 100
- convergence of a power series, 262
- convergent power series, 262
- converges absolutely, 262
- convolution, 256
- corresponding eigenfunction, 150
- cosine series, 178
- critical point, 37
- critically damped, 67
  
- d'Alembert solution to the wave equation, 208
- damped, 66
- damped motion, 62
- defect, 125
- defective eigenvalue, 125
- deficient matrix, 125
- dependent variable, 7
- determinant, 92
- diagonal matrix, 116
  - matrix exponential of, 130
- diagonalization, 131
- differential equation, 7
- direction field, 87
- Dirichlet boundary conditions, 180, 221
- Dirichlet problem, 215
- displacement vector, 116
- distance, 16
- divergent power series, 262
- dot product, 91, 158
- dynamic damping, 123

- eigenfunction, 150, 222
- eigenfunction decomposition, 221, 226
- eigenvalue, 103, 222
- eigenvalue of a boundary value problem, 150
- eigenvector, 103
- eigenvector decomposition, 138, 145
- ellipses (vector field), 112
- elliptic PDE, 190
- endpoint problem, 149
- envelope curves, 68
- equilibrium solution, 37
- Euler's equation, 50
- Euler's equations, 56
- Euler's formula, 53
- Euler's method, 42
- even function, 161, 176
- even periodic extension, 176
- existence and uniqueness, 20, 48, 57
- exponential growth model, 9
- exponential of a matrix, 129
- exponential order, 244
- extend periodically, 157
  
- first order differential equation, 7
- first order linear equation, 27
- first order linear system of ODEs, 99
- first order method, 43
- first shifting property, 246
- forced motion, 62
  - systems, 121
- Fourier series, 159
- fourth order method, 44
- Fredholm alternative
  - simple case, 154
  - Sturm-Liouville problems, 225
- free motion, 62
- free variable, 96
- fundamental matrix, 100
- fundamental matrix solution, 100, 130
  
- general solution, 10
  
- generalized eigenvectors, 125, 127
- Genius software, 5
- geometric multiplicity, 124
- geometric series, 266
- Gibbs phenomenon, 164
  
- half period, 167
- harmonic function, 214
- harvesting, 39
- heat equation, 190
- Heaviside function, 242
- Hermite's equation of order  $n$ , 272
- Hermite's equation of order 2, 50
- homogeneous equation, 34
- homogeneous linear equation, 47
- homogeneous side conditions, 191
- homogeneous system, 100
- Hooke's law, 62, 115
- hyperbolic PDE, 190
  
- identity matrix, 91
- imaginary part, 54
- implicit solution, 24
- inconsistent system, 96
- indefinite integral, 14
- independent variable, 7
- initial condition, 10
- initial conditions for a PDE, 190
- inner product, 91
- inner product of functions, 159, 225
- integral equation, 253, 258
- integrate, 14
- integrating factor, 27
- integrating factor method, 27
  - for systems, 136
- inverse Laplace transform, 245
- invertible matrix, 92
- IODE software, 5
  - Lab I, 18
  - Lab II, 42
  - Project I, 18



- Project II, 42
- Project III, 57
- Project IV, 167
- Project V, 167
- la vie, 72
- Laplace equation, 190, 214
- Laplace transform, 241
- Laplacian, 214
- leading entry, 96
- Leibniz notation, 15, 22
- linear combination, 48, 57
- linear equation, 27, 47
- linear first order system, 87
- linear operator, 48, 70
- linear PDE, 190
- linearity of the Laplace transform, 243
- linearly dependent, 57
- linearly independent, 49, 57
  - for vector valued functions, 100
- logistic equation, 38
  - with harvesting, 39
- mass matrix, 116
- mathematical model, 9
- mathematical solution, 9
- matrix, 90
- matrix exponential, 129
- matrix inverse, 92
- matrix valued function, 99
- method of partial fractions, 245
- Mixed boundary conditions, 221
- mks units, 65, 69, 185
- multiplication of complex numbers, 53
- multiplicity, 60
- multiplicity of an eigenvalue, 124
- natural (angular) frequency, 65
- natural frequency, 77, 118
- natural mode of oscillation, 118
- Neumann boundary conditions, 180, 221
- Newton's law of cooling, 31, 37
- Newton's second law, 62, 63, 86, 115
- nilpotent, 131
- normal mode of oscillation, 118
- odd function, 161, 176
- odd periodic extension, 176
- ODE, 8
- one-dimensional heat equation, 190
- one-dimensional wave equation, 201
- operator, 48
- ordinary differential equation, 8
- ordinary point, 269
- orthogonal
  - functions, 153, 159
  - vectors, 157
  - with respect to a weight, 224
- orthogonality, 153
- overdamped, 67
- parabolic PDE, 190
- parallelogram, 93
- partial differential equation, 8, 190
- partial sum, 261
- particular solution, 10, 70
- PDE, 8, 190
- period, 65
- periodic, 157
- phase diagram, 38
- phase portrait, 38, 88
- phase shift, 65
- Picard's theorem, 20
- piecewise continuous, 170
- piecewise smooth, 170
- power series, 261
- practical resonance, 82, 189
- practical resonance amplitude, 82
- practical resonance frequency, 81
- product of matrices, 91
- projection, 159
- proper rational function, 246

- pseudo-frequency, 68
- pure resonance, 79, 187
- quadratic formula, 52
- radius of convergence, 262
- ratio test for series, 263
- real part, 54
- real world problem, 8
- recurrence relation, 270
- reduced row echelon form, 96
- reduction of order method, 50
- regular Sturm-Liouville problem, 223
- reindexing the series, 265
- repeated roots, 59
- resonance, 79, 122, 187, 258
- RLC circuit, 62
- row vector, 90
- Runge-Kutta method, 46
- saddle point, 111
- sawtooth, 160
- scalar, 90
- scalar multiplication, 90
- second order differential equation, 11
- second order linear differential equation, 47
- second order method, 43
- second shifting property, 250
- semistable critical point, 39
- separable, 22
- separation of variables, 191
- shifting property, 246, 250
- side conditions for a PDE, 190
- simple harmonic motion, 65
- sine series, 178
- singular matrix, 92
- singular point, 269
- singular solution, 24
- sink, 110
- slope field, 18
- solution, 7
- solution curve, 88
- source, 110
- spiral sink, 113
- spiral source, 112
- square wave, 82, 162
- stable critical point, 37
- stable node, 110
- steady periodic solution, 81, 184
- steady state temperature, 199, 214
- stiff problem, 45
- stiffness matrix, 116
- Sturm-Liouville problem, 222
  - regular, 223
- superposition, 47, 57, 100, 191
- symmetric matrix, 153, 157
- system of differential equations, 85
- Taylor series, 264
- tedious, 72, 73, 80, 141
- thermal conductivity, 190
- three mass system, 115
- timbre, 230
- trajectory, 88
- transfer function, 252
- transient solution, 81
- transpose, 91
- trigonometric series, 159
- undamped, 64
- undamped motion, 62
  - systems, 115
- underdamped, 68
- undetermined coefficients, 71
  - for second order systems, 121, 144
  - for systems, 141
- unforced motion, 62
- unit step function, 242
- unstable critical point, 37
- unstable node, 110
- upper triangular matrix, 126
- variation of parameters, 73

- for systems, 143
- vector, 90
- vector field, 87
- vector valued function, 99
- velocity, 16
- Volterra integral equation, 258
- wave equation, 190, 201, 208
- weight function, 224