***Exercise* 1.4.3:** *Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.*

$$y' + y = e^{x^2-x} \qquad y(0) = 10.$$

Example 1.4.2: The following is a simple application of linear equations and this type of a problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water.

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kg / liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?

Let us come up with the equation. Let x denote the kg of salt in the tank, let t denote the time in minutes. Then for a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t$$

Taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out})$$

We have

$$\begin{aligned} \text{rate in} &= 5 \\ \text{concentration in} &= 0.1 \\ \text{rate out} &= 3 \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t} \end{aligned}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60 + 2t}\right)$$

Or in the form (1.3)

$$\frac{dx}{dt} + \frac{3}{60 + 2t} x = 0.5$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60 + 2t} dt\right) = \exp\left(\frac{3}{2} \ln(60 + 2t)\right) = (60 + 2t)^{3/2}$$

We multiply both sides of the equation to get

$$
\begin{aligned}
(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2} \\
\frac{d}{dt}\Big[(60+2t)^{3/2}x\Big] &= 0.5(60+2t)^{3/2} \\
(60+2t)^{3/2}x &= \int 0.5(60+2t)^{3/2}dt + C \\
x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}dt + C(60+2t)^{-3/2} \\
x &= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2} \\
x &= \frac{60+2t}{10} + C(60+2t)^{-3/2}
\end{aligned}
$$

Now to figure out C. We know that at $t = 0$, $x = 10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2}$$

or

$$C = 4(60^{3/2}) \approx 1859.03$$

We are interested in x when the tank is full. So we note that the tank is full when $60+2t = 100$, or when $t = 20$. So

$$x(20) = \frac{60+40}{10} + C(60+40)^{-3/2} \approx 10 + 1859.03(100)^{-3/2} \approx 11.86$$

The concentration at the end is approximately 0.1186 kg/liter and we started with $\frac{1}{6}$ or 0.167 kg/liter.

1.4.1 Exercises

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

***Exercise* 1.4.4:** *Solve* $y' + xy = x$.

***Exercise* 1.4.5:** *Solve* $y' + 6y = e^x$.

***Exercise* 1.4.6:** *Solve* $y' + 3x^2y = \sin(x)\, e^{-x^3}$, *with* $y(0) = 1$.

***Exercise* 1.4.7:** *Solve* $y' + \cos(x)y = \cos(x)$.

***Exercise* 1.4.8:** *Solve* $\frac{1}{x^2+1}\, y' + xy = 3$, *with* $y(0) = 0$.