

## 4.10 Dirichlet problem in the circle and the Poisson kernel

Note: 2 lectures, , §9.7 in [EP], §10.8 in [BD]

### 4.10.1 Laplace in polar coordinates

Perhaps a more natural setting for the Laplace equation  $\Delta u = 0$  is the circle rather than the square. On the other hand, what makes the problem somewhat more difficult is that we need to use polar coordinates.

Recall the polar coordinates for a plane

$$x = r \cos \theta, \quad y = r \sin \theta,$$

where  $r \geq 0$  and  $-\pi < \theta \leq \pi$ .

Now that we know our coordinates, let us give the problem that we wish to solve. We have a circular region of radius 1, and we are interested in the Dirichlet problem for the Laplace equation for this region. Let  $u(r, \theta)$  denote the temperature at the point  $(r, \theta)$  in polar coordinates. Then we have the problem:

$$\begin{aligned} \Delta u &= 0, & \text{for } r < 1, \\ u(1, \theta) &= g(\theta), & \text{for } -\pi < \theta \leq \pi. \end{aligned} \quad (4.25)$$

The first problem we face is that we do not know what the Laplacian is in polar coordinates. Normally we would find  $u_{xx}$  and  $u_{yy}$  in terms of the derivatives in  $r$  and  $\theta$ . We would need to solve for  $r$  and  $\theta$  in terms of  $x$  and  $y$ . While this is certainly possible. It happens to be more convenient to do this in reverse. Let us instead compute derivatives in  $r$  and  $\theta$  in terms of derivatives of  $x$  and  $y$  and then solve. The reason is that the computations are easier. First

$$x_r = \cos \theta, \quad x_\theta = -r \sin \theta, \quad y_r = \sin \theta, \quad y_\theta = r \cos \theta.$$

Next by chain rule we obtain

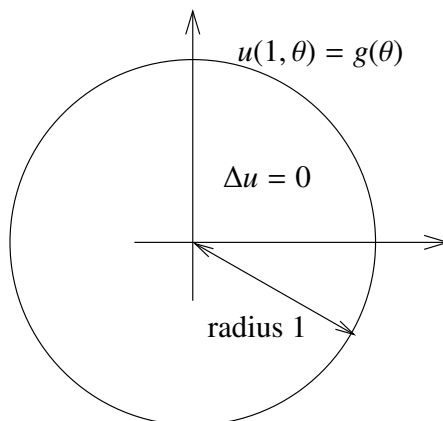
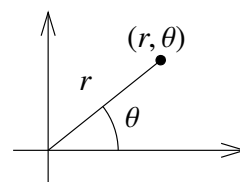
$$u_r = u_x x_r + u_y y_r = \cos(\theta) u_x + \sin(\theta) u_y,$$

$$u_{rr} = \cos(\theta)(u_{xx} x_r + u_{xy} y_r) + \sin(\theta)(u_{yx} x_r + u_{yy} y_r) = \cos^2(\theta) u_{xx} + 2 \cos(\theta) \sin(\theta) u_{xy} + \sin^2(\theta) u_{yy}.$$

Similarly for the  $\theta$  derivative. Note that we have to use product rule for the second derivative.

$$u_\theta = u_x x_\theta + u_y y_\theta = -r \sin(\theta) u_x + r \cos(\theta) u_y,$$

$$\begin{aligned} u_{\theta\theta} &= -r \cos(\theta) u_x - r \sin(\theta)(u_{xx} x_\theta + u_{xy} y_\theta) - r \sin(\theta) u_y + r \cos(\theta)(u_{yx} x_\theta + u_{yy} y_\theta) \\ &= -r \cos(\theta) u_x - r \sin(\theta) u_y + r^2 \sin^2(\theta) u_{xx} - r^2 2 \sin(\theta) \cos(\theta) u_{xy} + r^2 \cos^2(\theta) u_{yy}. \end{aligned}$$



Let us now try to solve for  $u_{xx} + u_{yy}$ . We start with  $\frac{1}{r^2}u_{\theta\theta}$  to get rid of those pesky  $r^2$ . If we add  $u_{rr}$  and use the fact that  $\cos^2(\theta) + \sin^2(\theta) = 1$ , we get

$$\frac{1}{r^2}u_{\theta\theta} + u_{rr} = u_{xx} + u_{yy} - \frac{1}{r}\cos(\theta)u_x - \frac{1}{r}\sin(\theta)u_y.$$

We're not quite there yet, but we notice that all we are lacking is  $\frac{1}{r}u_r$ . We therefore obtain the *Laplacian in polar coordinates*:

$$\frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + u_{rr} = u_{xx} + u_{yy} = \Delta u.$$

Notice that the Laplacian in polar coordinates no longer has constant coefficients.

### 4.10.2 Series solution

Let us separate variables as usual. That is let us try  $u(r, \theta) = R(r)\Theta(\theta)$ . Then

$$0 = \Delta u = \frac{1}{r^2}R\Theta'' + \frac{1}{r}R'\Theta + R''\Theta.$$

Let us put  $R$  on one side and  $\Theta$  on the other and then conclude that both sides must be constant.

$$\begin{aligned}\frac{1}{r^2}R\Theta'' &= -\left(\frac{1}{r}R' + R''\right)\Theta. \\ \frac{\Theta''}{\Theta} &= -\frac{rR' + r^2R''}{R} = -\lambda.\end{aligned}$$

We get two equations:

$$\begin{aligned}\Theta'' + \lambda\Theta &= 0, \\ r^2R'' + rR' - \lambda R &= 0.\end{aligned}$$

Let us first focus on  $\Theta$ . We know that  $u(r, \theta)$  ought to be  $2\pi$ -periodic in  $\theta$ , that is,  $u(r, \theta) = u(r, \theta + 2\pi)$ . Therefore, the solution to  $\Theta'' + \lambda\Theta = 0$  must be  $2\pi$ -periodic. We conclude that  $\lambda = 0, 1, 4, 9, \dots$ . That is,  $\lambda = n^2$  for a nonnegative integer  $n = 0, 1, 2, 3, \dots$ . The equation becomes  $\Theta'' + n^2\Theta = 0$ . When  $n = 0$  the equation is just  $\Theta'' = 0$ . For convenience let us write down its solution as

$$\Theta_0 = \frac{a_0}{2}$$

for some constant  $a_0$ . For positive  $n$ , the solution to  $\Theta'' + n^2\Theta = 0$  is

$$\Theta_n = a_n \cos(n\theta) + b_n \sin(n\theta),$$

for some constants  $a_n$  and  $b_n$ .

We next consider the equation for  $R$ ,

$$r^2 R'' + rR' - n^2 R = 0.$$

This equation has appeared in exercises before—we solved it in Exercise 2.1.6 and Exercise 2.1.7. The idea is to try a solution  $r^s$  and if that does not work out try a solution of the form  $r^s \ln r$ . When  $n = 0$  we obtain

$$R_0 = Ar^0 + Br^0 \ln r = A + B \ln r,$$

and if  $n > 0$ , we get

$$R_n = Ar^n + Br^{-n}.$$

The function  $u(r, \theta)$  must be finite at the origin, that is, when  $r = 0$ . Therefore,  $B = 0$  in both cases. Let us set  $A = 1$  in both cases as well, the constants in  $\Theta_n$  will pick up the slack so we do not lose anything. Therefore let

$$R_0 = 1, \quad \text{and} \quad R_n = r^n.$$

Hence our building block solutions are

$$u_0(r, \theta) = \frac{a_0}{2}, \quad u_n(r, \theta) = a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

Putting everything together our solution is:

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

We look at the boundary condition in (4.25),

$$g(\theta) = u(1, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Therefore, the solution to (4.25) is to expand  $g(\theta)$ , which is a  $2\pi$ -periodic function as a Fourier series, and then the  $n^{\text{th}}$  coordinate is multiplied by  $r^n$ . In other words, to compute  $a_n$  and  $b_n$  from the formula we can, as usual, compute

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta, \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta.$$

**Example 4.10.1:** Suppose we wish to solve

$$\begin{aligned} \Delta u &= 0, & 0 \leq r < 1, & \quad -\pi < \theta \leq \pi, \\ u(1, \theta) &= \cos(10\theta), & -\pi < \theta \leq \pi. \end{aligned}$$

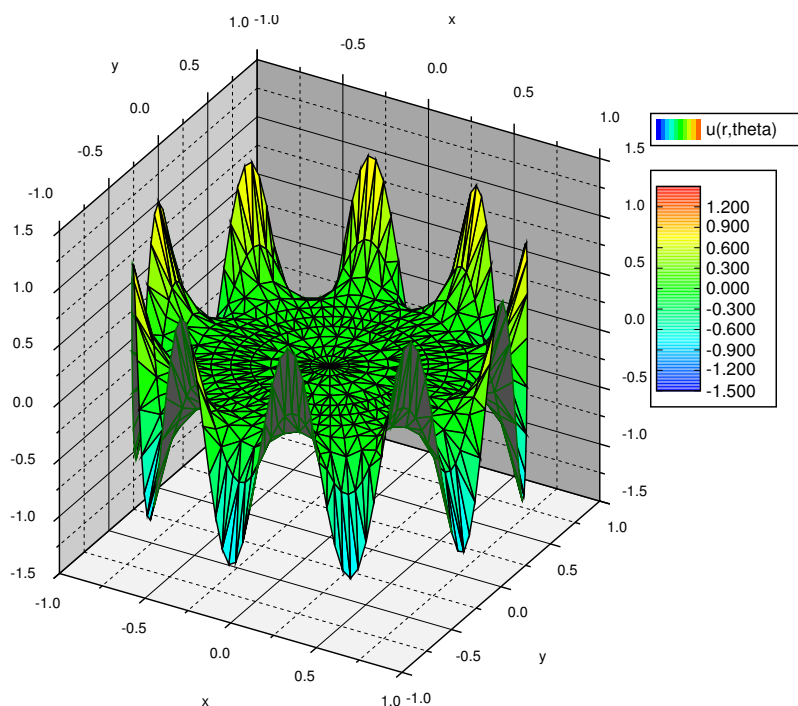


Figure 4.23: The solution of the Dirichlet problem in the disc with  $\cos(10\theta)$  as boundary data.

The solution is

$$u(r, \theta) = r^{10} \cos(10\theta).$$

See the plot in Figure 4.23. The thing to notice in this example is that the effect of a high frequency is mostly felt at the boundary. In the middle of the disc, the solution is very close to zero. That is because  $r^{10}$  becomes very small as  $r$  tends to 0.

**Example 4.10.2:** Let us solve a more difficult problem. Suppose we have a long rod with circular cross section of radius 1 and we wish to solve the steady state heat problem. If the rod is long enough we simply need to solve the Laplace equation in two dimensions. Let us put the center of the rod at the origin and we have exactly the region we are currently studying—a circle of radius 1. For the boundary conditions, suppose in Cartesian coordinates  $x$  and  $y$ , the temperature is fixed at 0 when  $y < 0$  and at  $2y$  when  $y > 0$ .

First we set the problem up. As  $y = r \sin(\theta)$ , then on the circle of radius 1 we have  $2y = 2 \sin(\theta)$ . The problem becomes

$$\Delta u = 0, \quad 0 \leq r < 1, \quad -\pi < \theta \leq \pi,$$

$$u(1, \theta) = \begin{cases} 2 \sin(\theta) & \text{if } 0 \leq \theta \leq \pi, \\ 0 & \text{if } -\pi < \theta < 0. \end{cases}$$

We must now compute the Fourier series for the boundary condition. By now the reader has plentiful experience in computing Fourier series and so we simply state that

$$u(1, \theta) = \frac{2}{\pi} + \sin(\theta) + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2 - 1)} \cos(2n\theta).$$

**Exercise 4.10.1:** Compute the series for  $u(1, \theta)$  and verify that it really is what we have just claimed. Hint: Be careful, make sure not to divide by zero.

We now simply write the solution (see Figure 4.24) by multiplying by  $r^n$  in the right places.

$$u(r, \theta) = \frac{2}{\pi} + r \sin(\theta) + \sum_{n=1}^{\infty} \frac{-4r^{2n}}{\pi(4n^2 - 1)} \cos(2n\theta).$$

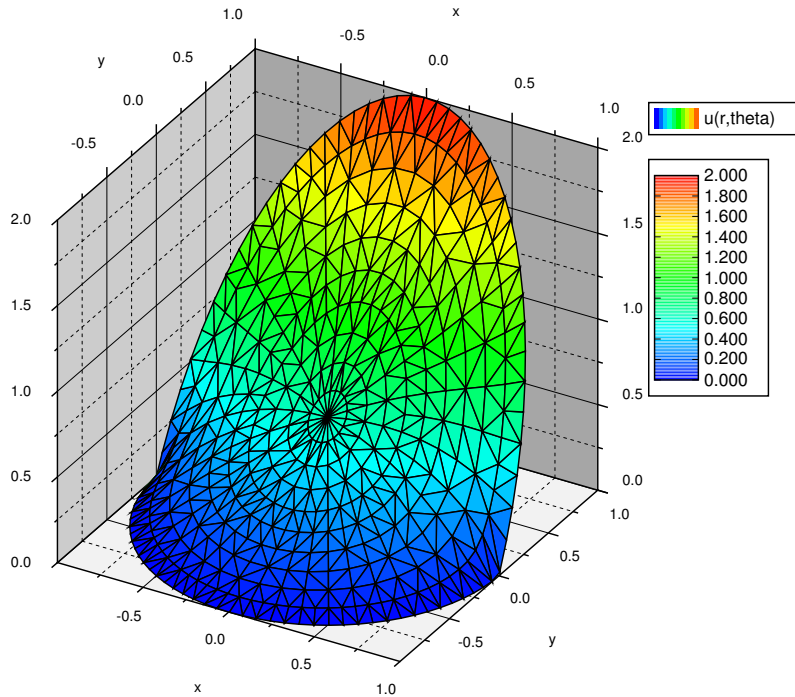


Figure 4.24: The solution of the Dirichlet problem with boundary data 0 for  $y < 0$  and  $2y$  for  $y > 0$ .

### 4.10.3 Poisson kernel

There is another way to solve the Dirichlet problem with the help of an integral kernel. That is, we will find a function  $P(r, \theta, \alpha)$  called the *Poisson kernel*\* such that

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, \theta, \alpha) g(\alpha) d\alpha.$$

While the integral will generally not be solvable analytically, it can be evaluated numerically. In fact, unless the boundary data is given as a Fourier series already, it will be much easier to numerically evaluate this formula as there is only one integral to evaluate.

The formula also has theoretical applications. For instance, as  $P(r, \theta, \alpha)$  will have infinitely many derivatives, then via differentiating under the integral we find that the solution  $u(r, \theta)$  has infinitely many derivatives, at least when inside the circle,  $r < 1$ . By infinitely many derivatives what you should think of is that  $u(r, \theta)$  has “no corners” and all of its partial derivatives too exist and have “no corners”.

We will compute the formula for  $P(r, \theta, \alpha)$  from the series solution, and this idea can be applied anytime you have a convenient series solution where the coefficients are obtained via integration. Hence you can apply this reasoning to obtain such integral kernels for other equations, such as the heat equation. The computation is long and tedious, but not overly difficult. Since the ideas are often applied in similar contexts, it is good to understand how this computation works.

What we do is start with the series solution and replace the coefficients with the integrals that compute them. Then we try to write everything as a single integral. We must use a different dummy variable for the integration and hence we use  $\alpha$  instead of  $\theta$ .

$$\begin{aligned} u(r, \theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\alpha) d\alpha \\ &\quad + \sum_{n=1}^{\infty} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} g(\alpha) \cos(n\alpha) d\alpha \right) r^n \cos(n\theta) + \left( \frac{1}{\pi} \int_{-\pi}^{\pi} g(\alpha) \sin(n\alpha) d\alpha \right) r^n \sin(n\theta) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( g(\alpha) + 2 \sum_{n=1}^{\infty} g(\alpha) \cos(n\alpha) r^n \cos(n\theta) + g(\alpha) \sin(n\alpha) r^n \sin(n\theta) \right) d\alpha \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( 1 + 2 \sum_{n=1}^{\infty} r^n (\cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta)) \right) g(\alpha) d\alpha \end{aligned}$$

OK, so we have what we wanted, the expression in the parentheses is the Poisson kernel,  $P(r, \theta, \alpha)$ . However, we can do a lot better. It is still given as a series, and we would really want to have a nice

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\*Named for the French mathematician Siméon Denis Poisson (1781 – 1840).

simple expression for it, so we work a little harder. The trick is to rewrite everything in terms of complex exponentials. Let us now work just on the kernel.

$$\begin{aligned}
 P(r, \theta, \alpha) &= 1 + 2 \sum_{n=1}^{\infty} r^n (\cos(n\alpha) \cos(n\theta) + \sin(n\alpha) \sin(n\theta)) \\
 &= 1 + 2 \sum_{n=1}^{\infty} r^n \cos(n(\theta - \alpha)) \\
 &= 1 + \sum_{n=1}^{\infty} r^n (e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}) \\
 &= 1 + \sum_{n=1}^{\infty} (re^{i(\theta-\alpha)})^n + \sum_{n=1}^{\infty} (re^{-i(\theta-\alpha)})^n.
 \end{aligned}$$

In the above expression we recognize the *geometric series*. That is, recall from calculus that as long as  $|z| < 1$  then

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Note that  $n$  starts at 1 and that is why we have the  $z$  in the numerator. It is the standard geometric series multiplied by  $z$ . Let us continue with the computation.

$$\begin{aligned}
 P(r, \theta, \alpha) &= 1 + \sum_{n=1}^{\infty} (re^{i(\theta-\alpha)})^n + \sum_{n=1}^{\infty} (re^{-i(\theta-\alpha)})^n \\
 &= 1 + \frac{re^{i(\theta-\alpha)}}{1 - re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{1 - re^{-i(\theta-\alpha)}} \\
 &= \frac{(1 - re^{i(\theta-\alpha)})(1 - re^{-i(\theta-\alpha)}) + (1 - re^{-i(\theta-\alpha)})re^{i(\theta-\alpha)} + (1 - re^{i(\theta-\alpha)})re^{-i(\theta-\alpha)}}{(1 - re^{i(\theta-\alpha)})(1 - re^{-i(\theta-\alpha)})} \\
 &= \frac{1 - r^2}{1 - re^{i(\theta-\alpha)} - re^{-i(\theta-\alpha)} + r^2} \\
 &= \frac{1 - r^2}{1 - 2r \cos(\theta - \alpha) + r^2}.
 \end{aligned}$$

Now that's a formula we can live with. The solution to the Dirichlet problem using the Poisson kernel is

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \alpha) + r^2} g(\alpha) d\alpha.$$

Sometimes the formula for the Poisson kernel is given together with the constant  $\frac{1}{2\pi}$ , in which case we should of course not leave it in front of the integral. Also, often the limits of the integral are given as 0 to  $2\pi$ ; everything inside is  $2\pi$ -periodic in  $\alpha$ , this does not change the integral.

Let us not leave the Poisson kernel without explaining its geometric meaning. Let  $\bar{r}$  be the distance from  $(r, \theta)$  to  $(1, \alpha)$ . You may recall from calculus that this distance  $\bar{r}$  in polar coordinates is given precisely by the square root of  $1 - 2r \cos(\theta - \alpha) + r^2$ . That is, the Poisson kernel is really the formula

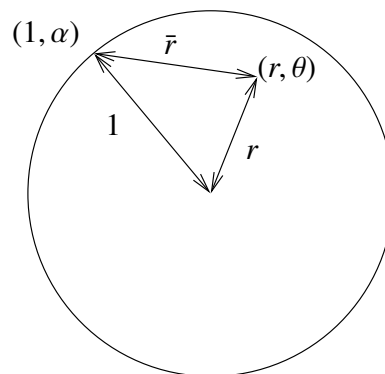
$$\frac{1 - r^2}{\bar{r}^2}.$$

One final note we make about the formula is to note that it is really a weighted average of the boundary values. First let us look at what happens at the origin, that is when  $r = 0$ .

$$\begin{aligned} u(0, 0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - 0^2}{1 - 2(0) \cos(\theta - \alpha) + 0^2} g(\alpha) d\alpha \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\alpha) d\alpha. \end{aligned}$$

This is precisely the average value of  $g(\theta)$  on the boundary. This is a general feature of harmonic functions, their value at some point  $p$  is equal to the average of its values on a circle centered at  $p$ .

What the formula says is that the value of the solution at any point in the circle is a weighted average of the boundary data  $g(\theta)$ . The kernel is bigger when  $(r, \theta)$  is closer to  $(1, \alpha)$ , and therefore we give more weight to the values  $g(\alpha)$  closer to  $(r, \theta)$  and less weight to the values  $g(\alpha)$  when  $(1, \alpha)$  far from  $(r, \theta)$ .



#### 4.10.4 Exercises

**Exercise 4.10.2:** Using series solve  $\Delta u = 0$ ,  $u(1, \theta) = |\theta|$ , for  $-\pi < \theta \leq \pi$ .

**Exercise 4.10.3:** Using series solve  $\Delta u = 0$ ,  $u(1, \theta) = 1/2 + 3 \sin(\theta) + \cos(3\theta)$ .

**Exercise 4.10.4:** Using the Poisson kernel, give the solution to  $\Delta u = 0$ , where  $u(1, \theta)$  is zero for  $\theta$  outside the interval  $[-\pi/4, \pi/4]$  and  $u(1, \theta)$  is 1 for  $\theta$  on the interval  $[-\pi/4, \pi/4]$ .

**Exercise 4.10.5:** a) Draw a graph for the Poisson kernel as a function of  $\alpha$  when  $r = 1/2$  and  $\theta = 0$ . b) Describe what happens to the graph when you make  $r$  bigger. c) Knowing that the solution  $u(r, \theta)$  is the weighted average of  $g(\theta)$  with Poisson kernel as the weight, explain what your answer to part b means.

**Exercise 4.10.6:** Take the function  $g(\theta)$  to be the function  $xy = \cos \theta \sin \theta$  on the boundary. Use the series solution to find a solution to the Dirichlet problem  $\Delta u = 0$ ,  $u(1, \theta) = g(\theta)$ . Now convert the solution to Cartesian coordinates  $x$  and  $y$ . Is this solution surprising? Hint: use your trig identities.

**Exercise 4.10.7:** Carry out the computation we needed in the separation of variables and solve  $r^2 R'' + rR' - n^2 R = 0$ , for  $n = 0, 1, 2, 3, \dots$



**Exercise 4.10.8** (challenging): *Derive the series solution to the Dirichlet problem if the region is a circle of radius  $\rho$  rather than 1. That is, solve  $\Delta u = 0$ ,  $u(\rho, \theta) = g(\theta)$ .*

**Exercise 4.10.101:** *Using series solve  $\Delta u = 0$ ,  $u(1, \theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\theta)$ .*

**Exercise 4.10.102:** *Using the series solution find the solution to  $\Delta u = 0$ ,  $u(1, \theta) = 1 - \cos(\theta)$ . Express the solution in Cartesian coordinates (that is, using  $x$  and  $y$ ).*

**Exercise 4.10.103:** *Try and guess a solution to  $\Delta u = -1$ ,  $u(1, \theta) = 0$ . Hint: try a solution that only depends on  $r$ . Also first, don't worry about the boundary condition.*

**Exercise 4.10.104** (challenging): *Derive the Poisson kernel solution if the region is a circle of radius  $\rho$  rather than 1. That is, solve  $\Delta u = 0$ ,  $u(\rho, \theta) = g(\theta)$ .*

**4.6.103:**  $u(x, t) = e^{lt} e^{\lambda x}$  for some  $\lambda$

**4.6.104:**  $u(x, t) = Ae^x + Be^t$

**4.7.101:**  $y(x, t) = \sin(x) (\sin(t) + \cos(t))$

**4.7.102:**  $y(x, t) = \frac{1}{5\pi} \sin(\pi x) \sin(5\pi t) + \frac{1}{100\pi} \sin(2\pi x) \sin(10\pi t)$

**4.7.103:**  $y(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \cos(n\sqrt{2}t)$

**4.7.104:**  $y(x, t) = \sin(2x) + t \sin(x)$

**4.8.101:**  $y(x, t) = \frac{\sin(2\pi(x-3t)) + \sin(2\pi(3t+x))}{2} + \frac{\cos(3\pi(x-3t)) - \cos(3\pi(3t+x))}{18\pi}$

**4.8.102:** a)  $y(x, 0.1) = \begin{cases} x - x^2 - 0.04 & \text{if } 0.2 \leq x \leq 0.8 \\ 0.6x & \text{if } x \leq 0.2 \\ 0.6 - 0.6x & \text{if } x \geq 0.8 \end{cases}$

b)  $y(x, 1/2) = -x + x^2$     c)  $y(x, 1) = x - x^2$

**4.8.103:** a)  $y(1, 1) = -1/2$     b)  $y(4, 3) = 0$     c)  $y(3, 9) = 1/2$

**4.9.101:**  $u(x, y) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n\pi x) \left( \frac{\sinh(n\pi(1-y))}{\sinh(n\pi)} \right)$

**4.9.102:**  $u(x, y) = 0.1 \sin(\pi x) \left( \frac{\sinh(\pi(2-y))}{\sinh(2\pi)} \right)$

**4.10.101:**  $u = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} r^n \sin(n\theta)$

**4.10.102:**  $u = 1 - x$

**4.10.103:**  $u = \frac{-1}{4} r^2 + \frac{1}{4}$

**4.10.104:**  $u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho - 2r\rho \cos(\theta - \alpha) + r^2} g(\alpha) d\alpha$

**5.1.101:**  $\lambda_n = \frac{(2n-1)\pi}{2}$ ,  $n = 1, 2, 3, \dots$ ,  $y_n = \cos\left(\frac{(2n-1)\pi}{2}x\right)$

**5.1.102:** a)  $p(x) = 1$ ,  $q(x) = 0$ ,  $r(x) = \frac{1}{x}$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_1 = 1$ ,  $\beta_2 = 0$ . The problem is not regular. b)  $p(x) = 1 + x^2$ ,  $q(x) = x^2$ ,  $r(x) = 1$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 0$ ,  $\beta_1 = 1$ ,  $\beta_2 = 1$ . The problem is regular.

**5.2.101:**  $y(x, t) = \sin(\pi x) \cos(4\pi^2 t)$

**5.2.102:**  $9y_{xxxx} + y_{tt} = 0$  ( $0 < x < 10, t > 0$ ),  $y(0, t) = y_x(0, t) = 0$ ,  $y(10, t) = y_x(10, t) = 0$ ,  $y(x, 0) = \sin(\pi x)$ ,  $y_t(x, 0) = x(10 - x)$ .

**5.3.101:**  $y_p(x, t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-4}{n^4 \pi^4} \left( \cos(n\pi x) - \frac{\cos(n\pi) - 1}{\sin(n\pi)} \sin(n\pi x) - 1 \right) \cos(n\pi t)$ .

**5.3.102:** Approximately 1991 centimeters

**6.1.101:**  $\frac{8}{s^3} + \frac{8}{s^2} + \frac{4}{s}$

**6.1.102:**  $2t^2 - 2t + 1 - e^{-2t}$