7.3 Singular points and the method of Frobenius

Note: 1 or 1.5 lectures , §8.4 and §8.5 in [EP], §5.4 – §5.7 in [BD]

While behaviour of ODEs at singular points is more complicated, certain singular points are not especially difficult to solve. Let us look at some examples before giving a general method. We may be lucky and obtain a power series solution using the method of the previous section, but in general we may have to try other things.

7.3.1 Examples

Example 7.3.1: Let us first look at a simple first order equation

$$2xy' - y = 0.$$

Note that $x = 0$ is a singular point. If we only try to plug in

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k) x^k. \end{aligned}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)a_k$ for $k = 1, 2, 3, \ldots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

Let us try $y = x^r$ for some real number r. Consequently our solution—if we can find one—may only make sense for positive x. Then $y' = rx^{r-1}$. So

$$0 = 2xy' - y = 2xrx^{r-1} - x^r = (2r-1)x^r.$$

Therefore $r = 1/2$, or in other words $y = x^{1/2}$. Multiplying by a constant, the general solution for positive x is

$$y = Cx^{1/2}.$$

Note that the solution is not even differentiable at $x = 0$. The derivative necessarily must "blow up" at the origin, so much is clear from the differential equation itself.

Not every problem at a singular point has solution of the form $y = x^r$, of course. But perhaps we can combine the methods. What we will do is to try a solution of the form

$$y = x^r f(x)$$

where $f(x)$ is an analytic function.

Example 7.3.2: Suppose that we have the equation

$$4x^2y'' - 4x^2y' + (1 - 2x)y = 0,$$

and again note that $x = 0$ is a singular point.

Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r},$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$\begin{aligned} y' &= \sum_{k=0}^{\infty} (k + r)\, a_k x^{k+r-1}, \\ y'' &= \sum_{k=0}^{\infty} (k + r)\,(k + r - 1)\, a_k x^{k+r-2}. \end{aligned}$$

Plugging into our equation we obtain

$$\begin{aligned} 0 &= 4x^2y'' - 4x^2y' + (1 - 2x)y \\ &= 4x^2 \left(\sum_{k=0}^{\infty} (k + r)\,(k + r - 1)\, a_k x^{k+r-2}\right) - 4x^2 \left(\sum_{k=0}^{\infty} (k + r)\, a_k x^{k+r-1}\right) + (1 - 2x)\left(\sum_{k=0}^{\infty} a_k x^{k+r}\right) \\ &= \left(\sum_{k=0}^{\infty} 4(k + r)\,(k + r - 1)\, a_k x^{k+r}\right) - \left(\sum_{k=0}^{\infty} 4(k + r)\, a_k x^{k+r+1}\right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r}\right) - \left(\sum_{k=0}^{\infty} 2a_k x^{k+r+1}\right) \\ &= \left(\sum_{k=0}^{\infty} 4(k + r)\,(k + r - 1)\, a_k x^{k+r}\right) - \left(\sum_{k=1}^{\infty} 4(k + r - 1)\, a_{k-1} x^{k+r}\right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r}\right) - \left(\sum_{k=1}^{\infty} 2a_{k-1} x^{k+r}\right) \\ &= 4r(r - 1)\, a_0 + a_0 + \sum_{k=1}^{\infty} \Big(4(k + r)\,(k + r - 1)\, a_k - 4(k + r - 1)\, a_{k-1} + a_k - 2a_{k-1}\Big)\, x^{k+r} \\ &= (4r(r - 1) + 1)a_0 + \sum_{k=1}^{\infty} \Big(\big(4(k + r)\,(k + r - 1) + 1\big)\, a_k - \big(4(k + r - 1) + 2\big)\, a_{k-1}\Big)\, x^{k+r}. \end{aligned}$$

Therefore to a solution we must first have $(4r(r - 1) + 1)a_0 = 0$. Supposing that $a_0 \neq 0$ we obtain

$$4r(r - 1) + 1 = 0.$$

This equation is called the *indicial equation*. We notice that this particular indicial equation has a double root at $r = 1/2$.

OK, so we know what r has to be. That we obtained simply by looking at what the coefficient of x^r. All other coefficients of x^{k+r} also have to be zero so

$$\big(4(k+r)\,(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1} = 0.$$

If we plug in $r = 1/2$ and solve for a_k we get

$$a_k = \frac{4(k+1/2-1)+2}{4(k+1/2)\,(k+1/2-1)+1}\,a_{k-1} = \frac{1}{k}\,a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1}a_0 = 1, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2}, \qquad a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}, \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}, \qquad \ldots$$

Extrapolating, we notice that

$$a_k = \frac{1}{k(k-1)(k-2)\cdots 3\cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = x^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^{1/2} e^x.$$

That was lucky! In general, we will not be able to write the series in terms of elementary functions.

We have one solution, let us call it $y_1 = x^{1/2}e^x$. But what about a second solution? If we want a general solution, we need two linearly independent solutions. Picking a_0 to be a different constant only gets us a constant multiple of y_1, and we do not have any other r to try; we only have one solution to the indicial equation. Well, there are powers of x floating around and we are taking derivatives, perhaps the logarithm (the antiderivative of x^{-1}) is around as well. It turns out we want to try for another solution of the form

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+r} + (\ln x) y_1,$$

which in our case is

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+1/2} + (\ln x) x^{1/2} e^x.$$

We would now differentiate this equation, substitute into the differential equation again and solve for b_k. A long computation would ensue and we would obtain some recursion relation for b_k. In fact, the reader can try this to obtain for example the first three terms

$$b_1 = b_0 - 1, \qquad b_2 = \frac{2b_1 - 1}{4}, \qquad b_3 = \frac{6b_2 - 1}{18}, \qquad \ldots$$

We would then fix b_0 and obtain a solution y_2. Then we write the general solution as $y = Ay_1 + By_2$.

7.3.2 The method of Frobenius

Before giving the general method, let us clarify when the method applies. Let

$$p(x)y'' + q(x)y' + r(x)y = 0$$

be an ODE. As before, if $p(x_0) = 0$, then x_0 is a singular point. If, furthermore, we have that

$$\lim_{x\to x_0} (x - x_0)\frac{q(x)}{p(x)} \qquad \text{and} \qquad \lim_{x\to x_0} (x - x_0)^2 \frac{r(x)}{p(x)}$$

both exist and are finite, then we say that x_0 is a *regular singular point*.

Example 7.3.3: Often, and for the rest of this section, $x_0 = 0$. Consider

$$x^2y'' + x(1 + x)y' + (\pi + x^2)y = 0.$$

Then we write

$$\lim_{x\to 0} x\frac{q(x)}{p(x)} = \lim_{x\to 0} x\frac{x(1 + x)}{x^2} = \lim_{x\to 0}(1 + x) = 1,$$

and

$$\lim_{x\to 0} x^2\frac{r(x)}{p(x)} = \lim_{x\to 0} x^2\frac{(\pi + x^2)}{x^2} = \lim_{x\to 0}(\pi + x^2) = \pi.$$

so 0 is a regular singular point.

On the other hand if we make the slight change:

$$x^2y'' + (1 + x)y' + (\pi + x^2)y = 0.$$

Then

$$\lim_{x\to 0} x\frac{q(x)}{p(x)} = \lim_{x\to 0} x\frac{(1 + x)}{x^2} = \lim_{x\to 0} \frac{1 + x}{x} = \text{DNE}.$$

Here DNE stands for *does not exist*. So the point 0 is a singular point, but not a regular singular point.

Let us now discuss the general *Method of Frobenius**. Let us only consider the method at the point $x = 0$ for simplicity. The main idea is the following theorem.

Theorem 7.3.1 (Method of Frobenius)**.** *Suppose that*

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{7.3}$$

has a regular singular point at $x = 0$*, then there exists at least one solution of the form*

$$y = x^r \sum_{k=0}^{\infty} a_k x^k.$$

*Named after the German mathematician Ferdinand Georg Frobenius (1849 – 1917).

Following the method is really a series of steps.

(i) We seek a solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

We plug this y into equation (7.3). A solution of this form is called a *Frobenius-type solution*.

(ii) We obtain a series which must be zero. Setting the first coefficient (the coefficient of x^r) to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.

(iii) If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then we obtain two linearly independent solutions

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} a_k x^k,$$

by solving all the relations that appear.

(iv) If the indicial equation has a doubled root r, then there we find one solution

$$y_1 = x^{r} \sum_{k=0}^{\infty} a_k x^k,$$

and then we obtain a new solution by plugging

$$y_2 = x^{r} \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into equation (7.3) and solving for the variables b_k.

(v) If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x) y_1,$$

where we plug y_2 into (7.3) and solve for the constants b_k and C.

(vi) Finally, if the indicial equation has complex roots, then solving for a_k in the solution

$$y = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

will result in a complex valued function—all the a_k will be complex numbers. We obtain our two linearly independent solutions* by taking the real and imaginary parts of y.

Note that the main idea is to find at least one Frobenius-type solution. If we are lucky and find two we are done. If we only get one, there's a variety of other methods (different from the above) to obtain a second solution. For example, reduction of order, see Exercise 2.1.8 on page 50.

7.3.3 Bessel functions

An important class of functions that arises commonly in physics are the *Bessel functions*†. For example, these functions arise as when solving the wave equation in two and three dimensions. First we have *Bessel's equation* of order p.

$$x^2 y'' + x y' + \left(x^2 - p^2\right) y = 0.$$

We allow p to be any number, not just an integer, although integers and multiples of $1/2$ are most important in applications.

When we plug

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

into Bessel's equation of order p we obtain the indicial equation

$$r(r-1) + r - p^2 = (r-p)(r+p) = 0.$$

Therefore we obtain two roots $r_1 = p$ and $r_2 = -p$. If p is not an integer following the method of Frobenius and setting $a_0 = 1$, we can obtain linearly independent solutions of the form

$$y_1 = x^p \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k+p)(k-1+p)\cdots(2+p)(1+p)},$$
$$y_2 = x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k-p)(k-1-p)\cdots(2-p)(1-p)}.$$

*See Joseph L. Neuringera, *The Frobenius method for complex roots of the indicial equation*, International Journal of Mathematical Education in Science and Technology, Volume 9, Issue 1, 1978, 71–77.

†Named after the German astronomer and mathematician Friedrich Wilhelm Bessel (1784 – 1846).

Exercise* 7.3.1*: *a) Verify that the indicial equation of Bessel's equation of order p is* $(r-p)(r+p) = 0$*. b) Suppose that p is not an integer. Carry out the computation to obtain the solutions* y_1 *and* y_2 *above.*

Bessel functions will be convenient constant multiples of y_1 and y_2. First we must define the *gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

The gamma function has a wonderful property

$$\Gamma(x+1) = x\Gamma(x).$$

From this property, one can show that $\Gamma(n) = (n-1)!$ when n is an integer. Furthermore we can compute that

$$\Gamma(k+p+1) = (k+p)(k-1+p)\cdots(2+p)(1+p)\Gamma(1+p),$$
$$\Gamma(k-p+1) = (k-p)(k-1-p)\cdots(2-p)(1-p)\Gamma(1-p).$$

Exercise* 7.3.2*: *Verify the above identities using* $\Gamma(x+1) = x\Gamma(x)$*.*

We define the *Bessel functions of the first kind* of order p and $-p$ as

$$J_p(x) = \frac{1}{2^p\Gamma(1+p)} y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p},$$
$$J_{-p}(x) = \frac{1}{2^{-p}\Gamma(1-p)} y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}.$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. The constants are picked for convenience.

When p is not an integer, J_p and J_{-p} are linearly independent. When n is an integer we obtain

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k+n}.$$

In this case it turns out that

$$J_n(x) = (-1)^n J_{-n}(x),$$

and so we do not obtain a second linearly independent solution. The other solution is the so-called *Bessel function of second kind*. These make sense only for integer orders n and are defined as limits of linear combinations of $J_p(x)$ and $J_{-p}(x)$ as p approaches n in the following way:

$$Y_n(x) = \lim_{p \to n} \frac{\cos(p\pi) J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

As each linear combination of $J_p(x)$ and $J_{-p}(x)$ is a solution to Bessel's equation of order p, then as we take the limit as p goes to n, $Y_n(x)$ is a solution to Bessel's equation of order n. It also turns out that $Y_n(x)$ and $J_n(x)$ are linearly independent. Therefore when n is an integer, we have the general solution to Bessel's equation of order n

$$y = AJ_n(x) + BY_n(x),$$

for arbitrary constants A and B. Note that $Y_n(x)$ goes to negative infinity at $x = 0$. Many mathematical software packages have these functions $J_n(x)$ and $Y_n(x)$ defined, so they can be used just like say $\sin(x)$ and $\cos(x)$. In fact, they have some similar properties. For example, $-J_1(x)$ is a derivative of $J_0(x)$, and in general the derivative of $J_n(x)$ can be written as a linear combination of $J_{n-1}(x)$ and $J_{n+1}(x)$. Furthermore, these functions oscillate, although they are not periodic. See Figure 7.4 for graphs of Bessel functions.

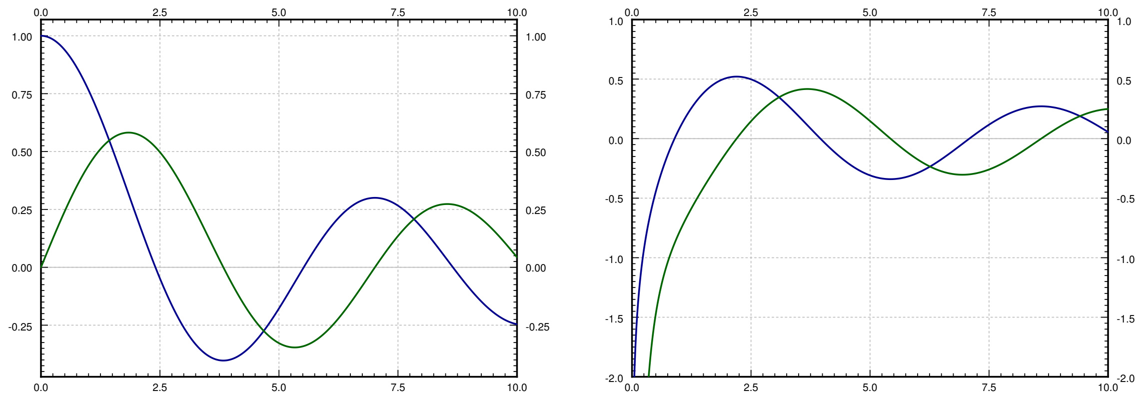

Figure 7.4: Plot of the $J_0(x)$ and $J_1(x)$ in the first graph and $Y_0(x)$ and $Y_1(x)$ in the second graph.

Example 7.3.4: Other equations can sometimes be solved in terms of the Bessel functions. For example for a positive constant λ,

$$xy'' + y' + \lambda^2 xy = 0,$$

can be changed to $x^2y'' + xy' + \lambda^2x^2y = 0$, and then changing variables $t = \lambda x$ we obtain via chain rule the equation in y and t:

$$t^2y'' + ty' + t^2y = 0,$$

which can be recognized as Bessel's equation of order 0. Therefore the general solution is $y(t) = AJ_0(t) + BY_0(t)$, or in terms of x:

$$y = AJ_0(\lambda x) + BY_0(\lambda x).$$

This equation comes up for example when finding fundamental modes of vibration of a circular drum, but we digress.

7.3.4 Exercises

Exercise* 7.3.3*: *Find a particular (Frobenius-type) solution of* $x^2y'' + xy' + (1 + x)y = 0$.

Exercise* 7.3.4*: *Find a particular (Frobenius-type) solution of* $xy'' - y = 0$.

Exercise* 7.3.5*: *Find a particular (Frobenius-type) solution of* $y'' + \frac{1}{x}y' - xy = 0$.

Exercise* 7.3.6*: *Find the general solution of* $2xy'' + y' - x^2y = 0$.

Exercise* 7.3.7*: *Find the general solution of* $x^2y'' - xy' - y = 0$.

Exercise* 7.3.8*: *In the following equations classify the point* $x = 0$ *as* ordinary, regular singular, *or* singular but not regular singular.

a) $x^2(1 + x^2)y'' + xy = 0$

b) $x^2y'' + y' + y = 0$

c) $xy'' + x^3y' + y = 0$

d) $xy'' + xy' - e^xy = 0$

e) $x^2y'' + x^2y' + x^2y = 0$

Exercise* 7.3.101*: *In the following equations classify the point* $x = 0$ *as* ordinary, regular singular, *or* singular but not regular singular.

a) $y'' + y = 0$

b) $x^3y'' + (1 + x)y = 0$

c) $xy'' + x^5y' + y = 0$

d) $\sin(x)y'' - y = 0$

e) $\cos(x)y'' - \sin(x)y = 0$

Exercise* 7.3.102*: *Find the general solution of* $x^2y'' - y = 0$.

Exercise* 7.3.103*: *Find a particular solution of* $x^2y'' + (x - 3/4)y = 0$.

Exercise* 7.3.104** (Tricky):*** *Find the general solution of* $x^2y'' - xy' + y = 0$.

6.3.101: $\frac{1}{2}(\cos t + \sin t - e^{-t})$

6.3.102: $5t - 5\sin t$

6.3.103: $\frac{1}{2}(\sin t - t\cos t)$

6.3.104: $\int_0^t f(\tau)(1 - \cos(t - \tau))\, d\tau$

6.4.101: $x(t) = t$

6.4.102: $x(t) = e^{-at}$

6.4.103: $x(t) = (\cos * \sin)(t) = \frac{1}{2}t\sin(t)$

6.4.104: $\delta(t) - \sin(t)$

6.4.105: $3\delta(t - 1) + 2t$

7.1.101: Yes. Radius of convergence is 10.

7.1.102: Yes. Radius of convergence is e.

7.1.103: $\frac{1}{1-x} = -\frac{1}{1-(2-x)}$ so $\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1}(x - 2)^n$, which converges for $1 < x < 3$.

7.1.104: $\sum_{n=7}^{\infty} \frac{1}{(n-7)!} x^n$

7.1.105: $f(x) - g(x)$ is a polynomial. Hint: Use Taylor series.

7.2.101: $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, recurrence relation (for $k \geq 5$): $a_k = -2a_{k-5}$, so:
$y(x) = a_0 + a_1x - 2a_0x^5 - 2a_1x^6 + 4a_0x^{10} + 4a_1x^{11} - 8a_0x^{15} - 8a_1x^{16} + \cdots$

7.2.102: a) $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = a_{k-3} + 1$, so
$y(x) = a_0+a_1x+\frac{1}{2}x^2+(a_0+1)x^3+(a_1+1)x^4+\frac{3}{2}x^5+(a_0+2)x^6+(a_1+2)x^7+\frac{5}{2}x^8+(a_0+3)x^9+(a_1+3)x^{10}+\cdots$
b) $y(x) = \frac{1}{2}x^2 + x^3 + x^4 + \frac{3}{2}x^5 + 2x^6 + 2x^7 + \frac{5}{2}x^8 + 3x^9 + 3x^{10} + \cdots$

7.2.103: Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.

7.3.101: a) ordinary, b) singular but not regular singular, c) regular singular, d) regular singular, e) ordinary.

7.3.102: $y = Ax^{\frac{1+\sqrt{5}}{2}} + Bx^{\frac{1-\sqrt{5}}{2}}$

7.3.103: $y = x^{3/2} \sum_{k=0}^{\infty} \frac{(-1)^{-1}}{k!(k+2)!} x^k$ (Note that for convenience we did not pick $a_0 = 1$)

7.3.104: $y = Ax + Bx\ln(x)$