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Singularities and Complexity in CR Geometry

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The dissertation of Jiří Lebl is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

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Markétě

My hovercraft is full of eels!
—Monty Python

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PUBLICATIONS

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ABSTRACT OF THE DISSERTATION

Singularities and Complexity in CR Geometry

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Professor Peter Ebenfelt, Chair

Several related questions in CR geometry are studied. First, the structure of the singular set of Levi-flat hypersurfaces is investigated. The singularity is completely characterized when it is a submanifold of codimension 1, and partial information is gained about higher codimension cases.

Second, a local uniqueness property of holomorphic functions on real-analytic nowhere minimal CR submanifolds of higher codimension is investigated. A sufficient condition called almost minimality is given and studied. A weaker property, not being contained inside a possibly singular real-analytic Levi-flat hypersurface is studied and characterized, and a sufficient and necessary condition is given in terms of normal coordinates.

One natural generalization of this problem is the classification of codimension 2 real-analytic CR submanifolds, which are locally the boundaries of C^∞ Levi-flat hypersurfaces. These submanifolds are completely classified in terms of their normal coordinate representation. In fact, an extension theorem is proved allowing smooth Levi-flat hypersurfaces to always be extended past CR submanifolds and in most cases forcing such hypersurfaces to be real-analytic. Examples are found that this extension result is optimal.

Finally, relation of the complexity of a mapping and the source and target dimensions is studied for proper holomorphic mappings between balls in different dimensions. A conjecture of John D'Angelo states that a mapping from n to N

dimensions has degree less than or equal to $\frac{N-1}{n-1}$, as long as $n \geq 3$, and $2N-3$ when $n=2$. The special, but highly nontrivial, case of monomial mappings and a related problem in real algebraic geometry is studied and a weaker bound is proved. The more general cases of polynomial and rational mappings is also treated. In the general rational case, this problem can be thought of as a generalization of the local uniqueness property studied before to vector valued holomorphic functions.

1 Introduction

We will be studying several related questions in CR geometry, and hence we will begin by giving a quick introduction to CR geometry. The background on CR geometry comes mostly from [BER99, Bog91, D'A93], and so the reader should consult those for more details and proofs of any results given here. For the background on complex-analytic subvarieties the reader should consult [Whi72].

In chapter 2 we will study the singularity set of a real-analytic Levi-flat hypersurface. Singularities of Levi-flat hypersurfaces have been previously studied by Burns and Gong [BG99], by Bedford [Bed77], and more recently by Brunella [Bru07]. By analogy to complex-analytic subvarieties, it is reasonable to believe that the singularity is either complex-analytic or Levi-flat. We will prove this is the case when the singularity is a submanifold of maximal dimension and we will get partial information about higher codimension cases.

In chapter 3 we will study a certain uniqueness property of holomorphic functions with respect to real submanifolds of codimension 2. Suppose that we have two holomorphic functions f and g, such that on some real submanifold M we have $|f| = |g|$. If this implies $f = cg$ everywhere for a constant c, we will say M has the modulus uniqueness property. When M does not have this uniqueness property, we know that there exists a certain possibly singular Levi-flat hypersurface, which contains M. We will study and characterize when M is contained in a Levi-flat hypersurface in terms of its normal coordinates. For this we will utilize our results from chapter 2. The results from chapters 2, 3 and 4 come from the paper [Leb07].

A natural related question is to ask when are submanifolds the boundaries of smooth Levi-flat hypersurfaces. Such a question has been recently studied by Dolbeault, Tomassini and Zaitsev [DTZ]. Related questions were also studied by Straube and Sucheston [SS03]. In $\mathbb{C}^2$ such a question has been studied for a long time, beginning with Bishop [Bis65], and further for example by Moser and Webster [MW83], Bedford and Gaveau [BG83], or more recently for example by Gong in [Gon04]. However, in $\mathbb{C}^2$ this question is uninteresting near CR points of the boundary.

We will be studying, in chapter 5, the local behavior near a generic real-analytic boundary of a C^∞ Levi-flat hypersurface. We will completely classify such boundaries locally in terms of their normal coordinates. We will also prove a local extension and uniqueness theorem. It turns out that if the boundary has nontrivial CR structure, there is a unique real-analytic Levi-flat hypersurface containing the submanifold, forcing our original hypersurface to be real-analytic. We will provide examples to show our theorems are optimal. These results have been published in [Leb].

In chapter 4, we will study a certain class of generic submanifolds we call *almost minimal*, which exhibit certain properties similar to those of minimal submanifolds in the sense of Tumanov [Tum88]. We will give examples of such submanifolds, which are nowhere minimal. We will prove a finite jet determination result for holomorphic mappings of such submanifolds and for infinitesimal holomorphisms. We will further provide connections to previous chapters, as almost minimal submanifolds can be useful examples of singular behavior of the CR structure in nowhere minimal submanifolds. For example, we prove that an almost minimal submanifold of codimension 2 cannot be contained in a Levi-flat hypersurface, while all algebraic nowhere minimal submanifolds are contained in such hypersurfaces. Therefore the example in section 4.4 is an example of a submanifold, which is not equivalent to any real-algebraic submanifold. Examples of such hypersurfaces both minimal and nonminimal can be found in [BER00] and [HJY01].

Finally, in chapter 6, we will be studying proper holomorphic mappings from the unit ball to the unit ball. Let $\mathbb{B}_n \subset \mathbb{C}^n$ be the unit ball. A topological mapping is proper if pullbacks of compact sets are compact. If the mapping extends continuously to the boundary this is equivalent to the mapping taking the boundary to the boundary. A basic question in CR geometry is therefore the following:

Suppose that $f\colon \mathbb{B}_n \to \mathbb{B}_N$ is a proper holomorphic mapping. How can we relate the "complexity" of f to the dimensions n and N? It is not hard to see that no such f can exist if $n > N$. Forstnerič [For89] proved that if f is sufficiently smooth up to the boundary, then f is a rational mapping, and further, that there exists a bound for the degree of f in terms of n and N. Hence, one measure of complexity of f is the degree of f. Given that the mapping z^d is a proper mapping from the unit disc to itself, the case $n = 1$ is uninteresting. When $n = N \geq 2$, Pinčuk [Pin75] proved that f must be a linear fractional transformation and thus of degree 1. Faran [Far82] proved that when $n = 2$ and $N = 3$, then $d \leq 3$, and further, he identified all examples up to automorphisms.

D'Angelo has made a systematic study of of these mappings, see for example [D'A03, D'A88, D'A93]. He made the following conjecture: If $n = 2$ then $\deg(f) \leq 2N - 3$, and if $n \geq 3$ then $\deg(f) \leq \frac{N-1}{n-1}$. There exist monomial examples (each component of the mapping is a monomial) that achieve both bounds, and the bound is proved for monomials when $n = 2$ in [DKR03]. In [DLP], together with D'Angelo and Peters, the author proved the sharp bound under additional conditions and and a weaker bound in general. We will explore these in chapter 6.

For general mappings, Huang and Ji [Hua99, HJ01] and Huang, Ji and Xu [HJX06] studied the low codimension case for all mappings with sufficient regularity at the boundary. Meylan [Mey06] recently improved the bound on the degree for all rational mappings if $n = 2$.

1.1 Complex variables background

First, we will fix some terminology. We will be working in $\mathbb{C}^N$, and we will frequently write the coordinates as $z = (z_1, \ldots, z_N)$. Note that if $z \in \mathbb{C}$ we can write $z = x + iy$, where $x, y \in \mathbb{R}$ are the real and imaginary parts of z. Therefore, we can think of $\mathbb{C}^N$ as $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ by writing $z_k = x_k + iy_k$. Complex conjugation is defined by $\bar{z}_k := x_k - iy_k$. We define the complex differentials

$$dz_k := dx_k + i dy_k, \quad \text{and} \quad d\bar{z}_k := dx_k - i dy_k. \tag{1.1}$$

Next we define the vectors $\frac{\partial}{\partial z_k}$ and $\frac{\partial}{\partial \bar{z}_k}$ in the only way possible such that $dz_k(\frac{\partial}{\partial z_k}) = 1$, $d\bar{z}_k(\frac{\partial}{\partial \bar{z}_k}) = 1$, and any other combination gives zero. That is:

$$\frac{\partial}{\partial z_k} := \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_k} := \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right). \tag{1.2}$$

For a C^1 function $f\colon \mathbb{C}^N \to \mathbb{C}$ we define

$$\partial f := \sum_{k=1}^{N} \frac{\partial f}{\partial z_k} dz_k \quad \text{and} \quad \bar{\partial} f := \sum_{k=1}^{N} \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k. \tag{1.3}$$

Note that if d is the standard exterior differentiation, then $df = \partial f + \bar{\partial} f$.

We will use the multiindex notation, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex and $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$), then $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. By $|\alpha|$ we will mean $\alpha_1 + \cdots + \alpha_n$.

A function $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$(or $\mathbb{C}$) is said to be *real-analytic* if near every point $a \in \mathbb{R}^n$ there exists a convergent power series

$$\sum_\alpha c_\alpha (x-a)^\alpha, \tag{1.4}$$

and where this power series converges (and where $f(x)$ is defined), it equals $f(x)$. If f is complex valued we let c_α be complex, otherwise c_α is real. As we are considering only local questions, we will from now on consider $a = 0$.

If we are working in $\mathbb{C}^N$, and think of this as $\mathbb{R}^{2N}$, then by substituting $z = x+iy$ and $\bar{z} = x - iy$, we get a power series in z and $\bar{z}$

$$\sum_{\alpha,\beta} c_{\alpha\beta} z^\alpha \bar{z}^\beta. \tag{1.5}$$

And if f is real valued then $c_{\alpha\beta} = \overline{c_{\beta\alpha}}$.

A C^1 function $f\colon U \subset \mathbb{C}^N \to \mathbb{C}$ is said to be *holomorphic* if $\bar{\partial} f = 0$ at all points in U. These are called the *Cauchy-Riemann* equations. A holomorphic function can be shown to be real-analytic and in fact have a power series representation

$$\sum_\alpha c_\alpha z^\alpha. \tag{1.6}$$

In particular we notice that we can treat z and $\bar{z}$ (formally) as independent variables when working with real-analytic functions. In fact, when f is not holomorphic, we will write $f(z)$ as $f(z, \bar{z})$ as $f(z)$ will be reserved for holomorphic functions. If a power series in z and $\bar{z}$ is convergent, then if we replace $\bar{z}$ with w, then the power series is still convergent. This idea is called *complexification* or *polarization*. So if the power series $P(z, \bar{z})$ converges on U, then $P(z, w)$ converges on $U \times {}^*U$ where ${}^*U$ is the image of U under the complex conjugation. Note however that if $f(z, \bar{z})$ is real-analytic in U, it need not be the case that $f(z, w)$ is defined on all of $U \times {}^*U$. If we are interested only in local properties we can usually assume that U is small enough such that f has a convergent power series in all of U and hence complexifies to $U \times {}^*U$.

When we consider mappings $f \colon \mathbb{C}^n \to \mathbb{C}^m$, then we say f is real-analytic (resp. holomorphic), whenever all the m components of f are real-analytic (resp. holomorphic). We will say that f is a biholomorphism if f has a holomorphic inverse.

We will use the following standard theorem. It is really a theorem about power series and hence there are both real variable and purely algebraic statements of it. We use the standard notation for coordinates in $\mathbb{C}^n$ that $z = (z_1, \ldots, z_n) = (z', z_n)$. That is z' is the first $n - 1$ coordinates.

Theorem 1.1 (Weierstrass preparation theorem)**.** *Let $f \colon \mathbb{C}^n \to \mathbb{C}$ be a function analytic in a neighbourhood U of the origin such that $\frac{f(0,z_n)}{z_n^m}$ extends to be analytic at the origin and is not zero at the origin for some positive integer m (in other words, as a function of z_n, the function has a zero of order m at the origin). Then there exists a polydisc $D \subset U$ such that every function g holomorphic and bounded in D can be written as*

$$g = qf + r, \tag{1.7}$$

where q is an analytic function and r is a polynomial in the z_n variable of degree less than m with the coefficients being holomorphic functions in z'.

A function such as r is called a *pseudopolynomial*, that is a polynomial in z_n such that the coefficients are holomorphic functions of z'. When a pseudopolynomial is monic and further all the coefficients vanish at $z' = 0$, then it is called a *Weierstrass polynomial*. We will really mostly be using the following corollary.

Corollary 1.2. *Let f be as above, then there is a unique representation of f as $f = hW$, where h is analytic in a neighbourhood of the origin and $h(0) \neq 0$ and W being a Weierstrass polynomial.*

1.2 Submanifolds and subvarieties

A subset $M \subset \mathbb{R}^n$ is said to be a C^k (real) *submanifold* ($k = 1, \dots, \infty, \omega$, where ω means real-analytic) of (real) *codimension* d, if near every point $p \in M$, there is a neighbourhood $U \subset \mathbb{R}^n$ of p, and a C^k submersion[1] $\rho \colon U \to \mathbb{R}^d$, such that $M \cap U = \rho^{-1}(\{0\})$. $n - d$ is said to be the (real) *dimension* of M.

Similarly, a *complex submanifold* of codimension d is a subset $M \subset \mathbb{C}^N$ of *codimension* d, if near every point $p \in M$, there is a neighbourhood $U \subset \mathbb{C}^n$ of p, and a holomorphic submersion $\varphi \colon U \to \mathbb{C}^d$, such that $M \cap U = \rho^{-1}(\{0\})$. $N - d$ is said to be the (complex) *dimension* of M.

The elements of the submersion $\rho = (\rho_1, \dots, \rho_d)$ are said to be the *defining functions* of M.

When we consider a submanifold of $M \subset \mathbb{R}^{2N}$ as a subset of $\mathbb{C}^N$ we will emphasize this by saying M is a *real submanifold*. Note that if M is a complex submanifold of $\mathbb{C}^N$ of codimension d (and dimension $N - d$), then it is also a real submanifold of real codimension $2d$ (and real dimension $2N - 2d$).

What we mean by submanifold some authors refer to as embedded (or imbedded) submanifold. This is to differentiate from *immersed submanifold*, which need not be locally closed. M is an immersed submanifold of codimension d if it is the image of an abstract manifold by an immersion.

Suppose $U \subset \mathbb{R}^n$ (resp. $\mathbb{C}^N$). Then $V \subset U$ is a real-analytic (resp. complex-analytic) *subvariety* of U, if V is closed in U, and for every point $p \in U$, there exists a neighbourhood N of p and a family $\mathcal{F}$ of real-analytic (resp. holomorphic) functions $f \colon N \to \mathbb{R}$ (resp. $\mathbb{C}$), such that $V \cap N = \{x \in N \mid f(x) = 0, \ \forall f \in \mathcal{F}\}$. Note that on an open dense subset of V, V is a real-analytic (resp. complex) submanifold. One can define the dimension of V at such a point and then the dimension of the whole subvariety is the maximum of the dimensions at all points near which V is a submanifold. Note that by unique continuation of analytic functions, if $U \subset \mathbb{R}^n$ (resp. $\mathbb{C}^N$) is connected and dimension of V is equal to n (resp. N), then $U = V$. It can also be proved that we can always pick $\mathcal{F}$ to be a

[1] ρ is a submersion if the matrix with elements $\frac{\partial \rho_j}{\partial x_k}$ is of maximal rank at all points.

finite family of functions. On the other hand, it is a hard question in general to decide exactly how many functions are needed.

Proposition 1.3. *The intersection of two (complex or real) subvarieties of an open set U is again a subvariety of U.*

If V is a complex-analytic subvariety (of U), the set of points at which V is not a complex submanifold, called the singular set *and denoted* $\operatorname{Sing}(V)$*, is a complex-analytic subvariety (of U).*

The second part of the above proposition need not be true for real-analytic subvarieties. However, the singular set of a real-analytic subvariety V is contained in a proper subvariety of V of lower dimension[2].

For convenience, we will usually denote by V^* the nonsingular points of V at which V is a submanifold of the same dimension as V and by V_s everything else. It is many times convenient to consider nonsingular points of a subvariety, which are of lower dimension than other points, to be somehow singular.

A real-analytic subvariety could be (or contain) a C^k submanifold of the same dimension even at a singular point. For example, in $\mathbb{R}^2$ the subvariety $y^3 = x^8$ is a C^2 submanifold (it is the graph of the C^2 function $y = |x|^{8/3}$), and it is singular at the origin. The following theorem of Malgrange says that in some sense this cannot happen for smooth submanifolds. See Malgrange [Mal67] Chapter VI, Proposition 3.11.

Theorem 1.4 (Malgrange). *Suppose that $M \subset V \subset \mathbb{R}^n$ where M is a C^∞ submanifold and V is a real-analytic subvariety and* $\dim(V) = \dim(M)$*. Then M is a real-analytic submanifold.*

A subvariety V of an open set U is said to be *irreducible* if for any decomposition of V into two subvarieties $V = V_1 \cup V_2$ implies that $V = V_1$ or $V = V_2$.

Proposition 1.5. *If V is a subvarieties of U, then there exists finitely many irreducible subvarieties $V_1, \ldots, V_k$ called the* irreducible components *or* branches *of V.*

[2]Proper subvariety of V is a subset of V that is again a subvariety of the ambient open set, but not equal to V.

Remark 1.6. To make the above precise as a definition of branches, we want to take the maximal such collection V_j. There are real-analytic examples where this matters.

A *germ* of a set V at a point p, denoted by (V, p), is the equivalence class of sets containing p such that $V_1 \sim V_2$ if and only if there exists a neighbourhood N of p such that $V_1 \cap N = V_2 \cap N$.

With this definition we define the germs of subvarieties and submanifolds. The definition of germ is especially useful for analytic subvarieties and submanifolds because of unique analytic continuation. I.e. if V_1 and V_2 are two irreducible complex-analytic subvarieties of the same open set U, and $(V_1, p) = (V_2, p)$ for some $p \in V_1 \cap V_2$, then $V_1 = V_2$. This is not true in general for real-analytic subvarieties. However, we can always complexify and get at least partial such results. For example, if V_1 and V_2 are hypersurfaces and p is a point of codimension 1. We define an *irreducible* germ in a similar way.

Note that even if a subvariety is irreducible in U the germ (V, p) may not be irreducible. However, we can choose a small enough neighbourhood such that the decomposition of U into irreducible components has such property.

Proposition 1.7. *Let V be a subset of U, and $p \in V$. There exists a neighbourhood N of p such that $N \cap V$ has a decomposition into irreducible branches $V_1, \ldots, V_k$ and (V_j, p) are irreducible as germs for $j = 1, \ldots, k$.*

Similarly we can define the *germ of a function* f, denoted (f, p). I.e. the equivalence class of functions under the relation $f \sim g$ if and only if there exists a neighbourhood U of p such that $f|_U = g|_U$.

1.3 Semialgebraic and subanalytic sets

For more details than we give in this section, see Bierstone and Milman [BM88]. Let $V \subset U_x \times U_y$ be a real subvariety. Suppose that x are the coordinates in U_x, and y are the coordinates in U_y. It is not always true that if π_x is the projection function onto the x coordinates, that $\pi_x(V)$ is a subvariety of U_x, nor even of an open subset of U_x.

Let us look at the semialgebraic case first. Consider the $A \subset \mathbb{R}^n$, defined by real polynomials $p_{j\ell}$, $j = 1, \ldots, k$, $\ell = 1, \ldots, m$, and the relations $\epsilon_{j\ell}$, where $\epsilon_{j\ell}$ is $>$, $=$, or $<$.

$$A = \bigcup_{\ell=1}^{m} \{x \in \mathbb{R}^n \mid p_{j\ell}(x)\ \epsilon_{j\ell}\ 0,\ j = 1, \ldots, k\}. \tag{1.8}$$

Sets of this form are said to be *semialgebraic*. Similarly as algebraic subvarieties, they are closed under finite union and intersection. However, they are also closed under complement. In general arbitrary intersection of semialgebraic sets is not semialgebraic.

On a dense open subset of A, A is (locally) a submanifold, and hence we can easily define the dimension of A to be the largest dimension at points at which A is a submanifold. It is not hard to see that a semialgebraic set lies inside an algebraic subvariety of the same dimension.

Theorem 1.8 (Tarski-Seidenberg)**.** *The set of semialgebraic sets is closed under projection.*

Emboldened by such success, we can try to do the same thing in the analytic setting. However, Tarski-Seidenberg does not hold here.

Let $U \subset \mathbb{R}^n$. Suppose $\mathcal{A}(U)$ is any ring of real valued functions on U. Define $\mathcal{S}(\mathcal{A}(U))$ to be the smallest set of subsets of U, which contain the sets $\{x \in U \mid f(x) > 0\}$ for all $f \in \mathcal{A}(U)$, and is closed under finite union, finite intersection and complement.

A set $V \subset \mathbb{R}^n$ is *semianalytic* if and only if for each $x \in \mathbb{R}^n$, there exists a neighbourhood U of x, such that $V \cap U \in \mathcal{S}(\mathcal{O}(U))$, where $\mathcal{O}(U)$ here denotes the real-analytic real valued functions.

Again on a dense open subset of V, V is (locally) a submanifold, and hence we can easily define the dimension of V to be the largest dimension at points at which V is a submanifold. It is not hard to see that a semianalytic set lies inside a subvariety of the same dimension.

A variation on Tarski-Seidenberg by Łojasiewicz is the following. Suppose that $\mathcal{A}(U)$ is again an arbitrary ring of functions on $U \subset \mathbb{R}^n$. Let $\mathcal{A}(U)[t]$ denote the ring of polynomials in $t \in \mathbb{R}^m$ with coefficients in $\mathcal{A}(U)$.

Theorem 1.9 (Tarski-Seidenberg-Łojasiewicz)**.** *Suppose that $V \subset U \times \mathbb{R}^m \subset \mathbb{R}^{n+m}$, is such that $V \in \mathcal{S}(\mathcal{A}(U)[t])$. Then the projection of V onto the first n variables is in $\mathcal{S}(\mathcal{A}(U))$.*

Since Tarski-Seidenberg does not hold in general, we say $V \subset \mathbb{R}^n$ is a *subanalytic* set if for each $x \in \mathbb{R}^n$, there exists a relatively compact semianalytic set $X \subset \mathbb{R}^{n+m}$ and a neighbourhood U of x, such that $V \cap U$ is the projection of X onto the first n coordinates.

Subanalytic sets are again on an open dense set submanifolds, and hence we can again define dimension. However, subanalytic sets are not in general contained in any subvariety of the same dimension. A useful theorem that we will need is the following.

Theorem 1.10. *A subanalytic set A can be written as a locally finite union of submanifolds.*

Subanalytic sets are still not completely closed under projections however. Note that a real-analytic subvariety that is not relatively compact can have a projection which is not a locally finite union of submanifolds, and hence is not subanalytic.

1.4 CR geometry

CR geometry is essentially studying the properties of objects invariant under biholomorphic transformations. We say a subset $S \subset \mathbb{C}^N$ is *biholomorphically equivalent* to $S' \subset \mathbb{C}^N$ at if there exists an open set U such that $S \subset U$ and a one to one holomorphic mapping $F\colon U \to \mathbb{C}^N$ such that $F(S) = S'$. We say S is *locally biholomorphically equivalent* (at $p \in S$) to S' if there exists a neighbourhood U of p, a one to one holomorphic mapping $F\colon U \to \mathbb{C}^N$ and a neighbourhood U' of $F(p)$, such that $F(S \cap U) = S' \cap U'$.

When we consider real-analytic subvarieties of $\mathbb{C}^N$ we are treating $\mathbb{C}^N$ as $\mathbb{R}^{2N}$ as above. From now on we will be working only in $\mathbb{C}^N$ and hence when talking about real objects we will make the identification $\mathbb{C}^N = \mathbb{R}^{2N}$.

Real analytic or complex submanifolds as defined above are also subvarieties of some open set. As we have complexified real-analytic functions on $\mathbb{C}^N$, we can complexify real-analytic submanifolds and subvarieties by taking their defining functions and complexifying those. Again note that just because M is a real-analytic submanifold of U does not mean that we can complexify M to $U \times {}^*U$. But since locally any submanifold or subvariety has only finitely many defining functions, we can pick a small neighbourhood N such that there exist finitely many power series converging in this neighbourhood, which are the defining functions for M in this neighbourhood. Then we can complexify $M \cap N$ to be a complex submanifold (or subvariety) of $N \times {}^*N$. Also note that to guarantee that the complexified M is still a submanifold we might need to take N even smaller to guarantee that the defining functions are still a submersion.

We define the *tangent space* at $p \in \mathbb{C}^N$

$$T_p\mathbb{C}^N = \operatorname{span}\left\{ \frac{\partial}{\partial x_k}\Big|_p, \frac{\partial}{\partial y_k}\Big|_p \;\Big|\; k = 1, \ldots, N \right\}, \tag{1.9}$$

and the *tangent bundle* $T\mathbb{C}^N = \bigcup_{p \in \mathbb{C}^N} T_p\mathbb{C}^N$.

We define the *complex structure* of $\mathbb{C}^N$ to be the mapping $J\colon T\mathbb{C}^N \to T\mathbb{C}^{\mathbb{N}}$, linear on each tangent space, and

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k} \quad \text{and} \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}. \tag{1.10}$$

Let M be a real submanifold and let ρ_k, $k = 1, \ldots, d$, be the defining functions for M. Then the tangent space to M is defined by

$$T_pM = \{v \in T_p\mathbb{C}^N \mid d\rho_k(v) = 0, k = 1, \ldots, d\}, \tag{1.11}$$

and the tangent bundle by $TM = \bigcup_{p \in M} T_pM$.

We define the *complex tangent space* [3]

$$T_p^cM := T_pM \cap J(T_pM). \tag{1.12}$$

We complexify TM to $\mathbb{C} \otimes TM$, that is we let $\mathbb{C}T_pM := \mathbb{C} \otimes_{\mathbb{R}} T_pM$, which really just means that we take the basis of the vector space T_pM, and allow complex coefficients. We can extend J linearly to $\mathbb{C} \otimes TM$ and we can look at $\mathbb{C} \otimes T^cM$.

[3] This is incredibly bad (but standard) notation. It is in fact a real vector space, not a complex one.

J as a linear mapping on $\mathbb{C} \otimes T_p^c M$ has two eigenvalues, $-i$ and i. We can take the two corresponding eigenspaces and call them $T_p^{1,0}M$ and $T_p^{0,1}M$ such that $\mathbb{C} \otimes T_p^c M = T_p^{1,0}M \oplus T_p^{0,1}M$. It turns out that $T_p^{1,0}M$ are those tangent vectors we can write in terms of $\frac{\partial}{\partial z_k}$, and $T_p^{0,1}M$ are the vectors we can write in terms of $\frac{\partial}{\partial \bar{z}_k}$. That is,

$$T_p^{1,0}M = \left\{ v \in \mathbb{C} \otimes T_p M \;\middle|\; v = \sum_{k=1}^{N} a_k \frac{\partial}{\partial z_k},\ a_k \in \mathbb{C} \right\}, \tag{1.13}$$

and similarly for $T_p^{0,1}M$. $T_p^{1,0}M$ are called the *holomorphic tangent vectors* and $T_p^{0,1}M$ are called the *antiholomorphic tangent vectors* or *CR vectors*.

If the dimension of $T_p^c M$ (and hence of $T_p^{1,0}M$ and $T_p^{0,1}M$) is constant along M, then M is said to be a *CR submanifold*. The complex dimension of $T_p^{0,1}M$ is called the *CR dimension* of M. In this case, we can define the vector bundles $T^c M$, $T^{1,0}M$, and $T^{0,1}M$ in the obvious way. Vectorfields (sections of the bundle) in $T^{0,1}M$ are called the *CR vectorfields*

If $M \subset \mathbb{C}^N$ is a real submanifold defined by $\rho_1, \ldots, \rho_d$, such that $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d \neq 0$, then M is said to be a *generic* submanifold. If we need more than one neighbourhood and more than one set of defining functions, M is generic if and only if it is generic in all of them. It is not hard to prove that this is equivalent to saying that $T_p M + J(T_p M) = T_p \mathbb{C}^N$ for all $p \in M$. A generic submanifold is automatically CR, the opposite is not true in general.

Proposition 1.11. *If $M \subset \mathbb{C}^N$ is a generic submanifold and $U \subset \mathbb{C}^N$ a connected open subset such that $U \cap M \neq \emptyset$. Then the smallest (the intersection of) complex-analytic subvariety of U which contains M is all of U.*

Note that Proposition 1.11 implies that if a holomorphic function is 0 on a generic submanifold, then it is zero everywhere. We also have the following proposition which allows us to concentrate on generic submanifolds, rather than all CR submanifolds.

Proposition 1.12. *Suppose M is a real-analytic CR submanifold, then there exists a germ of a complex-analytic submanifold V_0 at 0 such that $(M, 0) \subset V_0$. And $(M, 0)$ is germ of a generic submanifold inside V_0.*

$\mathcal{X}_p$ is said to be the *intrinsic complexification* of M. We can therefore study local properties of CR submanifolds inside their intrinsic complexifications and hence just consider generic submanifolds.

Let $M \subset \mathbb{C}^N$ be C^∞ or a real-analytic generic submanifold. The bundle T^cM need not be integrable, however the following is true by a theorem of Nagano [Nag66] in the real-analytic case and Sussmann [Sus73] in the smooth case.

Proposition 1.13. *Let M be a real-analytic (resp. smooth) generic submanifold, and $p \in M$. Then there exists a germ Orb_p of a real-analytic (resp. smooth) submanifold of M, of the same CR dimension as M, which is unique and smallest in the sense that if N_p is another germ of a submanifold of M with the same CR dimension, then $\operatorname{Orb}_p \subset N_p$.*

Orb_p will be referred to as the CR orbit of M at p. If p is real-analytic we will also quite frequently look at the germ $\mathcal{X}_p$ which we will define as the intrinsic complexification of Orb_p.

If $\operatorname{Orb}_p = (M, p)$, then we will say that M is *minimal* (in the sense of [Tum88]) at p. When Orb_p is a proper submanifold for all $p \in M$, then M is said to be *nowhere minimal.*

A CR submanifold with CR dimension 0, is said to be *totally real* and if it is generic, it is called *maximally totally real* submanifold. For example, if $\mathbb{R}^N \subset \mathbb{C}^N$ is the standard embedding, then $\mathbb{R}^N$ is a maximally totally real submanifold.

For a generic submanifold $M \subset \mathbb{C}^N$, we will consider M in *normal coordinates* $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$, where d is the real codimension of M and n is the CR dimension of M, and M is given by

$$w = Q(z, \bar{z}, \bar{w}), \tag{1.14}$$

where Q is a holomorphic mapping defined in a neighbourhood of the origin in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$, $Q(0, \zeta, \omega) \equiv Q(z, 0, \omega) \equiv \omega$, and $Q(z, \zeta, \bar{Q}(\zeta, z, w)) \equiv w$. We should note, however, that normal coordinates are not unique, but they are guaranteed to exist.

Proposition 1.14. *If M is a connected real-analytic generic submanifold through the origin, then there exist suitable local coordinates near the origin such that near the origin M is given by* (1.14)*.*

It is not hard to see that this is equivalent to the real equation

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w), \tag{1.15}$$

where φ is an $\mathbb{R}^d$-valued real-analytic mapping, such that $\varphi(0, \bar{z}, s) \equiv \varphi(z, 0, s) \equiv 0$.

A C^1 function $f \colon M \to \mathbb{C}$ is such that $df(X) = 0$ whenever X is a section of $T^{0,1}M$, then f is said to be a *CR function*. I.e. f is annihilated by the antiholomorphic vectors. If $f \colon U \to \mathbb{C}$ is a holomorphic function, and M is a submanifold of U, then $f|_M$ is a CR function on M, however the inverse need not be true in general. It is true in case M and f are real-analytic.

Proposition 1.15. *Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold and $f \colon M \to \mathbb{C}$ a real-analytic CR function. Then f extends to a unique holomorphic function on a neighbourhood of M in $\mathbb{C}^N$.*

1.5 Levi-flat submanifolds and subvarieties

A generic submanifold M is said to be *Levi-flat* when the bundle $T^c M$ is integrable. By a theorem of Newlander-Nirenberg [NN57], the foliation given by this bundle is then locally a foliation by complex submanifolds. This foliation is called the *Levi foliation* of M. If V is an irreducible real-analytic subvariety of codimension 1, we say it is *Levi-flat* if it is Levi-flat at all the points of V^*, that is all nonsingular points of codimension 1.

We will mostly be interested in Levi-flat hypersurfaces, i.e. submanifolds or subvarieties of codimension 1. When we say a possibly singular real-analytic hypersurface we will mean subvariety of codimension 1.

In [BG99] Burns and Gong prove the following two Lemmas.

Lemma 1.16. *Let K be a real-analytic subvariety of codimension 1 ($0 \in K$) defined by $r(z, \bar{z}) = 0$, for r an irreducible real-analytic real valued function. Then for some small neighbourhood U of 0, r complexifies (the Taylor series $r(z, w)$ converges for $z \in U$, $\bar{w} \in U$) and is irreducible as a holomorphic function.*

Lemma 1.17. *If a subvariety K defined by an irreducible function r, and let U be as in Lemma 1.16. Then if $K^* \cap U$ is Levi-flat at a single point, then $K^* \cap U$ is Levi-flat at all points.*

From this we show the following lemma for an arbitrary $U \subset \mathbb{C}^N$.

Lemma 1.18. *Let $H \subset U \subset \mathbb{C}^N$ (U connected open set) be an irreducible real-analytic subvariety of U of codimension 1. If there exists a point $p \in H^*$ and a neighbourhood $N \subset \mathbb{C}^N$ of p, such that $H^* \cap N$ is Levi-flat, then H is Levi-flat (i.e. H is Levi-flat at all points).*

Proof. By Lemma 1.16 we can find a collection of open neighbourhoods U_j and for each U_j we find irreducible branches $A_{j1}, \ldots, A_{jn}$ of H in U_j, and assume that each $A_{jk} \subset U_j$ satisfies the above property. Let H' be a union of those A_{jk} such that $A_{11} \subset H'$ and if $A_{jk} \subset H'$ and $A_{\ell m} \cap A_{jk}$ is of codimension 1, then $A_{\ell m} \subset H'$. It is clear that H' is a subvariety of U and since H is irreducible then $H' = H$. It is clear that all the A_{jk} are Levi-flat if and only if A_{11} is Levi-flat, and we are done. □

Note that if $M \subset \mathbb{C}^N$ is generic Levi-flat real-analytic submanifold of codimension d, then locally M is biholomorphically equivalent to the set defined by $\operatorname{Im} z_1 = \cdots = \operatorname{Im} z_d = 0$.

2 Singularities of Levi-flat hypersurfaces

2.1 Submanifolds of the singularity

In this chapter we will prove the following theorem, which is a corollary of the more technical Theorem 2.2 below.

Theorem 2.1. *Let $H \subset \mathbb{C}^N$ be a singular real-analytic Levi-flat hypersurface, and let $M \subset H_s \cap \overline{H^*}$ be a smooth submanifold, then for p on an open dense set of M, the germ of M at p is contained in some germ of a proper complex subvariety or generic real-analytic Levi-flat submanifold of real dimension $2N - 2$.*

We will consider a singular real-analytic Levi-flat hypersurface $H \subset U$, $0 \in H$, where U is an open neighbourhood of the origin in $\mathbb{C}^N$ and $H = \{z \in U \mid \rho(z, \bar{z}) = 0\}$, for a real valued real-analytic function ρ. As we are interested in local properties of H we will assume that U is small enough such that ρ can be complexified to $U \times {}^*U$, where ${}^*U = \{z \mid \bar{z} \in U\}$. Further, we will assume that U is connected. As before we will denote by H^* the nonsingular points of dimension $2N - 1$. Then we let $H_s := H \backslash H^*$. We note that it is not necessarily true that $\overline{H^*} = H$, even if H is irreducible. Since H is real-analytic, we say that H is Levi-flat, if near each $p \in H^*$ there are suitable holomorphic coordinates such that H is given by $\operatorname{Im} z_1 = 0$. By Lemma 1.18, if H is irreducible we only need to check this property at one $p \in H^*$.

Our main result about Levi-flat hypersurfaces is the following theorem.

Theorem 2.2. *Let $H \subset U \subset \mathbb{C}^N$ be singular real-analytic Levi-flat hypersurface, then*

$$H_s \cap \overline{H^*} \subset \bigcup_{j=1}^{\infty} M_j \tag{2.1}$$

where $M_j \subset U_j$ for some countable collection of open sets $U_j \subset U$, and where M_j is either a proper complex-analytic subvariety of U_j or a generic real-analytic Levi-flat submanifold of real dimension at most $2N-2$.

Theorem 2.1 follows from this technical result.

Proof of Theorem 2.1. By the above theorem, $M \subset \bigcup M_j$. Suppose that there is no point in M such that near that point $M \subset M_j$ (as germs) for some j. That means, $M \cap M_j$ is nowhere dense in M (it does not contain an open set). But there are only countably many such sets, and so by Baire category theorem they cannot cover all of M, which would be a contradiction. Thus there has to exist a point p where M is contained (as a germ at p) in some M_j. This holds on an open set near p as well, and furthermore, since it holds for all open U, by taking U smaller we can see that it has to hold on an open dense set of M. □

A useful weaker result, at a point where H_s is a submanifold of codimension one in H, is the following.

Corollary 2.3. *Let H be a singular real-analytic Levi-flat hypersurface defined in a neighbourhood of the origin in $\mathbb{C}^N$, and suppose that H_s is a manifold of dimension $2N-2$ and $H_s \subset \overline{H^*}$. Then H_s is either complex-analytic or Levi-flat.*

Proof. If H_s was of a different type, then all the Levi-flat and complex-analytic M_j's have an intersection of a lower dimension with H_s. By Baire category theorem again, this is not possible, as there are only countably many. □

Thus we have a complete categorization of singularities if they are of highest possible dimension and are in the closure of the nonsingular points. There are examples where the singular set is complex (e.g. $\{z \mid \operatorname{Im} z_1^2 = 0\}$) or Levi-flat (e.g. $\{z \mid (\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0\}$). More examples can be found in section 2.4 but it is not clear that an irreducible hypersurface can have a Levi-flat singularity.

A smooth CR submanifold is said to be of *finite type* at $p \in M$ if the CR vector fields, their complex conjugates and finitely many commutators of these, span the complexified tangent space $\mathbb{C}T_pM$ ($\mathbb{C}T_pM = \mathbb{C} \otimes_{\mathbb{R}} T_pM$). In case M is real-analytic, being finite type at p is equivalent to being minimal at p. It is not hard to see that if M is finite type at p, then there cannot exist a holomorphic function in a neighbourhood of p which is real valued on M. We can now rule out all smooth finite type generic submanifolds of any codimension being contained in Levi-flat hypersurfaces.

Corollary 2.4. *Let $H \subset U \subset \mathbb{C}^N$ be singular real-analytic Levi-flat hypersurface, and let $M \subset \overline{H^*}$ be a smooth generic submanifold. Then M is not of finite type at any point.*

Proof. Take a point $p \in M$. If $p \in M \cap H^*$, then there is some neighbourhood of p, where in suitable local coordinates H is given by $\operatorname{Im} z_1 = 0$, and thus z_1 is real valued on an open set of M. Since if M would be of finite type at p, it would be of finite type in a neighbourhood of p. If there exists a real valued holomorphic function on M near p, M cannot be of finite type at p. So let $p \in H_s$. Again if M would be of finite type at p then it would be so near p, and there would either be a point $q \in M \cap H^*$ where M was of finite type, which we now know cannot happen, or $M \subset H_s$ as germs at p. But then by Theorem 2.1 for some point $q \in M$, where M would be of finite type, it would be contained as germ in either a complex variety or a Levi-flat generic submanifold which is again impossible. Thus M cannot be of finite type. $\square$

2.2 Proof of Theorem 2.2

Before going into the proof of Theorem 2.2, let us fix some notation and background. Let Σ_z be the *Segre variety* of H at the point z, that is the set $\{\zeta \in U \mid \rho(\zeta, \bar{z}) = 0\}$, and let Σ'_z be the branches of Σ_z completely inside H. We say that Σ_z is *degenerate* if Σ_z contains an open set of $\mathbb{C}^N$, that is, if $\Sigma_z = U$ if U is connected.

We will the following lemma about Levi-flat hypersurfaces, this is proved in [BG99].

Lemma 2.5. *Let $H \subset U$ be as above and Levi-flat, and suppose $z \in H$ is such that Σ_z is non-degenerate. Then Σ'_z is non-empty, and further one branch of Σ'_z passes through z. If $z \in H^*$, then Σ'_z has only one branch through z, and this is the unique germ of a complex variety through z.*

Also, since we could pick U smaller and smaller, one branch of Σ'_z must therefore always pass through z.

If ρ is a defining function for H in a neighbourhood U, then at all points of H_s, ρ must have a vanishing gradient, since otherwise H would be a nonsingular hypersurface at that point. In fact, picking a possibly smaller U, $\{z \in H \mid \partial\rho(z,\bar{z}) = 0\}$ is a proper subvariety of H containing H_s (here ∂ means the exterior derivative in the z variables). Assume H is irreducible, complexify ρ into $U \times {}^*U$, and let $\mathcal{H} = \{(z,\zeta) \in U \times {}^*U \mid \rho(z,\zeta) = 0\}$. Then by Lemma 1.16, ρ is irreducible as a holomorphic function (in a possibly smaller neighbourhood), and thus generates the ideal of $\mathcal{H}$ by the Nullstellensatz at every point in $U \times {}^*U$. Therefore, the gradient of the complexified ρ does not vanish at all nonsingular points of $\mathcal{H}$. Near any $p \in H^*$ we have a local defining function with nonvanishing gradient near p, which when complexified divides ρ. That means, near p, H^* complexifies to a germ of a smooth complex hypersurface in $U \times {}^*U$ contained in $\mathcal{H}$. Since H^* is totally real in this complex hypersurface we know $\partial\rho$ cannot vanish identically on H^* (or it would vanish in all of $\mathcal{H}$ since it is irreducible). Hence, $\partial\rho = 0$ defines a proper lower dimensional subvariety of H which contains H_s. We cannot quite say it equals H_s, as a point p could be in H^*, but the point $(p,\bar{p})$ could a priory be a singular point of $\mathcal{H}$.

Lemma 2.6. *Let $H_1, H_2 \subset \mathbb{C}^N$ be two connected nonsingular real-analytic Levi-flat hypersurfaces, such that $0 \in H_1 \cap H_2$. If U is a sufficiently small neighbourhood of 0, and $H_1 \cap U \neq H_2 \cap U$, then there exists a possibly empty proper complex-analytic subvariety $A \subset U$ such that $(U \cap H_1 \cap H_2) \setminus A$ is either empty or a generic real-analytic Levi-flat submanifold of codimension 2.*

Proof. We let U be small enough such that $H_1 \cap U$ and $H_2 \cap U$ are closed in U and hence we can assume that $H_1, H_2 \subset U$. Further, let U be small enough such that there exist holomorphic coordinates in U where H_1 is given by $\operatorname{Im} z_1 = 0$ and H_2 is given by $\operatorname{Im} f = 0$, where f is holomorphic with nonvanishing differential. The set where the complex differentials of f and z_1 are linearly dependent is a complex-analytic subvariety. If the complex differentials are everywhere linearly dependent, then f depends only on z_1 and thus the intersection of H_1 and H_2 is complex-analytic. Suppose that outside a subvariety A, f and z_1 have linearly independent differentials. Locally, in an even smaller neighbourhood, we can change coordinates again to make $f = z_2$. Then the intersection is locally defined as $\operatorname{Im} z_1 = \operatorname{Im} z_2 = 0$, and we are done. $\square$

Proof of Theorem 2.2. Recall that to prove the Theorem, we will cover $H_s \cap \overline{H^*}$ by countably many Levi-flat submanifolds of codimension 2 and local complex-analytic subvarieties. These submanifolds and subvarieties need not lie in H itself, we just want their union as sets to contain $H_s \cap \overline{H^*}$.

Let $H_s' := H_s \cap \overline{H^*}$. The place in the proof where we fail to cover all of H_s, if $H_s \not\subset \overline{H^*}$, is in the application of Lemma 2.5.

Assume that H is irreducible. If it is reducible, and we prove the result for each branch, then it is also true for the union of those branches. This is because if K and L are branches of $H = K \cup L$, then $H_s = K_s \cup L_s \cup S$, where S is the set of points of $K^* \cap L^*$, where $K^* \cap L^*$ is not a hypersurface. Hence, if we have covered K_s and L_s, the only other points that need to be covered are points of S. If $p \in S$ we pick a small enough neighbourhood of p and apply Lemma 2.6. We can also assume H it is irreducible in arbitrarily small neighbourhoods of 0 as well for the same reason (so irreducible as a germ).

First we note that the points $z \in U$ where Σ_z is degenerate lie inside a complex-analytic variety, because $z \in \Sigma_w$ implies $w \in \Sigma_z$ by reality of ρ. So that means that if z is a degenerate point, then it is contained in Σ_w for all $w \in U$, and thus is inside a complex-analytic subvariety A. Because we only care about a countable union of local varieties and manifolds, we can just cover $U \setminus A$ by smaller neighbourhoods and work there. Thus we can assume that U contains no degenerate points.

Suppose $0 \in H$, and suppose that a branch of Σ_0, call it A again, is contained in H_s. Again, since we only care about a countable union of local varieties and manifolds, we can cover $U \setminus A$ by small neighbourhoods and work there. Thus we can assume that Σ_0 has no branch that is contained in H_s (and thus not in H_s').

By Lemma 2.5, Σ_0' is non-empty and we now know that no branch of it is contained completely in H_s'. So we know that there exists a point $\zeta \in \Sigma_0'$ such that $\zeta \in H^*$. As Σ_0' at $\zeta \in H^*$ is the unique complex variety (again by Lemma 2.5) passing through ζ we know that Σ_ζ' shares this branch with Σ_0'.

We can of course pick this ζ in a topological component of $(\Sigma_0')^* \cap H^*$, where $(\Sigma_0')^*$ is the nonsingular part of Σ_0', such that 0 is in the closure of this component. As no branch of Σ_0' lies inside H_s' and there is at least one branch through 0, then at least one topological component of $(\Sigma_0')^* \cap H^*$ will be such that 0 is in its closure.

We look at a small neighbourhood V of ζ such that $H \cap V$ is connected and nonsingular, and further, such that H is defined in V by $\operatorname{Im} f(z) = 0$, for some f holomorphic in V where the gradient of f does not vanish in V.

Pick a nonsingular real-analytic curve $\gamma\colon (-\epsilon, \epsilon) \to H$ such that $\gamma(0) = \zeta$, $\{\gamma\} \subset V$, and furthermore, that γ is transverse to the Levi foliation of H^*. We can do this by just changing coordinates in V such that $z_n = f$, and then our curve might be $t \mapsto t\alpha$ where $\alpha \in C^n$ and α_n is not real. Once we have γ we can look at the sets $\Sigma_{\gamma(t)}$ for various t. These are given by $\{z \mid \rho(z, \overline{\gamma(t)}) = 0\}$. However, we can just look at the zero set of the function $(z,t) \mapsto \rho(z, \bar{\gamma}(t))$ as t is real. Further, we can pick γ such that $\rho(0, \bar{\gamma}(t))$ is not identically zero since if it were for all choices of γ (by varying α above), then Σ_0 would contain an open set in H^* and thus would be degenerate, and we assumed it was not. We can complexify t and look at the zero set of $\rho(z, \bar{\gamma}(t))$ in $U \times D_\epsilon$ (where D_ϵ is the disk of radius $\epsilon > 0$).

Next apply the Weierstrass preparation theorem, which we can do in some neighbourhood of $(0,0)$ in $U' \times D_{\epsilon'} \subset U \times D_\epsilon$ and we get a polynomial

$$F(z,t) = t^m + \sum_{j=0}^{m-1} a_j(z) t^j, \tag{2.2}$$

whose zero set is the zero set of $\rho(z, \bar{\gamma}(t))$. Outside of the discriminant set of F, $\Delta \subset U'$, we have (locally) m holomorphic functions $\{e_j\}_1^m$ which give us the solutions to $F(z, e_j(z)) = 0$. We look at the places where these solutions are real, that is the points in U' where $e_j - \bar{e}_j = 0$. To be able to complexify we look at the function

$$i^m \prod_{j,k=1}^{m} (e_j(z) - \overline{e_k(z)}). \tag{2.3}$$

It is easy to see that this is a real function. Furthermore, it is symmetric both in the $e_j(z)$ and the $\overline{e_k(z)}$, this means that after complexification we have a well defined holomorphic function in $(U' \times {}^*U') \setminus (\Delta \times {}^*\Delta)$, and continuous in all of $(U' \times {}^*U')$ and thus holomorphic in $(U' \times {}^*U')$. (see [Whi72] for more). Thus we have a real-analytic function, say $\hat{\rho}\colon U' \to \mathbb{R}$ that is locally outside of Δ given by $i^m \prod_{j,k=1}^{m}(e_j(z) - \overline{e_k(z)})$.

We let $\hat{H} := \{\hat{\rho} = 0\}$. We need to now see that $(H \cap U') \cap \hat{H}$ is open in H, because then $(H \cap U') \subset \hat{H}$ as H is irreducible in U' and we can apply Lemma 1.16 as we can always pick a smaller U'.

It is obvious that $\Sigma'_{\gamma(0)} \cap U'$ is in both H and $\hat{H}$. The trouble is for other t, as $V \cap U'$ may in fact be empty. Because of how we picked ζ, we note that the topological component of $(\Sigma'_{\gamma(0)})^*$ where ζ lies is connected to 0. So we can find a nonsingular point ζ' of $\Sigma'_{\gamma(0)}$ on this component that is arbitrarily close to 0, and thus inside U'. We can pick a finite sequence of overlapping neighbourhoods $\{V_j\}$ from ζ to ζ' such that inside each V_j, H is given by $\operatorname{Im} f_j(z) = 0$ (for some f_j holomorphic in V_j). We call the final neighbourhood V' and assume $V' \subset U'$ and there H is given by $\operatorname{Im} f'(z) = 0$ (for some f' holomorphic in V'). It is easy to see that the Levi foliation is given by $f_j(z) = r$ for some real r, and that these sets must agree on $V_j \cap V_k$. Thus for some $\epsilon'' > 0$, for all $|t| < \epsilon''$, we have a component of $\Sigma_{\gamma t}$ passing through V which also passes thorough V' which contains ζ'. But $V' \subset U'$ and $\hat{H}$ and H both contain all points $\{z \mid f'(z) = t, |t| < \epsilon''\}$ and that is an open set in H.

Now that we know that H is contained in $\hat{H}$, we can remove Δ, which is complex-analytic, and work only in small neighbourhoods where $\hat{H}$ is given by $\prod(e_j(z) - \overline{e_k(z)})$. Since $e_j(z) - \overline{e_k(z)}$ is pluriharmonic, and thus its real and imaginary parts are pluriharmonic, meaning that we can represent them as the imaginary part of a holomorphic function, that is $\operatorname{Im} f_{jk}(z) + i \operatorname{Im} g_{jk}(z)$. Thus we get locally that

$$\hat{\rho}(z,\bar{z}) = i^m \prod_{j,k=1}^{m} (\operatorname{Im} f_{jk}(z) + i \operatorname{Im} g_{jk}(z)). \tag{2.4}$$

If $\operatorname{Im} f_{jk}(z) + i \operatorname{Im} g_{jk}(z)$ is zero then $(\operatorname{Im} f_{jk}(z))(\operatorname{Im} g_{jk}(z))$ is also zero. Thus we can make yet a larger surface by looking at the zero set of

$$\hat{\hat{\rho}}(z,\bar{z}) = \prod_{j,k=1}^{m} (\operatorname{Im} f_{jk}(z))(\operatorname{Im} g_{jk}(z)). \tag{2.5}$$

That is just a product of the imaginary parts of holomorphic functions. We can now take out the set where the gradient of f_{jk} and g_{jk} vanish, which is a complex-analytic set and work in smaller neighbourhoods outside this set. We can take these neighbourhoods small enough such that each $\operatorname{Im} f_{jk} = 0$ and $\operatorname{Im} g_{jk} = 0$ defines a nonsingular, connected hypersurface. The singular set of H must be contained in the intersection of at least two of these surfaces (if there is more than one left). This intersection is a generic real-analytic Levi-flat submanifold of codimension 2 outside a complex-analytic subvariety by Lemma 2.6. $\square$

2.3 Invariants of the singularity

In this section we will discuss some local biholomorphic invariants of the singularity of a singular real-analytic Levi-flat hypersurface. Since a nonsingular Levi-flat hypersurface is defined locally by $\operatorname{Im} z_1 = 0$ in suitable coordinates, there are no local invariants. If, however, we allow singularities, the story changes. In this section we will discuss several local invariants of singular real-analytic Levi-flat hypersurfaces.

Let an irreducible Levi-flat hypersurface H be defined in U by a single real-analytic function ρ as before, which complexifies to $U \times {}^*U$. By Lemma 1.16, we know ρ is irreducible as a holomorphic function. This means that if ρ' is another defining function for H such that it complexifies to $U \times {}^*U$, it is also irreducible and defines the same complexification. Let Σ_p again be the Segre variety in U of H at p defined by $\{z \mid \rho(z,\bar{p}) = 0\}$. We therefore have:

Proposition 2.7. *Let $H \subset U$ be as above, then Σ_p is independent of the defining function of H.*

Of course in the above we only consider those defining functions which complexify to $U \times {}^*U$, otherwise the Segre variety is not well defined for that defining function. We could drop the requirement that H is irreducible since we can just apply the proposition to each irreducible branch as there are only finitely many.

Let $f\colon U \subset \mathbb{C}^N \to \tilde{U} \subset \mathbb{C}^N$ be a biholomorphism. $\rho \circ f^{-1}$ is a defining function for $\tilde{H} := f(H)$, and it complexifies to $\tilde{U} \times {}^*\tilde{U}$, because ρ complexifies to $U \times {}^*U$. Then we have the following proposition. We let $\tilde{\Sigma}_q$ be the Segre variety of $\tilde{H}$ in $\tilde{U}$ at q.

Proposition 2.8. *Let f, H, U, $\tilde{H}$ and $\tilde{U}$ be as above. Then $f(\Sigma_p) = \tilde{\Sigma}_{f(p)}$.*

In particular we have the dimension of the Segre variety at a point is a local invariant. Hence a hypersurface with a degenerate Segre variety is not locally biholomorphic to one which has a non-degenerate Segre variety.

Another invariant which is suggested by the above discussion and the examples of Levi-flat hypersurfaces in general is the following. Let $p \in H$.

$$\begin{aligned} n(H,p) = \min\{n \in \mathbb{N} \ \mid \ & \exists \text{ neighbourhood } V \text{ of } p, \\ & \text{a holomorphic } f\colon V \to \mathbb{C}^n, \\ & \text{and a Levi-flat hypersurface } \tilde{H} \subset \mathbb{C}^n, \\ & \text{such that } H \subset f^{-1}(\tilde{H}) \text{ and } f^{-1}(\tilde{H}) \text{ is a hypersurface}\} \end{aligned} \tag{2.6}$$

Note that if $f^{-1}(\tilde{H})$ is a hypersurface, it is necessarily Levi-flat. We need to pull back the Levi-foliation of $\tilde{H}$ by f to see this. $n(H,p)$ is obviously a local invariant at p. To see that it is not trivial we note that there are lots of hypersurfaces such that $n(H,p) = 1$ at all points. For example, if a hypersurfaces is defined by $\operatorname{Im} g = 0$ for a holomorphic function g. However, we have the following proposition.

Proposition 2.9. *Let $H \subset U \subset \mathbb{C}^N$ be a Levi-flat hypersurface, where U is small enough as before. If Σ_p is degenerate, then $n(H,p) \geq 2$.*

Proof. Suppose not, then there is a holomorphic function $f\colon V \to \mathbb{C}$ such that the inverse image of some real-analytic line $\tilde{H}$ by f is a hypersurface and contains H. This means that f cannot be constant. In fact near p, if $\tilde{H}$ is defined by some real-analytic function ρ near $f(p)$, then $\rho(f(z), \overline{f(z)}) = 0$ defines a hypersurface containing H. We assume $0 \in \tilde{H}$. $\rho \circ f$ may not be a defining function, but it is divisible by a defining function. Hence the set $\rho(f(z), \overline{f(p)}) = 0$ contains the Segre variety of this hypersurface and hence Σ_p. So we just need to see that $\rho \circ f$ is not identically zero. If this is so, then $\rho(z, \bar{z})$ is divisible by $\bar{z}$, therefore also by z, and thus by $|z|^2$. Hence ρ was not a defining function of $\tilde{H}$. $\square$

Proposition 2.10. *Suppose H is defined by the vanishing of the imaginary part of a holomorphic function, then $n(H, p) = 1$ for all $p \in H$.*

Proof. If H is given by $\operatorname{Im} f = 0$, then f is the required mapping and the real line is $\tilde{H}$. $\square$

The hypersurface in $\mathbb{C}^2$ defined by $z\bar{w} - \bar{z}w = 0$ is a set with $n(H, 0) = 2$ by Proposition 2.9 and $n(H, p) = 0$ at all other $p \in H$. In fact we have:

Proposition 2.11. *Suppose H is defined by the vanishing of the imaginary part of a meromorphic function and $0 \in H$, then $n(H, 0) \leq 2$. Further, the set where $n(H, p) = 2$ is contained in a complex-analytic subvariety of codimension 2.*

Proof. What we mean by H being defined by the vanishing of the imaginary part of f/g is $f\bar{g} - \bar{f}g = 0$. So (f, g) is the required mapping into $\mathbb{C}^2$ and $\{z\bar{w} - \bar{z}w = 0\}$ is the required $\tilde{H}$. The set where $n(H, p)$ is 2 must be contained in the indeterminacy set of f/g (if in lowest terms at p, $\{f = g = 0\}$). Elsewhere either f/g or g/f is holomorphic and hence we can apply Proposition 2.10. $\square$

Let $z_1 = x_1 + iy_1$. Then the surface $H \subset \mathbb{C}^N$ defined by $x_1^2 - y_1^3 = 0$ is Levi-flat (obviously) and $n(H, p) = 1$ for all $p \in H$. This is the example given by Burns and Gong [BG99] as an example of a hypersurface not given by the vanishing of the imaginary part of a meromorphic function.

A different invariant, that may in fact be related to the Segre variety, is the attaching of analytic discs. For simplicity and without loss of generality, we will work in a polydisc $\mathbb{D}^N \subset \mathbb{C}^N$, i.e. the set $\{z \in \mathbb{C}^N \mid |z_k| < 1, k = 1, \ldots, N\}$. We will also only consider Levi-flat hypersurface H, defined by a single real-analytic defining function ρ which complexifies to $\mathbb{D}^N \times {}^*\mathbb{D}^N$, and such that $0 \in H$.

We define the set

$$H_D := \bigcup \{\Delta \subset \mathbb{D}^N \mid \Delta \text{ an analytic disc attached to } H\}. \tag{2.7}$$

By an *analytic disc*, we mean the image of a continuous mapping $\delta \colon \bar{\mathbb{D}} \to \mathbb{C}^N$, which is holomorphic in $\mathbb{D}$ (the unit disc in $\mathbb{C}$). I.e., $\Delta = \delta(\bar{\mathbb{D}})$. We say Δ is *attached to* H if $\delta(\partial\mathbb{D}) \subset H$.

Since $\mathbb{D}^N$ is pseudoconvex, then if $\delta(\partial\mathbb{D}) \subset \mathbb{D}^N$, then $\delta(\mathbb{D}) \subset \mathbb{D}^N$.

Proposition 2.12. *If $H \subset \mathbb{D}^N$ is defined by $\operatorname{Im} f = 0$ for a holomorphic f, then $H_D = H$. In fact, if Δ is an analytic disc attached to H, it lies in a set defined by $f = c$ for some real constant c.*

Proof. The composed holomorphic function $f \circ \delta \colon \mathbb{D} \to \mathbb{C}$ is a holomorphic function that extends to the boundary of the unit disc and is real valued there, hence it is constant. $\square$

Such an outcome is not true in general, even if the hypersurface H is defined by the vanishing of the imaginary part of a meromorphic function. The simplest example in $\mathbb{C}^2$, with coordinates z and w, is the hypersurface defined by $|z|^2 - |w|^2 = 0$. This is the same hypersurface as $z\bar{w} - \bar{z}w = 0$ after a holomorphic change of coordinates. We will just assume we are working in $\mathbb{D}^2$.

Proposition 2.13. *If $H \subset \mathbb{D}^2$ is defined by $|z|^2 - |w|^2 = 0$, then $H_D = \mathbb{D}^2$.*

Proof. We can get explicit formulas for the analytic disc in terms of Möbious mappings, but we will give a simpler proof and leave the computation to the reader. We just note that the hyperplane defined by $w = w_0$ for some fixed $w_0 \neq 0$ intersects H in the circle $|z|^2 = |w_0|^2$. There exists an analytic disc $\{(z, w_0) \mid |z|^2 \leq |w_0|^2\}$. Similar argument can be given by fixing $z = z_0$. The union of these discs is then obviously $\mathbb{D}^2$, that is the set $\{(z, w) \mid |z|^2 < 1, |w|^2 < 1\}$. So we have that $\mathbb{D}^2 \subset H_D$. $\square$

As it is not clear that pullbacks of analytic discs are analytic discs, we cannot generalize this result easily to all hypersurfaces defined by the vanishing of the imaginary part of a meromorphic function. As analytic discs have been used by Tumanov [Tum88] and others to fill the Segre sets and hence the intrinsic complexification of a CR orbit of submanifold, we may hope for a similar result here. Hence it is reasonable to expect that if a Levi-flat hypersurface has a degenerate Segre variety at a point, then analytic discs attached to H inside a small polydisc around the point fill an open set.

One final invariant we will discuss is motivated by the proof of Theorem 2.2. So suppose that $H \subset U$, $0 \in H$, with the usual requirements on the defining function of H. Further, assume that Σ_0 is nondegenerate, and as in the proof of Theorem 2.2, that Σ_0' does not lie completely in H_s. Then pick a real-analytic curve $\gamma(t)$ with image inside H^*, such that $\Sigma_{\gamma(0)}$ has a branch that goes through the origin, and with the same requirements on $\gamma(0)$ as in the proof. We can then write an equation $\rho(z, \bar{\gamma}(t)) = 0$, and for real t this sweeps out an open set inside H. We can apply the Weierstrass preparation theorem in some smaller neighbourhood of 0 to get a Weierstrass polynomial in t, $F(z,t)$ and again as in the proof of Theorem 2.2 note that the set $\tilde{H} = \{z \mid F(z,t) = 0, t \in \mathbb{R}\}$ contains an open set of H.

Similarly if H is algebraic, i.e. ρ is a polynomial, then we could find an algebraic (polynomial) mapping $\varphi \colon \mathbb{R} \to H$, such that φ does not map into a single Segre variety and such that the image of φ includes the nonsingular part of H. This can be done by cutting H in the proper place with a complex line L such that $H \cap L$ is a real-analytic line in $\mathbb{C}$ and then parametrizing this line. The Segre varieties $\Sigma_{\varphi(t)}$ then cover an open set of H (because near t where $\varphi(t) \in H^*$ the Segre varieties locally agree with the Levi-foliation of H). Then $F(z,t) = \rho(z, \bar{\varphi}(t))$ is the required pseudopolynomial in t, though it is not necessarily a Weierstrass polynomial. Note that we may have to repeat the procedure if this one F is not enough.

In the following we will call $F(z,t)$ a pseudopolynomial, if it is a polynomial in t with coefficients holomorphic functions of z. By the degree, we mean the degree of the variable t. Let us study the minimum degree of F that is required. When there is only one F that is needed, and if F is of degree 1, then the situation is simple.

Proposition 2.14. *If H is such that there exists a pseudopolynomial $F(z,t)$ of degree 1 such that $H \subset \{z \mid F(z,t) = 0, t \in \mathbb{R}\}$. Then there exists a hypersurface $\tilde{H}$ defined by the vanishing of the imaginary part of a meromorphic function f, and $H \subset \tilde{H}$. If F is also monic then f can be taken to be holomorphic.*

Proof. If $F(z,t) = t + a(z)$, then let $f = a$. If $F(z,t) = a(z)t + b(z)$, then let $f = b/a$. $\square$

We can define the following invariant.

$$\begin{aligned} d(H,p) = \min\{d \in \mathbb{N} \mid \exists \text{ neighbourhood } U \text{ of } p, \\ &\text{nonzero pseudopolynomials } F_k(z,t) \text{ defined on } U \times \mathbb{R}, \\ &\deg F_k(z,\cdot) \leq d,\ k = 1,\ldots,K \\ &H \cap U \subset \{z \mid F_k(z,t) = 0,\ t \in \mathbb{R},\ k = 1,\ldots,K\}\}. \end{aligned} \tag{2.8}$$

A priory $d(H,p)$ could also be infinity. For example, if H has a degenerate Segre variety at p or if Σ'_p lies in the singularity, H is not defined by the vanishing of the imaginary part of a meromorphic function and H is not algebraic (defined by a polynomial). However, no examples where $d(H,p) = \infty$ are known to the author.

If H is defined by the vanishing of the imaginary part of a meromorphic (or holomorphic) function, then of course $d(H,p) = 1$ as is obvious from the proof of Proposition 2.14. To see that there exists an H with a $d(H,p)$ that is greater than 1, we look at the example of a Levi-flat hypersurface not defined by a meromorphic function. Let $z_1 = x_1 + iy_1$ and define H by $x_1^2 - y_1^3 = 0$. Then let $F(z,t) := z_1 - (t^3 + it^2)$. Hence $d(H,0) = 2$ or 3.

While not every hypersurface with $d(H,p) < \infty$ is defined by the vanishing of the imaginary part of a meromorphic function, we have the following proposition however. Fix a neighbourhood U, and let $\mathcal{A}(U)$ be the ring of functions of the form

$$\sum_{j=1}^{k} (|f_j(z)|^2 - |g_j(z)|^2), \tag{2.9}$$

where f_j and g_j are any holomorphic functions on U. This ring contains all the functions of the form $\operatorname{Re} f$ for some f holomorphic on U. In fact it is not hard to check that this is the smallest such subring of real-analytic functions on U (and that it is a proper subring). Then we see that if $F(z,t)$ is a pseudopolynomial on $U \times \mathbb{R}$, then the projection of the set $F = 0$ onto the z coordinates is a set in $\mathcal{S}(\mathcal{A}(U))$ by Łojasiewicz's variation of Tarski-Seidenberg, Theorem 1.9. Since this set must be of real codimension 1 in U, then we can see that there exists one function $\varphi \in \mathcal{A}(U)$ that vanishes on this set. Hence:

Proposition 2.15. *Let H be a possibly singular Levi-flat hypersurface, $0 \in H$ and $0 \in H$, $d(H,0) < \infty$. Then there exists a neighbourhood U of 0, and a nonconstant $\varphi \in \mathcal{A}(U)$ such that $H \cap U \subset \{z \mid \varphi(z) = 0\}$.*

The tangent cone of a hypersurface is another invariant, but we will not discuss this here. Burns and Gong [BG99] have studied those hypersurfaces with quadratic tangent cones. They have proved that the tangent cone of a Levi-flat hypersurface is Levi-flat, and have classified fully the quadratic hypersurfaces (those defined by quadratic equations).

2.4 Examples

Burns and Gong [BG99] have classified all singular quadratic[1] Levi-flat hypersurfaces in $\mathbb{C}^N$ to be as follows.

Theorem 2.16 (Burns-Gong)**.** *If $H \subset \mathbb{C}^N$ is a quadratic Levi-flat hypersurface, then it is biholomorphically equivalent to a hypersurface with one of the following five defining functions.*

(i) $\operatorname{Im}(z_1^2 + \cdots + z_k^2) = 0$, $k = 1, \ldots, N$,

(ii) $\operatorname{Im} z_1 = 0$,

[1]Hypersurfaces defined by the vanishing of a quadratic polynomial.

(iii) $z_1^2 + 2\lambda z_1 \bar{z}_1 + \bar{z}_1^2 = 0$*, where* $\lambda \in (0,1)$*,*

(iv) $(\operatorname{Im} z_1)(\operatorname{Im} z_2) = 0$*,*

(v) $|z_1|^2 - |z_2|^2 = 0$*.*

Hypersurfaces (i), (ii), (v) are defined by the vanishing of the imaginary part of a meromorphic function. Hypersurface (iii) can sometimes be defined in such a way as well, depending on λ. Hypersurfaces (iii) and (iv) are reducible and each branch is a nonsingular Levi-flat hypersurface. In fact if we only look at the z_1 variable, then (iii) is just two intersecting lines, intersecting in different angles depending on λ.

On the other hand, the hypersurface defined by $x_1^2 - y_1^3 = 0$, for $z_1 = x_1 + iy_1$ is irreducible and not defined by the vanishing of the imaginary part of a meromorphic function. To see this not that we need only work in one dimension, and there we can just assume that the hypersurface would be defined by the vanishing of a holomorphic function. After a holomorphic change of coordinates we have a line with a cusp such that z^k is real valued on it, which is not possible.

More complicated examples can be derived by pulling back a Levi-flat hypersurface by a holomorphic mapping. That is, if $H \subset \mathbb{C}^n$ is a hypersurface defined by $\rho = 0$, and $f \colon \mathbb{C}^N \to \mathbb{C}^n$ is a nontrivial holomorphic mapping, then the set $\tilde{H} \subset \mathbb{C}^N$, defined by $\rho \circ f = 0$ is a Levi-flat hypersurface. This can be seen by pulling back the Levi foliation of H which becomes the Levi foliation of $\tilde{H}$. In view of section 2.3 however, such examples need not be, in a certain sense, any more complicated than H was. The singularity of $\tilde{H}$ will consist of the pullback H_s, union with any singularities introduced by f. Any singularities introduced by f have to lie in the set where $\det Df = 0$. Hence,

$$\tilde{H}_s \subset f^{-1}(H_s) \cup \{ z \mid \det Df(z) = 0 \}. \tag{2.10}$$

Further, it is not hard to see that if H cannot be defined by the vanishing of the imaginary part of a meromorphic function, then neither can $\tilde{H}$.

Brunella [Bru07] found the following example which shows that one can have an irreducible Levi-flat hypersurface $H \subset \mathbb{C}^2$ such that $\overline{H^*} \neq H$, and further, where H_s is a totally real submanifold. If we let $z = x + iy$ and $w = s + it$, then we define the hypersurface by

$$4(y^2 + s)y^2 - t^2 = 0. \tag{2.11}$$

The curves $w = (z + c)^2$ for some constant $c \in \mathbb{R}$ give the Levi-foliation of H^*. Furthermore, $H_s = \{t = y = 0\}$ and $\overline{H^*} \cap H_s = \{t = y = 0, s \geq 0\}$. This hypersurface also cannot be defined by the vanishing of the imaginary part of a meromorphic function, since H_s is not contained in a complex analytic subvariety. By the same argument, we have that $n(H, 0) = 2$, that is, it is not a pullback of a one-dimensional line in $\mathbb{C}$. This is an example of a hypersurface with $n(H, 0) = 2$, such that Σ_0 is nondegenerate, as Σ_0 is defined by $4w^2 - z^2(2w - z^2) = 0$.

3 Submanifolds inside Levi-flat hypersurfaces

3.1 Uniqueness property for holomorphic functions

For a generic submanifold M through the origin in $\mathbb{C}^N$, we wish to investigate when does there exist a meromorphic function near the origin which is real valued on M. By composing with a Möbius mapping of the real line onto the unit circle we see that this is equivalent to the existence of a meromorphic function which is unimodular on M, which in turn means that there are two relatively prime holomorphic functions f and g such that on M, $|f| = |g|$. We will thus define:

Definition 3.1. M has the *modulus uniqueness property* if $|f| = |g|$ on M, for holomorphic f and g defined in a neighbourhood of M, implies $f = cg$ for a unimodular constant c. We will say that M has the modulus uniqueness property at $p \in M$, if $M \cap U$ has the modulus uniqueness property for every connected neighbourhood U of $p \in M$.

In the following we will denote the local CR orbit at a point p by Orb_p. The motivation for our problem is the following theorem.

Theorem 3.2 (see [BER98])**.** *Let $M \subset U \subset \mathbb{C}^N$ be a generic real-analytic nowhere minimal submanifold of codimension d. Let $p \in M$ be such that Orb_p is of maximal dimension. Then there are coordinates $(z, w', w'') \in \mathbb{C}^n \times \mathbb{C}^{d-q} \times \mathbb{C}^q = \mathbb{C}^N$, where q denotes the codimension of Orb_p in M, vanishing at p such that near p, M is* *defined by*

$$\begin{aligned} \operatorname{Im} w' &= \varphi(z, \bar{z}, \operatorname{Re} w', \operatorname{Re} w'') \\ \operatorname{Im} w'' &= 0, \end{aligned} \tag{3.1}$$

φ is a real valued real-analytic function with $\varphi(z, 0, s', s'') \equiv 0$. Moreover, the local CR orbit of the point $(z, w', w'') = (0, 0, s'')$, for $s'' \in \mathbb{R}^q$, is given by

$$\begin{aligned} \operatorname{Im} w' &= \varphi(z, \bar{z}, \operatorname{Re} w', s'') \\ w'' &= s''. \end{aligned} \tag{3.2}$$

So a natural question is to ask what happens at points where Orb_p is not of maximal dimension. In general there do not exist local normal coordinates such that $\operatorname{Im} w'' = 0$ is one of the equations for M, but it is natural to ask when can we get a meromorphic function f such that $\operatorname{Im} f = 0$ on M.

Before looking at this case we summarize the results for the easy cases.

Proposition 3.3. *Let M be a connected real-analytic CR submanifold through the origin. Then M does not have the modulus uniqueness property at the origin if any of the following holds,*

(i) M is not generic,

(ii) M is totally real,

(iii) M is nowhere minimal and Orb_0 has the maximal dimension.

On the other hand M has the modulus uniqueness property at any point $p \in M$ if

(iv) M is generic and minimal at some point.

Proof. The first three cases are clear. For the last one we just note that if M is minimal at some point, it is minimal on a dense open subset. If we had a nonconstant meromorphic function real valued on M, then on some small neighbourhood we would have that M is minimal and there would exist a holomorphic function with nonvanishing gradient which was real valued on M and this would give local foliation of M by smaller submanifolds of same CR dimension and this would violate minimality. $\square$

We also note that if M is not generic, but it is minimal, then M has the modulus uniqueness property inside the intrinsic complexification of M. So since Orb_p is always minimal then if we call $\mathcal{X}_p$ the intrinsic complexification of Orb_p, then any meromorphic function real valued on M is constant in $\mathcal{X}_p$ for any CR manifold.

It is clearly useful to be able to construct $\mathcal{X}_p$ and study its properties. The following constructions are described in [BER99]. We will look at a generic submanifold M defined in normal coordinates (z, w) in some neighbourhood U of the origin, and we will assume that U is small enough such that the defining equations for M complexify into $U \times {}^*U$, and we can take U to be connected. If $M = \{z \in U \mid r(z, \bar{z}) = 0\}$ we let $\mathcal{M} := \{(z, \zeta) \in U \times {}^*U \mid r(z, \zeta) = 0\}$. We define the *Segre manifolds* for $p \in U$

$$\begin{aligned} \mathfrak{S}_{2j+1}(p, U) := \{&(z, \zeta^1, z^1, \ldots, \zeta^j, z^j) \in U \times {}^*U \times U \times \ldots \times {}^*U \times U \mid \\ &(z, \zeta^1), (z^1, \zeta^1), \ldots, (z^j, \zeta^j), (z^1, \zeta^2), \ldots, (z^{j-1}, \zeta^j), (z^j, \bar{p}) \in \mathcal{M}\} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathfrak{S}_{2j}(p, U) := \{&(z, \zeta^1, z^1, \ldots, z^{j-1}, \zeta^j) \in U \times {}^*U \times U \times \ldots \times U \times {}^*U \mid \\ &(z, \zeta^1), (z^1, \zeta^1), \ldots, (z^{j-1}, \zeta^{j-1}), (z^1, \zeta^2), \ldots, (z^{j-1}, \zeta^j), (p, \zeta^j) \in \mathcal{M}\} \end{aligned} \tag{3.4}$$

Where $\mathfrak{S}_1(p, U) = \{z \in U \mid (z, \bar{p}) \in \mathcal{M}\}$. If we define $\pi \colon \mathbb{C}^N \times \ldots \times \mathbb{C}^N \to \mathbb{C}^N$ be the projection to the first coordinate, then we can define the *Segre sets* for $p \in U$ by $S_k(p, U) := \pi(\mathfrak{S}_k(p, U))$. Note that both $S_k(p, U)$ and $\mathfrak{S}_k(p, U)$ depend on both the point p and the neighbourhood U.

We have the following proposition, first part is proved in [BER99] (Proposition 10.2.7), second part is then immediate.

Proposition 3.4. *For $k \geq 1$ we have*

$$S_k(p, U) = \bigcup_{q \in S_{k-1}(p, U)} S_1(q, U), \tag{3.5}$$

and if $k \geq 2$ we have

$$S_k(p, U) = \bigcup_{q \in S_{k-2}(p, U)} S_2(q, U). \tag{3.6}$$

Further, for normal coordinates where $U = U_z \times U_w$ we have the following (again proved in [BER99] as part of Proposition 10.4.1):

Proposition 3.5. *Let M be given by $w = Q(z, \bar{z}, \bar{w})$ in normal coordinates in U and let $p = (z^0, w^0)$. Then there exists an open set $V \subset {}^*U_z$ ($0 \in V$) such that $(z, w) \in U$ is in $S_2(p, U)$ if and only if there exist $\zeta \in V$ such that $w = Q(z, \zeta, \bar{Q}(\zeta, z^0, w^0))$.*

The set V above is the set of all $\zeta \in {}^*U_z$ such that $\bar{Q}(\zeta, z^0, w^0) \in {}^*U_w$. In particular $0 \in V$. With this we prove the following useful lemma.

Lemma 3.6. *Suppose that $M \subset U \subset \mathbb{C}^N$ is a generic submanifold given by normal coordinates defined near the origin for a suitable U. Then for any point $p = (z^0, w^0) \in U$, the variety $\{(z, w) \in U \mid w = w^0\}$ is contained inside $S_2(p, U)$ (the second Segre set at p).*

Proof. Let M be given by $\{(z, w) \mid w = Q(z, \bar{z}, \bar{w})\}$ in normal coordinates. Thus $S_2(p, U) = \{(z, w) \mid w = Q(z, \zeta, \bar{Q}(\zeta, \bar{z}^0, \bar{w}^0)), \zeta \in V\}$, where V is as in Lemma 3.5. In particular $0 \in V$ and thus since we are in normal coordinates, $Q(z, 0, w) \equiv Q(0, z, w) \equiv w$. Thus $\{(z, w) \mid w = w^0\} \subset S_2(p, U)$. $\square$

To be able to use this we note the following theorem given and proved in [BER99] (Theorems 10.5.2 and 10.5.4).

Theorem 3.7. *If M is as above, then there exists a number j_0 such that for every sufficiently small neighbourhood U of p where $p \in M$, $S_{2j_0}(p, U)$ coincides with $\mathcal{X}_p$ as germs at p, the complexification of Orb_p.*

The number j_0 is called the Segre number of M at p, but we are only interested in the fact that such a number exists and not how it is arrived at. Another useful proposition from [BER99] (Proposition 10.2.28) is the following.

Proposition 3.8. *Let $p \in M \subset U$ and an integer $k_0 \geq 1$, then there exist neighbourhoods $U'' \subset U' \subset U$ of p such that for all $q \in U''$, $\mathfrak{S}_k(q, U')$ is connected for all $k \leq k_0$.*

Next we assume that U is a polydisc (in the normal coordinates).

Lemma 3.9. *Given $M \subset U$ in normal coordinates, then there is a small neighbourhood of the origin V such that for $p \in M \cap V$, $\mathcal{X}_p$ contains $\{(z,w) \in U \mid w = w^0\}$ as germs at any $(z^0, w^0) \in \mathcal{X}_p$. If $\mathcal{Z}_p$ is the smallest complex-analytic subvariety of U which contains $\mathcal{X}_p$, then $\mathcal{Z}_p$ contains $\{(z,w) \in U \mid w = w^0\}$ for any $(z^0, w^0) \in \mathcal{Z}_p$.*

Proof. Let M be in normal coordinates. We can always take U to be even smaller, so by Proposition 3.8 for a small enough neighbourhood of the origin U, there is a yet smaller neighbourhood of the origin V such that for $p \in V$, $\mathfrak{S}_k(p, U)$ is connected, for $k \leq 2(d+1)+2$, d being the codimension of M. Note that the Segre number of M at any point is always less than or equal to $d+1$. By Theorem 3.7 we know $S_{2(d+1)}(p, W) = \mathcal{X}_p$ as germs for some small neighbourhood W of p. Hence $S_{2(d+1)+2}(p, W) = \mathcal{X}_p = S_{2(d+1)}(p, W)$ as germs at p. Let $k = 2(d+1)$. By Proposition 3.4, $S_{k+2}(p, U)$ is a union of $S_2(q, U)$ for $q \in S_k(p, U)$, and by Lemma 3.6 each $S_2(q, U)$ contains the set $\{(z,w) \mid w = w(q)\}$. In particular $S_{k+2}(p, U)$ contains the set $\{(z,w) \mid w = w(q)\}$ for each $q \in \mathcal{X}_p$ (for some small enough representative of the germ $\mathcal{X}_p$). We note that $\mathfrak{S}_{k+2}(p, W)$ is an open submanifold of $\mathfrak{S}_{k+2}(p, U)$, which is connected. We pull back the mapping $(z, w) \mapsto z$ to $\mathfrak{S}_{k+2}(p, U)$ and look at its rank to conclude that for $q \in S_{k+2}(p, W)$ we have $\{(z,w) \mid w = w(q)\} \subset S_{k+2}(p, W)$ as germs at q. This proves the first part.

To see the second part suppose that $\mathcal{Z}_p$ did depend on z. Then we can intersect $\mathcal{Z}_p$ with $\{(z,w) \mid z = z^0\}$ and the intersection must still contain $\mathcal{X}_p$ projected on the w coordinate (it is of the form $\mathcal{X}_p = \mathbb{C}_z \times (\mathcal{X}_p)_w$). So we would get a different complex variety $\mathcal{Z}_p'$ which contains $\mathcal{X}_p$. Intersection of $\mathcal{Z}_p$ and $\mathcal{Z}_p'$ would violate minimality of $\mathcal{Z}_p$. $\square$

Theorem 3.10. *Suppose that M is a connected generic submanifold given in normal coordinates. Suppose that f and g are two holomorphic functions such that $|f| = |g|$ on M. Then f/g depends only on w. Or if h is a meromorphic function which is real valued on M, then h depends only on w.*

Proof. Obviously we only need to prove the first part as the second part follows. We can work in arbitrarily small neighbourhood U of the origin. As we noted before since Orb_p is minimal in $\mathcal{X}_p$ we know that $f = cg$ in $\mathcal{X}_p$ for any point p (where c

depends on p of course). That is the function f/g is constant on $\mathcal{X}_p$ (if we take p outside the zero set of g). Since we know that as germs $\{(z,w) \mid w = w^0\} \subset \mathcal{X}_p$, then for any $1 \leq j \leq n$ we have $\frac{\partial}{\partial z_j}(f/g) = 0$ at p. Since M is generic and since $g = 0$ is a proper subvariety of M, then this holds for an open set of p in M, and then it holds for an open subset of U and thus for all of U. $\square$

3.2 Submanifolds inside Levi-flat hypersurfaces

Since the question of the modulus uniqueness property of M (or alternatively of existence of a meromorphic function which is real valued on M) is the same as a question of M being contained in a certain kind of possibly singular real-analytic Levi-flat hypersurface, we can ask a weaker question; when is M contained in any possibly singular real-analytic Levi-flat hypersurface? We will consider M to be inside a hypersurface H if $M \subset \overline{H^*}$.

Our main result is the following theorem. It is surprising considering that the normal coordinates of M are not unique. See also Theorem 3.13 for algebraic submanifolds.

Theorem 3.11. *Let M be a germ of a generic real-analytic codimension 2 submanifold through the origin given in normal coordinates (z,w) and let M be nowhere minimal. Then $M \subset \overline{H^*}$, where H is a germ of a possibly singular real-analytic Levi-flat hypersurface if and only if the projection of M onto the second factor in (z,w) is contained in a germ of a possibly singular real-analytic hypersurface. Moreover, if M is not Levi-flat, then H is unique.*

Note that this theorem also gives a test for generic real-analytic submanifolds being nowhere minimal. If we can compute a hypersurface containing the projection of M to the w coordinate, we need only check if it is Levi-flat or not. Let us prove the following useful proposition before proving the theorem.

Proposition 3.12. *Suppose M is a connected generic real-analytic submanifold of codimension 2 in normal coordinates (z,w) and $M \subset \overline{H^*}$ where H is a irreducible possibly singular real-analytic Levi-flat hypersurface. Then in a possibly* *smaller neighbourhood of the origin, there exists a Levi-flat hypersurface $\hat{H}$ defined by $\{(z,w) \mid \rho(w,\bar{w}) = 0\}$ such that $M \subset \overline{\hat{H}^*}$ as germs at 0. Furthermore, if M is not Levi-flat then $H = \hat{H}$ as germs at 0.*

Proof. If Orb_p is constantly of codimension 2 in M or constantly of codimension 1 in M, then by Theorem 3.2 we have a holomorphic function near the origin which is real valued on M and thus by Theorem 3.10 the defining equation for that already does not depend on z.

By Corollary 2.4, M cannot be minimal at any point. So suppose that M is not minimal and Orb_p is not of constant dimension. This means that it is not Levi-flat and thus by Corollary 2.3 it cannot be contained in $H_s \cap \overline{H^*}$ and thus must intersect H^*. This means that it must in fact intersect H^* on a dense open set in M (as H_s is contained in a proper real-analytic subvariety of H). Suppose H is defined in U by $\{\rho(z,w,\bar{z},\bar{w}) = 0\}$, in particular H is closed in U. Then for $p \in M \cap H^*$ we can see that $\mathcal{X}_p \subset H$, since in small enough neighbourhood of p, such as we have by Theorem 3.7, the kth Segre set of M is contained in the kth Segre set of H, and the Segre sets of H all lie in H for small enough neighbourhood of a nonsingular point of H. By Lemma 2.5, the Segre variety of H at p agrees with the Levi foliation of H at p, and since this (the Segre variety of H) is a proper subvariety of U, then if $\mathcal{Z}_p$ is the smallest complex-analytic subvariety of U which contains $\mathcal{X}_p$, then $\mathcal{Z}_p \subset H$. This means in particular that $\big(\mathbb{C}_z \times \pi_w(M \cap H^*)\big) \cap U \subset H$ (where π_w is the projection onto second factor in the normal coordinates (z,w)), since $\mathcal{Z}_p$ contains all the $(z,w) \in U$ for fixed w by Lemma 3.9. As H is closed and $M \cap H^*$ is dense in M, then $\mathbb{C}_z \times \pi_w(M) \subset H$. Fix z^0 such that $\rho(z^0, w, \bar{z}^0, \bar{w}) = 0$ defines a hypersurface in $\mathbb{C}_w$, then this hypersurface is Levi-flat in $\mathbb{C}_w$. Define $\hat{H}$ by $\{(z,w) \mid \rho(z^0,w,\bar{z}^0,\bar{w}) = 0\}$, this is Levi-flat again and further $M \subset \hat{H}$.

It is then clear that since $\mathcal{X}_p \subset H$, then $\mathcal{X}_p \subset \hat{H}$, thus near points p where Orb_p is of codimension 1 in M, these locally give a branch of a nonsingular Levi-flat hypersurface which must be contained in $\hat{H}$, thus $M \subset \overline{\hat{H}^*}$.

If M is not Levi-flat then uniqueness of H comes from the fact that if M would be contained in two different Levi-flat hypersurfaces say H and H' it would be contained in their intersection and thus would be contained in the singular set of

$H \cup H'$ and this is impossible by Corollary 2.3. □

Proof of Theorem 3.11. The forward direction and uniqueness is proved by the preceding proposition. So suppose that $M \subset H$ where $H = \mathbb{C}_z \times H_w$ is a possibly singular hypersurface. We can assume that H is irreducible.

First suppose that Orb_0 is of maximal dimension, then by Theorem 3.2 there exists (near 0) a holomorphic function real valued on M which thus defines a Levi-flat hypersurface (nonsingular one in fact). Also by Theorem 3.10 this function only depends on the w coordinate, this means that it really defines a Levi-flat hypersurface in $\mathbb{C}^d$ (the w variables) and this contains $\pi_w(M)$.

Next suppose that Orb_0 is not of maximal dimension. Fix a certain neighbourhood U where M is defined in the given, fixed, normal coordinates. By Proposition 3.8 we can then pick a smaller $0 \in U' \subset U$ such that for all $p \in U'$, the Segre manifold $\mathfrak{S}_k(q, U)$ is connected. Making U' smaller we can assume it is of the form $U'_z \times U'_w$ where both U'_z and U'_w are polydiscs. We will pick a point $p \in M \cap U'$ where Orb_p is of maximal dimension (of codimension 1 in M).

By choosing U small enough above we can ensure that $\pi_w(M)$ is subanalytic (see section 1.3 or [BM88]). We look at a nonsingular point of this projection of highest dimension in $\pi_w(M) \cap U'_w$. Obviously this is either a hypersurface or codimension 2 point since it is contained in H_w. If $\pi_w(M)$ was a codimension 2 submanifold near some point, then it would be totally real, and thus M above it would be Levi-flat which is not the case. Thus there must be nonsingular points of hypersurface dimension. Further, since the $\mathcal{X}_q$ really only depend on the w variables, it is clear that there is a point $p \in M \cap U'$, such that $\pi_w(p)$ is a nonsingular point of $\pi_w(M) \cap U'_w$, and such that Orb_p is of maximal dimension. Next, pick a small enough neighbourhood $V \subset U'$ of p, such that $\pi_w(M \cap V)$ is a nonsingular hypersurface. Then $\pi_w(M \cap V)$ agrees with one of the branches of H_w at $\pi_w(p)$.

Locally in V (possibly taking smaller V) again we have a holomorphic function f in a neighbourhood of p that is real valued on M. We notice that in the proof of Lemma 3.9 the only reason why we restrict to a smaller neighbourhood is so that we can apply Proposition 3.8, and hence we could have picked a neighbourhood of any point in U. So we see that in the proof of Theorem 3.10 we did not need to pick a neighbourhood of the origin, but we could have just used V as given above (possibly making it smaller). Hence f only depends on w, and thus again $\operatorname{Im} f = 0$ defines a Levi-flat hypersurface near p which contains M near p. So in $\mathbb{C}^d$ (the w coordinates) this hypersurface contains $\pi_w(M \cap V)$ and thus agrees with a branch of H_w near $\pi_w(p)$. By Lemma 1.18, H_w must be a Levi-flat hypersurface, and we are done. $\square$

3.3 Algebraic submanifolds

A submanifold is real-algebraic if it is contained in a real-algebraic variety of the same dimension. Our main result about such submanifolds is the following.

Theorem 3.13. *Let M be a germ of a real-algebraic nowhere minimal generic submanifold of codimension 2. Then there exists a germ of a Levi-flat real-algebraic singular hypersurface H such that $M \subset \overline{H^*}$. Moreover, if M is not Levi-flat, then H is unique.*

Proof. If Orb_p is of constant codimension 2 in M, then we note that since normal coordinates are obtained by implicit function theorem and there exists an algebraic implicit function theorem, then we can find algebraic normal coordinates where M is given by $w = Q(z, \bar{z}, \bar{w})$. See [BER99] for the construction of the normal coordinates. Since Orb_p is of constant dimension 2 in M, it agrees locally with its intrinsic complexification which is then given by keeping w constant. Thus the vector fields $\frac{\partial}{\partial z_k}$ and $\frac{\partial}{\partial \bar{z}_k}$ for all $1 \leq k \leq n$ annihilate the defining equations for M (on M and since M is generic, in a neighbourhood). Thus M is given by $w_1 = Q_1(\bar{w}_1, \bar{w}_2)$ and $w_2 = Q_2(\bar{w}_1, \bar{w}_2)$. From this we can easily construct two algebraic holomorphic functions which are real valued on M, and we are done.

So assume that Orb_p is of codimension 1 in M on an open and dense set. Fix a certain representative of the germ of M. Pick a point $p \in M$ near the origin where Orb_p is of constant dimension 1 in M. Let U be a suitable neighbourhood of p. And let $p \in U' \subset U$ be a smaller neighbourhood such that If $\mathfrak{S}_k(q, U)$ is the kth Segre manifold at $q \in U'$, $\mathfrak{S}_k(q, U)$ is connected. We will call $\mathcal{U}$ the ambient space of $\mathfrak{S}_k(q, U)$, that is the $U \times {}^*U \times U \times \ldots \times {}^*U \times U$ or $U \times {}^*U \times U \times \ldots \times U \times {}^*U$ depending on whether k is even or odd. Then again denote by $\pi\colon \mathbb{C}^N \times \ldots \times \mathbb{C}^N \to \mathbb{C}^N$ the projection onto the first factor, but we will define π on the space $\mathcal{U} \times U'$. Then define

$$\mathfrak{S}_k(M \cap U', U) := \{(\chi, q) \in \mathcal{U} \times U' \mid \chi \in \mathfrak{S}_k(q, U), q \in M \cap U'\}. \tag{3.7}$$

$\mathfrak{S}_k(M \cap U', U)$ is a real-algebraic set in $\mathcal{U} \times U'$ and thus $\pi(\mathfrak{S}_k(M \cap U', U))$ is semialgebraic by Tarski-Seidenberg (see section 1.3 or [BM88]). We know that if U is small enough and k is large enough then $\mathcal{X}_q$ will lie in $\pi(\mathfrak{S}_k(M \cap U', U))$ and further these give a nonsingular Levi-flat hypersurface at that point. Since a semialgebraic set is contained in an algebraic set of the same dimension, that is, there exists a polynomial p defining a hypersurface $H = \{\xi \in \mathbb{C}^N \mid p(\xi, \bar{\xi}) = 0\}$ that contains $\pi(\mathfrak{S}_k(M \cap U', U))$. Since $\pi(\mathfrak{S}_k(M \cap U', U))$ locally agrees with a nonsingular Levi-flat hypersurface we can take H to be irreducible. Then H is Levi-flat at p and by Lemma 1.18 it is Levi-flat.

As germs at p we can see that $M \subset \overline{H^*}$. Further, since this happens at every point where Orb_p is of codimension 1 in M, and these are open and dense in M, then this must happen in some neighbourhood of the origin and hence as germs at the origin. Uniqueness was proved previously already in section 3.2. $\square$

Further properties of real-algebraic generic submanifolds will be discussed in section 4.2.

4 Almost minimal submanifolds

4.1 Almost minimal submanifolds and Levi-flat hypersurfaces

As we have already seen, if $M \subset \overline{H^*}$ and M is a generic nowhere minimal codimension 2 real-analytic submanifold and H is a real-analytic possibly singular Levi-flat hypersurface, then at a point $p \in M \cap H^*$, where Orb_p is of codimension 1 in M, $\mathcal{X}_p \subset H^*$, that is, $\mathcal{X}_p$ gives the Levi foliation of H. By Lemma 2.5, we have that locally the Segre variety Σ_p of H in U contains $\mathcal{X}_p$, and for $p \in H^*$, Σ_p is a proper analytic subvariety of U. So an obvious condition for M to be contained in a Levi-flat hypersurface is that $\mathcal{X}_p$ is contained in a proper subvariety of U. Since $\mathcal{X}_p$ is the smallest germ of a complex variety containing Orb_p, we let $\mathcal{Z}_p = \mathcal{Z}_{U,p}$ be the smallest complex-analytic subvariety of U that contains Orb_p (and thus $\mathcal{X}_p$).

Definition 4.1. Let $M \subset U \subset \mathbb{C}^N$ be a generic submanifold. We will say that M is *almost minimal in* U, if there exists a point p such that $\mathcal{Z}_{U,p}$ contains an open set, and we will say that p makes M almost minimal in U. We will say that a generic submanifold M is *almost minimal at* p, if it is almost minimal in every neighbourhood of p.

If M is minimal at $p \in U$, then it is, of course, almost minimal in U. And if a connected M is real-analytic and minimal at one point, it is minimal on an open dense set, and thus it is almost minimal at every point.

An example of a nowhere minimal submanifold that is almost minimal is the M_λ family given in the introduction for λ irrational. See section 4.4 for this example worked out. It should be noted that if M is nowhere minimal, then the points where it is almost minimal are contained in a proper real-analytic subvariety in M. This is because if M is almost minimal at p and nowhere minimal, then Orb_p must not be of maximal dimension.

Theorem 4.2. *Suppose that $M \subset \mathbb{C}^N$ is a germ of a real-analytic generic submanifold of codimension 2 through 0, and suppose $M \subset \overline{H^*}$ where H is a germ of a possibly singular real-analytic Levi-flat hypersurface, then M is not almost minimal at 0.*

Proof. Let U a small enough connected neighbourhood of the origin such that both M and H are closed in U and further such that their defining equations are complexifiable in U. M cannot be minimal at any point by Theorem 2.4. Further, if M is Levi-flat then Orb_p is constantly of codimension 2 in M. This means that Orb_p is in fact complex-analytic and is contained in the Segre variety (the first Segre set of M in U) and thus cannot be almost minimal.

So suppose on a dense open set of points of M, Orb_p is of codimension 1 in M, and in fact, if p makes M almost minimal in U then Orb_p has to be of codimension 1 in M. Further, $M \cap H^*$ is non-empty (since M is not Levi-flat) and as noted before is thus open and dense in M. Also as noted above, the p that makes M almost minimal cannot lie in $M \cap H^*$.

So pick a small neighbourhood of any $p \in M \cap H_s$ where Orb_p is of codimension 1 in M. Then by Theorem 3.2, there is a small neighbourhood V of p where there exist normal coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^2$ vanishing at p, such that M is given by $\operatorname{Im} w_1 = \rho(z, \bar{z}, \operatorname{Re} w)$ and $\operatorname{Im} w_2 = 0$, and further, that the $\mathcal{X}_q$ are then given by $w_2 = s$ (we will denote this set as $\{w_2 = s\}$) for some $s \in (-\epsilon, \epsilon)$. We can take V to be a polydisc in the (z, w) coordinates. If $M \cap \{w_2 = s\}$ (which is the CR orbit) contains a point which is in H^*, then as we reasoned above $\{w_2 = s\} \subset H$ since it agrees with the Levi foliation of H at some point in H^*. As $M \cap H^*$ is dense in M, then $\{w_2 = s\} \subset H$ for all $s \in (-\epsilon, \epsilon)$. This means that in V, $\operatorname{Im} w_2$ divides the defining function of H in U. Thus the Segre variety of H in U contains the Segre variety of $\{\operatorname{Im} w_2 = 0\}$ at all points in $\{\operatorname{Im} w_2 = 0\}$. We wish to show that Orb_p is contained in a proper complex-analytic subvariety. Either it is contained in a nondegenerate Segre subvariety of H in U or the Segre variety of H in U is degenerate at all points of $\operatorname{Orb}_p = M \cap \{w_2 = 0\}$, but the set of points where the Segre variety of H is degenerate is a proper analytic subset as we remarked before. In any case p does not make M almost minimal in U, and thus M is not almost minimal in U. $\square$

Corollary 4.3. *Suppose that $M \subset \mathbb{C}^N$ is a connected real-analytic generic submanifold of codimension 2 through 0, and M is almost minimal at* 0*, then M has the modulus uniqueness property at* 0*.*

4.2 Algebraic submanifolds

Recall that a real submanifold is real-algebraic if it is contained in a real-algebraic variety of the same dimension.

The following theorem is basically proved in [BER99] (Theorem 13.1.10). It is also easily seen as a direct consequence of Tarski-Seidenberg (see section 1.3 or [BM88]) and of the Chevalley theorem (see for example [Łoj91]). That is, projections of real or complex algebraic varieties are either semi-algebraic (in the real case) or constructible (in the complex case) but in both cases they are contained in a real or complex algebraic variety of the same dimension. And since $\mathcal{X}_p$ is locally given as projection of a Segre manifold which is complex-algebraic if M is real-algebraic, we have the following.

Theorem 4.4. *Let $M \subset \mathbb{C}^N$ be a real-algebraic generic submanifold and $p \in M$, then Orb_p is real-algebraic and similarly $\mathcal{X}_p$ is contained in a complex algebraic variety of the same dimension.*

So Orb_p is contained in a variety in $\mathbb{C}^N$ and thus cannot make M almost minimal for any U. We therefore have the following result.

Corollary 4.5. *Suppose $M \subset \mathbb{C}^N$ is a connected real-algebraic generic submanifold, then M is almost minimal at $p \in M$ if and only if M is minimal at some point.*

As a consequence we have a test for a submanifold being real-algebraic.

Corollary 4.6. *Let $M \subset \mathbb{C}^N$ be nowhere minimal real-analytic generic submanifold which is almost minimal at $p \in M$. Then M is not biholomorphic to a real-algebraic generic submanifold.*

This is because almost minimality would be preserved under biholomorphisms. The M_λ for λ irrational defined in the introduction is therefore an example of a submanifold not biholomorphic to a real-algebraic one.

4.3 Infinitesimal CR automorphisms

We will now look at the dimension of $\operatorname{hol}(M,p)$, the space of infinitesimal holomorphisms at p if M is almost minimal at p.

The space of *infinitesimal holomorphisms* at p is the Lie algebra generated by germs at p of real-analytic vector fields X on M defined in some neighbourhood U of p, such that for each $q \in U$ there is another neighbourhood $q \in V \subset U$ and the mapping $z \mapsto \exp tX \cdot z$ for $|t| \leq \epsilon$ is a CR diffeomorphism of $M \cap V$ (a diffeomorphism that preserves the CR vector bundle of M).

A vector field X in $\mathbb{C}^N$ is called a *holomorphic vector field*, if we can write it locally as $X = \sum_{k=1}^N a_k(z) \frac{\partial}{\partial z_k}$, where the a_k are holomorphic in $z \in \mathbb{C}^N$. A submanifold M is said to be *holomorphically nondegenerate* at $p \in M$, if there does not exist any germ at p of a nonzero holomorphic vector field tangent to M. If M is connected, real-analytic and generic it turns out, that if it is holomorphically nondegenerate at one point it is so at all points. Being holomorphically nondegenerate is a necessary condition for $\dim_{\mathbb{R}} \operatorname{hol}(M,p) < \infty$. In the case M is a hypersurface Stanton [Sta96] proved that this is in fact a sufficient condition. For higher codimension submanifolds, Baouendi, Ebenfelt and Rothschild [BER98] proved Theorem 4.7 below.

Theorem 4.7 (Baouendi-Ebenfelt-Rothschild see [BER98])**.** *Let $M \subset \mathbb{C}^N$ be a connected real-analytic CR submanifold that is holomorphically nondegenerate. If M is minimal at any point $p \in M$, then* $\dim_{\mathbb{R}} \operatorname{hol}(M,q) < \infty$ *for all $q \in M$. If M is nowhere minimal then* $\dim_{\mathbb{R}} \operatorname{hol}(M,q) = 0$ *or* $\dim_{\mathbb{R}} \operatorname{hol}(M,q) = \infty$ *for q in a dense open subset of M.*

Thus it remains to see at exactly what points is $\operatorname{hol}(M, q)$ finite dimensional in case M is nowhere minimal. Our main result of this section is that it turns out that the points where M is almost minimal are such points. We restate Theorem 4.3 from the introduction for convenience.

Theorem. *Let $M \subset \mathbb{C}^N$ be a connected, real-analytic holomorphically nondegenerate generic submanifold and suppose $p \in M$ and M is almost minimal at p. Then*

$$\dim_{\mathbb{R}} \operatorname{hol}(M, p) < \infty. \tag{4.1}$$

The proof is essentially the same as in [BER98] or [BER99] for minimal submanifolds, although we will require Lemma 4.9 to modify this proof. It would not be needed, if we had a more general way of showing that certain CR orbits (of the highest dimension for example) were holomorphically nondegenerate whenever M was. Alas, this is not so. For example, the manifold defined in $(z_1, z_2, w_1, w_2) \in \mathbb{C}^4$, by

$$\begin{aligned} \operatorname{Im} w_1 &= |z_1|^2 + (\operatorname{Re} w_2)\, |z_2|^2 , \\ \operatorname{Im} w_2 &= 0, \end{aligned} \tag{4.2}$$

is holomorphically nondegenerate. The CR orbit at 0 is defined by $\operatorname{Im} w_1 = |z_1|^2$ and $w_2 = 0$, and so $\frac{\partial}{\partial z_2}$ is a holomorphic vector field tangent to it. We can, however, prove the following result for almost minimal submanifolds.

Lemma 4.8. *Suppose $M \subset U \subset \mathbb{C}^N$ is a holomorphically nondegenerate generic submanifold, and $p \in M$ is such that $\mathcal{Z}_{U,p} = U$, that is, p makes M almost minimal in U. Then Orb_p is holomorphically nondegenerate.*

The proof is essentially contained the proof of Theorem 4.3 below, and uses the following technical lemma.

Lemma 4.9. *Let $M \subset U_z \times U_w \subset \mathbb{C}^N$ be a generic submanifold given in normal coordinates (z, w) in a sufficiently small $U = U_z \times U_w$ by $w = Q(z, \bar{z}, \bar{w})$. Suppose there exists holomorphic function $f \colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \to \mathbb{C}$ defined in a neighbourhood of the origin such that $f(z, \bar{z}, w)$ is defined in U, there exists a point $p \in M$ and*

$f(z, \bar{z}, w) = 0$ *on* Orb_p. *Then there exists a holomorphic function* $g\colon U_w \subset \mathbb{C}^d \to \mathbb{C}$ *such that* $g(w) = 0$ *on* Orb_p.

Proof. First note that Lemma 3.9 implies that locally near p, we can find a germ of a holomorphic function φ such that $\varphi(w) = 0$ defines $\mathcal{X}_p$. Thus we can do a local change of coordinates in w only, setting $w' = \psi(w)$ and $w'' = \varphi(w)$ for some function ψ. So locally we have Orb_p defined in the coordinates z, w', w'' (which are no longer normal coordinates) by $w' = \tilde{Q}(z, \bar{z}, \bar{w}')$ and $w'' = 0$, for some function $\tilde{Q}$ defined in a neighbourhood of p. We can now also write f in the z, w', w'' coordinates by abuse of notation as $f(z, \bar{z}, w', w'')$. Assuming U is small enough and the neighbourhood where w', w'' are defined is also small enough we can define a complexified version of Orb_p by setting $\bar{w}' = \xi$ and $\bar{z} = \zeta$ by $\xi = \bar{\tilde{Q}}(\zeta, z, w')$ and call this $\mathcal{C}$. Since $f(z, \bar{z}, w', 0) = 0$ on Orb_p, then as Orb_p is maximally real in $\mathcal{C}$ we have that $f(z, \zeta, w', 0) = 0$ on $\mathcal{C}$ and as z, ζ, w' are free variables on $\mathcal{C}$ we know that $f(z, \bar{z}, w', w'') = 0$ when $w'' = 0$, but $w'' = 0$ defines $\mathcal{X}_p$, so f is identically zero on all of $\mathcal{X}_p$. Since $\mathcal{X}_p$ is defined by an equation which is independent of z, then if we fix z^0 where $(z^0, w^0) = p \in M$, and we take $g(w) := f(z^0, \bar{z}^0, w)$, then $g(w)$ as a function of (z, w) but independent of z is zero on $\mathcal{X}_p$ and thus on Orb_p. And g is defined in all of U_w and thus we are done. □

We need to characterize $\mathrm{hol}(M, p)$ in a more natural way for the proof and the following proposition is proved in [BER99] (Proposition 12.4.22).

Proposition 4.10. *Let* $M \subset \mathbb{C}^N$ *be a real-analytic generic submanifold,* $p \in M$*, and* X *a germ at* p *of a real, real-analytic vector field on* M*, then* $X \in \mathrm{hol}(M, p)$ *if and only if there exists a germ* $\mathcal{X}$ *at* p *of a holomorphic vector field in* $\mathbb{C}^N$ *such that* $\operatorname{Re} \mathcal{X}$ *is tangent to* M *and* $X = \operatorname{Re} \mathcal{X}|_M$.

It is not hard to see that if $\mathcal{X}$ is a holomorphic vector field as above, $\tilde{\varphi}(z, \tau)$ is the holomorphic flow of $\mathcal{X}$ and $X = \operatorname{Re} \mathcal{X}$, and $\varphi(z, \bar{z}, t)$ is the flow of X, then φ and $\tilde{\varphi}$ coincide when $t = \tau \in \mathbb{R}$.

We will need the notion of k-nondegeneracy, but instead of giving the definition of being k-nondegenerate at a point, we can just take the following proposition from [BER99] (Corollary 11.2.14) and treat it as a definition.

Proposition 4.11. *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold of codimension d and CR dimension n given in normal coordinates $Z = (z, w) \subset U \subset \mathbb{C}^n \times \mathbb{C}^d$ by $w = Q(z, \bar{z}, \bar{w})$. Then M is k-degenerate at $p = (z_p, w_p)$ (sufficiently close to 0) if and only if*

$$\operatorname{span}\left\{ \left(\frac{\partial}{\partial \bar{z}}\right)^{\alpha} \frac{\partial \bar{Q}_j}{\partial Z}(\bar{z}_p, z_p, w_p) \mid j = 1, \ldots, d, 0 \leq |\alpha| \leq k \right\} = \mathbb{C}^N. \tag{4.3}$$

We must prove a result about finite jet determination of biholomorphisms of almost minimal submanifolds, which may be of interest on its own. As before let $\mathcal{Z}_{U,p}$ be the smallest complex-analytic variety containing Orb_p. So if M is almost minimal in U and p is the point that makes it almost minimal then we have the following proposition.

Proposition 4.12. *Let $M, M' \subset \mathbb{C}^N$ be real-analytic generic submanifolds of codimension d defined in open sets U and U' respectively. Let f and g be two holomorphic mappings taking U to U' and M to M'. Let $p \in M$ be such that $\mathcal{Z}_{U,p} = U$ and suppose M is k_0-nondegenerate at p. Also suppose that $f(p) = g(p) = p'$, $f_*(T_p^c M) = T_{p'}^c M$ and $g_*(T_p^c M) = T_{p'}^c M$. Then if $j_p^{(d+1)k_0} f = j_p^{(d+1)k_0} g$ then $f = g$.*

Proof. Follows by Corollary 12.3.8 in [BER99] which is a slightly stronger result than the above, but which says that $f = g$ in Orb_p only. As $\mathcal{Z}_{U,p} = U$, then of course $f = g$ everywhere on U. □

To be able to use Proposition 4.12 we need to know that M is k-nondegenerate at the right points. From [BER99] we have the following lemma (part of Theorem 11.5.1).

Lemma 4.13. *Suppose $M \subset \mathbb{C}^N$ is a connected real-analytic generic submanifold of CR dimension n that is holomorphically nondegenerate. Then there exists a proper real-analytic subvariety $V \subset M$ such that M is ℓ-nondegenerate for all $p \in M \backslash V$ for some $1 \leq \ell \leq n$.*

The $\ell = \ell(M)$ is the *Levi-number* of M.

Proof of Theorem 4.3. First suppose that $X^1, \ldots, X^m \in \mathrm{hol}(M, p)$ are linearly independent over $\mathbb{R}$. Suppose that $x = (x_1, \ldots, x_r)$ be local coordinates for M

vanishing at p. Here we may write $X^j = \sum_{k=1}^r X_k^j(x)\frac{\partial}{\partial x_k}$, or for short $X^j \cdot \frac{\partial}{\partial x}$. We let $y \in \mathbb{R}^m$ and denote by $\varphi(t,x,y)$ the flow of the vector field $y_1X^1 + \cdots + y_mX^m$, that is the solution of

$$\frac{\partial \varphi}{\partial t}(t,x,y) = \sum_{j=1}^m y_j X^j(\varphi(t,x,y)), \tag{4.4}$$

$$\varphi(0,x,y) = x. \tag{4.5}$$

Since $\varphi(st,x,y) = \varphi(t,x,sy)$ (which follows from the uniqueness of the solution), we can choose $\delta > 0$ small enough such that there exists $c > 0$ such that the flow is smooth for (t,x,y) where $|t| \leq 2$, $|x| \leq c$ and $|y| \leq \delta$. We look at the time-one mappings denoted by

$$F(x,y) := \varphi(1,x,y). \tag{4.6}$$

We have the following lemma proved in [BER98] and [BER99] (Lemma 12.5.10).

Lemma 4.14. *There exists $\gamma > 0$ such that $\gamma < \delta$ such that for any fixed $y^1, y^2 \in \mathbb{R}^m$ where $|y^j| \leq \gamma$, $j = 1,2$, if $F(x,y^1) \equiv F(x,y^2)$ for $|x| \leq c$ then necessarily $y^1 = y^2$.*

Suppose that X^j are as above and are in $\operatorname{hol}(M,p)$. Denote by $V \subset M$ the neighbourhood of p given by $|x| < c$ where x and c are as above. Let $\gamma > 0$ be picked as in Lemma 4.14. From Proposition 4.10 (and discussion afterward) it follows that for a fixed y such that $|y| < \gamma$ there exists a biholomorphism $z \mapsto \tilde{F}(z,y)$ defined in some connected open neighbourhood $U \subset \mathbb{C}^N$ of $V \subset M$ taking M into M (we can take γ smaller if necessary) and if $z(x)$ is the parametrization of M near p these satisfy $F(x,y) = \tilde{F}(z(x),y)$ where F are the time-one mappings defined above.

As M is holomorphically nondegenerate, then by Lemma 4.13 we have that outside a real-analytic set it is ℓ-nondegenerate. Note that by Proposition 4.11 we have that this set is actually contained in a set defined by the vanishing of a function of the form $\varphi(z,\bar{z},w)$, that is a real-analytic function in z, but holomorphic in w. Since M is almost minimal at p and if $\mathcal{Z}_{U,q} = U$, then $\mathcal{Z}_{U,q'} = U$ for all $q' \in \mathrm{Orb}_q$, we know by Lemma 4.9 that there must exist a $q \in M$ such that $\mathcal{Z}_{U,q} = U$ and M is ℓ-nondegenerate at q.

We have satisfied requirements of Proposition 4.12, and by applying Lemma 4.14 we see that we have an injective mapping

$$y \mapsto j_q^{(d+1)\ell} \tilde{F}(\cdot, y) \in J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q, \tag{4.7}$$

where $J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$ is the jet space at q of germs of holomorphic mappings from $\mathbb{C}^N$ to $\mathbb{C}^N$. As $J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$ is finite dimensional, then obviously $m \leq \dim_{\mathbb{R}} J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$, thus $\dim_{\mathbb{R}} \operatorname{hol}(M, p) \leq \dim_{\mathbb{R}} J^{(d+1)\ell}(\mathbb{C}^N, \mathbb{C}^N)_q$. $\square$

4.4 Example

Let M_λ, $\lambda \in \mathbb{R}$, be the generic, nowhere minimal submanifold of $\mathbb{C}^3$, with holomorphic coordinates (z, w_1, w_2) defined by

$$\begin{aligned} \bar{w}_1 &= e^{iz\bar{z}} w_1, \\ \bar{w}_2 &= e^{i\lambda z\bar{z}} w_2. \end{aligned} \tag{4.8}$$

As we will show below M_λ is almost minimal at 0 when λ is irrational. Hence as we have argued in the introduction, by the results of chapter 3 and chapter 4, M_λ cannot be locally biholomorphically equivalent to a generic real-algebraic submanifold.

We wish to classify the λ's for which M_λ has the modulus uniqueness property at the origin. That is, we will wish to find out when does there exist a nontrivial meromorphic function which is real valued on M_λ. Note that we can always find a multi-valued function which is real valued on M_λ, and that is

$$(z, w_1, w_2) \mapsto \frac{w_1^\lambda}{w_2}. \tag{4.9}$$

In fact, this proves that M_λ is nowhere minimal. Further, if λ is rational, say $\lambda = a/b$, then $(z, w_1, w_2) \mapsto w_1^a / w_2^b$ is a meromorphic function that is real valued on M. Thus M_λ does not have the modulus uniqueness property, and further, since it is of codimension 2, it is not almost minimal at the origin.

Let us check that M_λ is almost minimal at 0 when λ is irrational. For this we need to compute the Segre sets. We can compute the third Segre set at (z^0, w_1^0, w_2^0), where $w_1^0 \neq 0$ and $w_2^0 \neq 0$, by the following mapping (see [BER99])

$$(t_1, t_2, t_3) \mapsto (t_3, \overline{w_1^0} e^{i(t_3 t_2 - t_2 t_1 + t_1 \overline{z^0})}, \overline{w_2^0} e^{i\lambda(t_3 t_2 - t_2 t_1 + t_1 \overline{z^0})}). \tag{4.10}$$

We can pick t_3 to be anything we want, and we can pick t_2 and t_1 such that the second component is anything we want since w_1^0 is non zero. By adding multiples of 2π, we can add a dense set of rotations of the third component because λ is irrational. This means, that the closure of this set will be 5-dimensional, and thus we will not be able to fit it inside a proper complex-analytic subset and so M_λ is almost minimal.

We give an alternative more direct proof that M_λ does not have the modulus uniqueness property at the origin, and in fact prove a slightly more general theorem that can be used for generating further examples.

Proposition 4.15. *Suppose that M is a real-analytic, generic submanifold of codimension d inside $\mathbb{C}^{n+d}$ passing through the origin that can be defined by normal coordinates of the form*

$$w_j = Q_j(z, \bar{z}) \bar{w}_j \tag{4.11}$$

and further suppose that for any integer K the functions $Q_1^{k_1} \cdot Q_2^{k_2} \cdot \ldots \cdot Q_d^{k_d}$ for $0 \leq k_1, \ldots, k_d \leq K$ are linearly independent as functions. Then there does not exist a non-constant meromorphic (nor a holomorphic) function h defined in a neighbourhood of 0 which is real valued on M.

Proof. For easier notation we will assume $n = 1$ and $d = 2$. So suppose that $h = f/g$ is real valued on M, meaning that on M we have $f\bar{g} - \bar{f}g = 0$. We have proved before that h does not depend on z. Suppose that f and g are defined by Taylor series expansions about 0. Thus

$$f(w_1, w_2) = \sum_{k,l \geq 0} f_{kl} w_1^k w_2^l, \tag{4.12}$$

$$g(w_1, w_2) = \sum_{n,p \geq 0} g_{np} w_1^n w_2^p. \tag{4.13}$$

On M we therefore have (as $\bar{w}_i = \bar{Q}_i w_i$)

$$
\begin{aligned}
0 &= f\bar{g} - \bar{f}g \\
&= \left(\sum_{k,l,n,p \geq 0} f_{kl}\bar{g}_{np} w_1^k w_2^l \bar{w}_1^n \bar{w}_2^p \right) - \left(\sum_{k,l,n,p \geq 0} \bar{f}_{kl} g_{np} w_1^n w_2^p \bar{w}_1^k \bar{w}_2^l \right) \\
&= \sum_{s,t \geq 0} \left(\sum_{k+n=s,\ l+p=t} f_{kl}\bar{g}_{np} (\bar{Q}_1)^p (\bar{Q}_2)^n - \bar{f}_{kl} g_{np} (\bar{Q}_1)^l (\bar{Q}_2)^k \right) w_1^s w_2^t.
\end{aligned}
\tag{4.14}
$$

For a fixed z we have a holomorphic function in w_1 and w_2 that is 0 on a generic manifold (restriction of M_λ to the (w_1, w_2) space) and is thus identically zero. This means that each coefficient is 0, and since by assumption these are linear combinations of powers of Q_1 and Q_2, we get

$$
f_{(s-k)(t-l)}\bar{g}_{kl} - \bar{f}_{kl} g_{(s-k)(t-l)} = 0. \tag{4.15}
$$

The above is true for all $s, t \geq 0$ and all $k \leq s, l \leq t$. This implies that either $g \equiv 0$ or that $f_{kl} = C g_{kl}$ for all k, l for some constant C. Meaning there is no nonconstant meromorphic function which is real valued on M. $\square$

5 Boundaries of Levi-flat hypersurfaces

5.1 Main results

The question we wish to ask is when is a generic codimension 2 submanifold $M \subset \mathbb{C}^N$ locally the boundary of a Levi-flat hypersurface H. In particular, we will ask the following. When does H extend as a Levi-flat hypersurface past M? When is H unique? How does the regularity of H depend on the regularity of M? We will answer these questions fully when M is real-analytic and H is smooth. As we said before, the results here are motivated by Dolbeault, Tomassini and Zaitsev [DTZ], who consider the global situation under additional assumptions on M.

A set $H \subset \mathbb{C}^N$ is a C^k *hypersurface with boundary*, if there is a subset $\partial H \subset H$, such that $\partial H \subset H$, $H \setminus \partial H$ is a C^k hypersurface (submanifold of codimension 1), and for each point $p \in \partial H$, there exists a neighbourhood $p \in U \subset \mathbb{C}^N$, a C^k diffeomorphism $\varphi \colon U \to \mathbb{R}^{2N}$, such that $\varphi(H \cap U) = \{x \in \mathbb{R}^{2N} \mid x_{2N-1} \geq 0,\ x_{2N} = 0\}$, and such that $\varphi(\partial H \cap U) = \{x \in \mathbb{R}^{2N} \mid x_{2N-1} = 0,\ x_{2N} = 0\}$. Hence, ∂H is a C^k submanifold of codimension 2. We will call $H^o := H \setminus \partial H$ the interior of H. As we are concerned with only local questions, we can assume that there exists just one such U and such that $\partial H, H \subset U$. We can further assume that ∂H and H are closed subsets of U. We can extend H to $\tilde{H}$, a full C^k submanifold without boundary near 0, by just pulling back a neighbourhood of $0 \in \mathbb{R}^{2N}$ by φ.

A C^k ($k \geq 2$) hypersurface H is said to be *Levi-flat* if the bundle T^cH is involutive. An equivalent definition is to say that near every point of H, there exists a one parameter local foliation of H by complex hypersurfaces, which is called the *Levi foliation*. To see why these are equivalent, note that if T^cH is involutive the Frobenius theorem gives us a C^{k-1} foliation with the leaves being complex hypersurfaces (they are locally the graphs of holomorphic functions). If H is a hypersurface with boundary as defined above, then we will say it is Levi-flat when H^o is Levi-flat. If H is a real-analytic subvariety of codimension 1, then we say it is Levi-flat, if it is Levi-flat as a submanifold at all the nonsingular points. We can now state our main result, which we will prove in section 5.2.

Theorem 5.1. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic generic submanifold of codimension 2 through the origin, such that not all local CR orbits of M are of codimension 2. Suppose that there exists a connected Levi-flat C^∞ hypersurface H with boundary, where $M \subset \partial H$. Then there exists a neighbourhood U of the origin and a nonsingular real-analytic Levi-flat hypersurface $\mathcal{H}$ such that $H \cap U \subset \mathcal{H}$.*

Further, the germ $(\mathcal{H}, 0)$ is unique in the sense that if $(\mathcal{H}', 0)$ is a germ of an irreducible real-analytic Levi-flat subvariety of codimension 1 such that $(M, 0) \subset (\mathcal{H}', 0)$, then $(\mathcal{H}', 0) = (\mathcal{H}, 0)$.

First, note that the condition that M is real-analytic is necessary for the extension to hold. See Example 5.13 in section 5.4 for a counterexample in case M is C^∞.

The condition on the local CR orbits is necessary for the conclusion that the extension $\mathcal{H}$ is unique and real-analytic. If M is the boundary of a Levi-flat hypersurface, then all local CR orbits must be of positive codimension, see Lemma 5.5. If all the local CR orbits are of codimension 1, then the theorem follows easily by known results, see Lemma 5.7. Finally, if all local CR orbits would be of codimension 2, then $M \subset \mathbb{C}^N$ would be locally biholomorphic to $\mathbb{C}^{N-2} \times \mathbb{R}^2$, and we will give (Example 5.11 in section 5.4) an example of a bona fide C^∞ (i.e. not contained in a real-analytic subvariety) Levi-flat hypersurface which contains such an M. Hence the theorem is, in this respect, optimal. In section 5.3, we will prove the following weaker extension theorem for such submanifolds, which is also optimal in view of the above examples. In the sequel, when we consider $\mathbb{C}^{N-2} \times \mathbb{R}^2$ as a subset of $\mathbb{C}^N$, we mean the natural embedding.

Theorem 5.2. *Suppose $H \subset \mathbb{C}^N$ is a C^∞ Levi-flat hypersurface with boundary, and $0 \in \partial H \subset \mathbb{C}^{N-2} \times \mathbb{R}^2$. Then for some neighbourhood U of the origin, there exists a C^∞ Levi-flat hypersurface $\mathcal{H}$ (without boundary) such that $H \cap U \subset \mathcal{H}$.*

Further, the germ $(\mathcal{H}, 0)$ is unique in the sense that if $(\mathcal{H}', 0)$ is another a germ of a C^∞ Levi-flat hypersurface such that $(H, 0) \subset (\mathcal{H}', 0)$, then $(\mathcal{H}', 0) = (\mathcal{H}, 0)$.

Note that the uniqueness in Theorem 5.2 is much weaker as $\mathcal{H}$ depends on H, whereas in Theorem 5.1 $\mathcal{H}$ depends only on M.

Theorem 5.1 says that in particular, there exists a holomorphic function defined near the origin with nonzero gradient that is real valued on M. In other words, M is locally the boundary of a Levi-flat C^∞ hypersurface if and only if M has local defining functions in $(z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2$ of the form:

$$\begin{aligned} \operatorname{Im} w_1 &= \varphi(z, \bar{z}, \operatorname{Re} w), \\ \operatorname{Im} w_2 &= 0, \end{aligned} \tag{5.1}$$

for some φ such that $\varphi(0, \bar{z}, s) \equiv \varphi(z, 0, s) \equiv 0$ (i.e. these are normal coordinates). The classification of Levi-flat boundaries that are generic and real-analytic is therefore simple.

Corollary 5.3. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic generic submanifold of codimension 2 through the origin. The following are equivalent:*

(i) There exists a Levi-flat C^∞ hypersurface H with boundary, such that $0 \in \partial H \subset M$.

(ii) There exists a real-analytic Levi-flat hypersurface (submanifold) H defined in a neighbourhood U of the origin such that $M \cap U \subset H$.

(iii) There exist local holomorphic coordinates (near the origin) such that M is defined by an equation of the form (5.1)*.*

(iv) There exists a real-analytic foliation of codimension 1 in M, defined in a neighbourhood of the origin, such that the leaves are unions of (representatives of) local CR orbits of M.

When the Levi-flat hypersurface is only C^2 rather than smooth, then we will be able to prove that the individual leaves of the Levi foliation extend across M. See Lemma 5.14. However, we will not be able to ensure that the extended leaves are nonsingular, nor that their union is a nonsingular hypersurface. As an application of this lemma we prove the following theorem in section 5.5.

Theorem 5.4. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic generic submanifold of codimension 2 through the origin, which is almost minimal at the origin. Let H be a connected C^2 hypersurface with boundary and $M \subset \partial H$. Then H is not Levi-flat.*

If H would be C^∞ then the above result follows at once from Theorem 5.1. Further, not being almost minimal is a necessary, but not sufficient, condition to being a boundary of a C^2 Levi-flat hypersurface.

5.2 Locally flat boundaries

We prove some basic results about locally flat boundaries, before we prove theorem 5.1. For the rest of this section, we assume that H is a hypersurface with boundary, that $M = \partial H$, and that M is a generic submanifold through the origin.

Lemma 5.5. *Let M be C^∞ and H be C^2, and suppose that H is Levi-flat, then M is nowhere minimal.*

Proof. We can just extend H to $\tilde{H}$ as in the introduction and assume ρ is a defining function for $\tilde{H}$. Then $\theta = i(\partial\rho - \bar{\partial}\rho)$ is a real C^1 one-form that vanishes on T^cH. On H, as H is Levi-flat, $d\theta \wedge \theta = 0$ and by continuity this happens on M as well if we restrict θ to M. θ cannot vanish on M as that would make M have a complex tangency (it would be tangent to $T^c\tilde{H}$). Hence there exists (locally at near every point) a foliation of M by CR submanifolds of smaller dimension with the same CR dimension as M, and so M cannot be minimal at any point. □

Lemma 5.6. *Let M and H be C^k $(2 \leq k \leq \infty)$, and suppose that H is Levi-flat, then the Levi foliation of H^o extends to a foliation of H. That is, in a perhaps a smaller neighbourhood of the origin, there exists a C^{k-1}, real valued, function*

f on H (including M) with nonvanishing differential ($f|_M$ also has nonvanishing differential), such that f is constant along leaves of the Levi foliation of H^o. If M and H are C^∞, then f is C^∞.

Proof. If $k = \infty$, then by C^{k-1} we will mean C^∞ below. For convenience we change notation slightly. We straighten out the boundary, and assume H is the upper half plane $\{x \in \mathbb{R}^n \mid x_1 \geq 0\}$ and M is defined by $x_1 = 0$ (where $n = 2N - 1$). The C^{k-1} 1-form that vanishes on the vectors in $T_p^c H$ induces a C^{k-1} 1-form θ on the upper half plane in $\mathbb{R}^n$, and does not vanish on the tangent vectors to $x_1 = 0$ (else M would have a complex tangency). We can easily extend θ to all of $\mathbb{R}^n$ (or at least a neighbourhood of the origin) as a C^{k-1} 1-form. We now follow the proof of the Frobenius theorem in [Fla89], to show that there exists a real valued function with nonvanishing differential at 0 that is constant on the Levi foliation of H^o. That is, we just need to show that we can modify θ on the set $x_1 < 0$, such that the modification is completely integrable. We have that $d\theta \wedge \theta = 0$ for $x_1 \geq 0$. It is not hard to see that there exists a C^{k-2} 1-form α defined near the origin such that $d\theta = \theta \wedge \alpha$ for $x_1 \geq 0$.

As θ does not vanish near the origin (and does not vanish identically on $T_0 M$), we may assume that $\theta = dx_n + \sum_{j=1}^{n-1} A_j dx_j$. Fix a point a in x' space, where $x' = (x_1, \ldots, x_{n-1})$. We consider the equation $\theta = 0$ on the hyperplane $x_j = a_j t$ for $t \in \mathbb{R}$. We solve this ODE for x_n, with the initial condition $x_n(0) = c$, for some constant c. That is, we find the unique solution of

$$\begin{aligned} &\frac{\partial F}{\partial t}(t, a, c) = -\sum_j A_j(at, F(t, a, c))a_j, \\ &F(0, a, c) = c. \end{aligned} \tag{5.2}$$

We note that we can change scale $F(t, a, c) = F(kt, a/k, c)$, and hence setting $k = 1/t$, we get $F(t, a, c) = F(1, ta, c)$. We change variables to $(u, v) \in \mathbb{R}^{n-1} \times \mathbb{R}$ by

$$\begin{aligned} &x' = u, \\ &x_n = F(1, u, v). \end{aligned} \tag{5.3}$$

It is not hard to check that this is a change of coordinates. In these new coordinates we write

$$\theta = \sum_j P_j du_j + B dv. \tag{5.4}$$

We define

$$\tilde{\theta} = B dv. \tag{5.5}$$

If we show that the P_j vanish for $x_1 \geq 0$ ($u_1 \geq 0$), then we are done. We know that $\sum P_j(ta, v)a_j = 0$. This implies that if we consider the mapping $\varphi(t, a, v) := (ta, v)$, we get

$$\varphi^*\theta = \sum \tilde{P}_j(t, a, v) da_j + \tilde{B}(t, a, v) dv. \tag{5.6}$$

In particular, $\varphi^*\theta$ does not depend on dt. Further $\tilde{P}_j(t, a, v) = tP_j(ta, v)$ so $\tilde{P}_j(0, a, v) = 0$. Now suppose that $a_1 \geq 0$ and $t \geq 0$, then we have that $d(\varphi^*\theta) = (\varphi^*\alpha) \wedge (\varphi^*\theta)$. We set D so that $\varphi^*\alpha = D(t, a, v)dt + \ldots$. From this equation we obtain

$$\frac{\partial \tilde{P}_j}{\partial t} = D\tilde{P}_j. \tag{5.7}$$

By the uniqueness theorem for ODEs and the fact that $\tilde{P}_j(0, a, v) = 0$ this implies that $\tilde{P}_j$ is identically zero, and hence P_j is identically zero. This was true for $a_1 \geq 0$ ($t \geq 0$) and hence on the upper half plane and hence on H. We therefore have $\theta = \tilde{\theta}$ on the upper half plane and $\tilde{\theta}$ is closed and thus exact. We get our f of class C^{k-1} (or C^∞ if $k = \infty$) by Poincaré Lemma. □

Lemma 5.7. *Let M be real-analytic and H be C^2, and suppose that the local CR orbits of M are all of codimension 1. Then there exists a neighbourhood U of the origin such that $(U \cap H) \subset \mathcal{H}$, where $\mathcal{H}$ is the unique Levi-flat real-analytic hypersurface in U that contains M.*

Note that $\mathcal{H}$ is the union of the intrinsic complexifications of the local CR orbits of M. Where the intrinsic complexification is the smallest complex submanifold containing the local CR orbit.

Proof. Since M is real-analytic and the local CR orbits are all of codimension 1, we can therefore apply the analytic Frobenius theorem to get a real-analytic real valued function on some small neighbourhood U of the origin in M with nonvanishing differential that is constant along the local CR orbits of M. Such a function is CR and hence extends to be holomorphic and the vanishing of its imaginary part defines a Levi-flat hypersurface $\mathcal{H}$.

Assume that $H \subset U$. We must show that $H \subset \mathcal{H}$. By Lemma 5.6, we have that the Levi foliation of H^o extends to M (by perhaps making U smaller still). That is, we have complex submanifolds of $\mathbb{C}^n$ with boundary on M. It is not hard to see by the arguments used above that a leaf $L \subset H$ extended to the boundary intersects M precisely on a local CR orbit (by dimension). The function that defines the corresponding leaf of the Levi foliation of $\mathcal{H}$ is of course holomorphic on L and zero on the boundary of L, hence $L \subset \mathcal{H}$, and so $H \subset \mathcal{H}$. □

Proof of Theorem 5.1. Let L_k, $k = 1, \ldots, 2N - 4$, be a basis of real-analytic vectorfields spanning $T_p^c M$ defined near the origin. As not all local CR orbits are of codimension 2, then there must exist an iterated commutator K of the L_k, which is not identically zero. As M is nowhere minimal (by Lemma 5.5), then by dimension, K together with L_k span the tangent space of the CR orbit whenever K is nonzero.

By Lemma 5.6 we have a C^∞ codimension 1 foliation on M. Hence, by forgetting for a moment the CR structure of M, we can reduce to a situation where we have a C^∞ codimension 1 foliation on a small neighbourhood $U \subset \mathbb{R}^{2N-2}$, given by a C^∞ submersion $\varphi \colon U \to \mathbb{R}$, and real-analytic vector fields L_k and K, which are tangent to the leaves of the foliation, L_k never vanish and K does not vanish identically. To see that the foliation must be real-analytic, we only need to look at TU, the tangent bundle of U, and look at the normal bundle of the foliation:

$$\{(x, v) \in U \times \mathbb{R}^{2N-2} = TU \mid \nabla\varphi(x) = tv, t \in \mathbb{R}\}, \tag{5.8}$$

which is a C^∞ submanifold of dimension $2N - 1$. We define a larger real-analytic subvariety of the same dimension:

$$\{(x, v) \in U \times \mathbb{R}^{2N-2} = TU \mid K(x) \cdot v = 0, L_k(x) \cdot v = 0, k = 1, \ldots, 2N - 4\}, \tag{5.9}$$

where we view L_k and K as an $\mathbb{R}^{2N-2}$ valued function, and the dot is the usual dot product. Hence by Malgrange (Theorem 1.4), we see that the normal bundle to the foliation must be a real-analytic submanifold. Therefore there must exist (locally near the origin, by Frobenius) a real valued, real-analytic submersion $f\colon M \to \mathbb{R}$ defining the foliation. This submersion is constant along the local CR orbits of M and hence must be a CR function. All real-analytic CR functions extend uniquely to holomorphic functions in $\mathbb{C}^N$. Thus f is really a holomorphic function with a nonvanishing gradient on M, which is real valued on M. Hence the equation $\operatorname{Im} f = 0$ defines a real-analytic Levi-flat hypersurface $\mathcal{H}$, which contains M. $\mathcal{H}$ must contain H since it must contain the leaves of the Levi foliation of H by Lemma 5.7, and the leaves of H are given by the foliation given by Lemma 5.6. Actually, Lemma 5.7 only tells us about leaves that pass through points of M where the codimension of the local CR orbit is 1. However, the remaining points lie on a real-analytic subvariety of M, and hence leaves that only pass through these points are isolated and thus must also lie in $\mathcal{H}$, since it is locally closed.

The uniqueness of $\mathcal{H}$ is one of the conclusions of Theorem 3.11. □

5.3 Extension across flat boundaries

When the local CR orbits of M are all of codimension 2, the situation is different. In this section we will prove Theorem 5.2. First we will prove this result in $\mathbb{C}^2$, and then reduce the general case to this. In section 5.4, we will see that a C^∞ extension is the best we can do. Suppose that τ is the complex conjugation function.

Theorem 5.8. *Suppose that $U \subset \mathbb{C}^2$, $U = \tau(U)$ and $H \subset U$ is a Levi-flat C^∞ hypersurface with boundary, with $\partial H \subset \mathbb{R}^2$. Then $H \cup \tau(H)$ is a C^∞ Levi-flat hypersurface (without boundary).*

The idea is to extend the leaves of the Levi foliation of H across $\mathbb{R}^2$. Because H has a boundary on $\mathbb{R}^2$, the leaves must be subvarieties of $U \setminus \mathbb{R}^2$, and since H is C^∞ this will imply that these subvarieties, once extended, cannot have an isolated singularity on $\mathbb{R}^2$.

Proof. We can assume that H is closed in U. By Lemma 5.6 the foliation extends up to $\mathbb{R}^2$. In particular the leaves are closed subsets and hence the leaves are complex-analytic subvarieties of $U \setminus \mathbb{R}^2$. Let L be a leaf of the foliation. Since L has boundary on $\mathbb{R}^2$, we find the complex tangent line and can think of L as a graph of a holomorphic function. Thus by Schwarz reflection principle in one variable, $\overline{L \cap \tau(L)}$ is a subvariety of U. We now look at a leaf of the foliation restricted to $\mathbb{R}^2$. A priory, this is a C^∞ submanifold, however as the leaves of the Levi foliation on H extend as complex-analytic subvarieties across $\mathbb{R}^2$, this means that the leaf of the foliation on $\mathbb{R}^2$ must be contained in a real-analytic subvariety of the same dimension and by Theorem 1.4, must be a real-analytic submanifold. This means that the leaves of the Levi foliation of H cannot be singular on $\mathbb{R}^2$ when extended to U. Since the leaves of this foliation are complex submanifolds (and hence C^∞) and are not tangent to $\mathbb{R}^2$, it is not hard to check that $H \cup \tau(H)$ must be a C^∞ submanifold. □

To finish the proof of Theorem 5.2 we can just apply the following lemma. We will use coordinates $(z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2$.

Lemma 5.9. *Suppose that $U = U_z \times U_w \subset \mathbb{C}^{N-2} \times \mathbb{C}^2$ is a connected neighbourhood, and $\tau(U_w) = U_w$. $H \subset U$ is a connected Levi-flat C^∞ hypersurface with boundary, with $\partial H \subset \mathbb{C}^{N-2} \times \mathbb{R}^2$. Then $H \subset \mathbb{C}^{N-2} \times H_w \subset \mathbb{C}^{N-2} \times \mathbb{C}^2$, where $H_w \subset \mathbb{C}^2$ is a C^∞ Levi-flat hypersurface with boundary such that $\partial H_w \subset \mathbb{R}^2$.*

Proof. We have already seen that the leaves of the foliation induced on ∂H are unions of CR orbits. Here the CR orbits are just given by $\{(z, w) \mid w = w^0\}$ for a fixed w^0. So take one leaf L of the Levi foliation on H extended to the boundary. It is then easy to see that $L \cap \partial H$ is equal to (after perhaps extending in the z direction) to $\mathbb{C}^{N-2} \times A$ for some submanifold $A \subset \mathbb{R}^2$.

Fix some $p = (z^0, w^0) \in \mathbb{C}^{N-2} \times \mathbb{R}^2$, such that $p \in L$. It is then not hard to see that if we let $L_w := L \cap \{(z, w) \mid z = 0\}$, then $L_w \setminus \mathbb{R}^2$ is a codimension 1 complex-analytic subvariety of $V \setminus \mathbb{R}^2$, for some small neighbourhood V of w^0. Further, $L_w \cap \partial H = A$, and one component of L_w is path connected to A. This is because of how L is defined. If $\tilde{H}$ is any C^∞ submanifold extending H (as noted in the introduction), then L can be extended to a real C^∞ submanifold of $\tilde{H}$. Further, this extension meets $\mathbb{C}^{N-2} \times \mathbb{R}^2$ transversely in H, and all the derivatives in the z and $\bar{z}$ directions of the defining functions must vanish. Hence, L_w is a submanifold with boundary in some small neighbourhood of w^0. By dimension, L is then equal to $\mathbb{C}^{N-2} \times L_w$ in some small neighbourhood of p. So near some point, L can be defined by an equation not depending on z. Since L is a connected complex-analytic submanifold, this is true everywhere on L. H is a union of such L and the lemma follows. $\square$

The uniqueness in Theorem 5.2 is obvious in view of the fact that the extension (near the origin) is given by extension of the leaves of the Levi foliation and complex submanifolds have unique continuation.

5.4 Counterexamples

In this section we will give examples to show that the assumptions in Theorem 5.1 and Theorem 5.2 are indeed optimal.

Example 5.10. It is obvious that Levi-flat hypersurfaces which contain $\mathbb{R}^2 \subset \mathbb{C}^2$ cannot be unique since for example if we have coordinates $(z, w) \in \mathbb{C}^2$, then both the hypersurfaces $\operatorname{Im} z = 0$ and $\operatorname{Im} w = 0$ contain $\mathbb{R}^2$.

Example 5.11. We can find a C^∞ Levi-flat hypersurface in $\mathbb{C}^2$ which contains $\mathbb{R}^2$, but which is not real-analytic (not contained in a real-analytic subvariety of the same dimension). First let

$$\varphi(x) := \begin{cases} e^{-1/x} & x > 0, \\ 0 & x \leq 0. \end{cases} \tag{5.10}$$

Then define H by looking at

$$\rho_t(z, w) := \varphi(t) z^2 + t - w. \tag{5.11}$$

On $\mathbb{R}^2$ this defines a C^∞ (but not real-analytic) family of real-analytic curves, and it therefore cannot be induced by a real-analytic Levi-flat hypersurface. We need to show that as (z, w) range over some neighbourhood of the origin in $\mathbb{C}^2$, and t ranges over a small interval, $\rho_t = 0$ defines a Levi-flat hypersurface. It suffices to show that it is a submanifold near zero. It is automatically Levi-flat since it is then given by a 1 parameter family of complex-analytic subvarieties. First, we check that if z, w, and t are kept small, then the complex-analytic subvarieties do not intersect for different t. By direct calculation this can be seen to be the case as long as $|z| < 1$. We look at $\operatorname{Re} \rho_t$ and $\operatorname{Im} \rho_t$, and notice $\operatorname{Re} \rho_t$ as a function of $(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w, t)$ satisfies the real-analytic implicit function theorem at 0 and hence we can find a real-analytic solution $t = \alpha(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w)$, then we have a smooth hypersurface defined by

$$0 = \operatorname{Im} \rho_\alpha = \varphi(\alpha(\operatorname{Re} z, \operatorname{Im} z, \operatorname{Re} w)) \operatorname{Im}(z^2) - \operatorname{Re} w. \tag{5.12}$$

Thus the requirement in Theorem 5.1 that not all local CR orbits are of codimension 2 in M is necessary. This is because the above example extends to $\mathbb{C}^N$ by just letting $M = \mathbb{C}^{N-2} \times \mathbb{R}^2$.

Example 5.12. The methods of this chapter revolve around extending the Levi foliation of the hypersurface and thereby extending H. Such methods are bound to fail in general when M has a complex tangent and therefore is not a CR submanifold. In the following example, we show that even if we can extend a Levi-flat hypersurface past a CR singular boundary, the extension need not be unique, even in the sense of Theorem 5.2.

Let $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$ be our coordinates. For a fixed t, let H_t be a Levi-flat hypersurface defined by

$$\operatorname{Im} w = t\varphi(-\operatorname{Re} w), \tag{5.13}$$

where φ is as before. Then define M by

$$\operatorname{Re} w = |z_1|^2 + |z_2|^2 \quad \text{and} \quad \operatorname{Im} w = 0. \tag{5.14}$$

Outside of the origin, M is a CR submanifold, where the codimension of the CR orbits must be 1, as M contains no complex-analytic subvarieties. But then we have a whole family of Levi-flat hypersurfaces which contain M.

Example 5.13. If M would be only C^∞, then no general extension theorem like Theorem 5.1 nor Theorem 5.2 holds. First, let $\sqrt{\cdot}$ denote the principal branch of the square root, and note that the function $\xi \mapsto e^{-1/\sqrt{\xi}}$, holomorphic for $\operatorname{Re} \xi > 0$, can be extended to be C^∞ on $\operatorname{Re}\xi \geq 0$. Suppose that in coordinates $(z, w_1, w_2) \in \mathbb{C}^3$ we define a C^∞ Levi-flat hypersurface with boundary by

$$\operatorname{Re} w_1 \geq |z|^2 \quad \text{and} \quad \operatorname{Re} w_2 = \operatorname{Re} e^{-1/\sqrt{w_1}}. \tag{5.15}$$

M is defined similarly by $\operatorname{Re} w_1 = |z|^2$ and $\operatorname{Re} w_2 = \operatorname{Re} e^{-1/\sqrt{w_1}}$. It is easy to check that M is a generic C^∞ submanifold. Further, since M contains no complex-analytic subvarieties, the CR orbits of M can be seen to be of codimension 1. At an interior point, H is given by a vanishing of the real part of a holomorphic function and so H is Levi-flat.

However, H cannot possibly extend across M since that would mean that the leaves of the Levi foliation of H would have to extend. The leaf of H that goes through the origin is given by $w_2 = e^{-1/\sqrt{w_1}}$. Since this subvariety is given as a graph, if we could possibly extend this complex-analytic subvariety across the origin, we could extend the function $e^{-1/\sqrt{w_1}}$ across $w_1 = 0$, and we know this is not possible.

5.5 Almost minimal submanifolds

We will now prove Theorem 5.4. Recall that a real-analytic generic submanifold M is almost minimal at 0 if for every neighbourhood U of 0, there exists a point $p \in M \cap U$ such that (some representative of) the local CR orbit at p is not contained in a proper complex-analytic subvariety of U. Let us restate Theorem 5.4 for reader convenience.

Theorem. *Let M be a connected real-analytic generic submanifold of codimension 2 through the origin, which is almost minimal at the origin. Let H be a connected C^2 hypersurface with boundary and $M \subset \partial H$. Then H is not Levi-flat.*

Theorem 5.4 is a consequence of the following more general result.

Lemma 5.14. *Let M be a connected real-analytic generic codimension 2 submanifold through the origin and let H be a connected C^2 hypersurface with boundary, and $M \subset \partial H$. Suppose that there exists a point on M where the local CR orbits are of codimension 1. Then there exists some neighbourhood U of the origin* *such that the leaves of the Levi foliation of H^o extend to be closed complex-analytic subvarieties of U.*

Recall $\mathcal{X}_p$ is the intrinsic complexification of the local CR orbit at p, that is, the smallest germ of a complex-analytic submanifold that contains the local CR orbit at p.

We now prove Lemma 5.14 and therefore Theorem 5.4. The method of this proof together with Theorem 5.8 could be used to give a different (but longer) proof of Theorem 5.1.

Proof of Lemma 5.14. We first write M in terms of normal coordinates $(z, w) \in \mathbb{C}^{N-2} \times \mathbb{C}^2$, and take U to be the neighbourhood small enough to apply Lemma 3.9 (that is $U = V$ in the Lemma).

If the local CR orbits are of codimension 1 somewhere on M, they are of codimension 1 outside a proper real-analytic subvariety of M. Let p be one of the points where local CR orbits of M are of codimension 1.

We note that if L is a leaf of the Levi foliation of H (we extend this foliation to M as above) such that $p \in L$, then by Lemma 5.7, applied in a suitably small neighbourhood of p, we see that as germs $(L, p) \subset \mathcal{X}_p$. Hence we can extend L to a small neighbourhood of p, and it will agree with some representative of $\mathcal{X}_p$. By Lemma 3.9, we see that near p, L is defined by equations independent of z. Since L is a connected complex submanifold of U, then at each point it is defined by equations independent of z. Hence there exists a submanifold $\tilde{L}$ of the same dimension, such that $L \subset \tilde{L}$ and $\tilde{L} = \mathbb{C}^{N-2} \times \tilde{L}_w$ where $\tilde{L}_w$ is a complex hypersurface of $\mathbb{C}^2$. If we fix $z = z^0$ and look at $M \cap \{z = z_0\}$, we see that this is a maximally totally real submanifold of $\mathbb{C}^2$, and hence locally biholomorphic to $\mathbb{R}^2$. We can apply the same reasoning as in the proof of Theorem 5.8 to apply Schwarz reflection principle to extend this complex hypersurface across $\mathbb{R}^2$. We can therefore assume that $\tilde{L}_w$ is a subvariety of $U \cap \{z = z_0\}$ (for a perhaps smaller U) and hence $\tilde{L}$ is a complex-analytic subvariety of U. $\square$

5.6 Almost minimal example

As in section 4.4, let M_λ, $\lambda \in \mathbb{R}$, be the generic, nowhere minimal submanifold of $\mathbb{C}^3$, with holomorphic coordinates (z, w_1, w_2) defined by

$$\begin{aligned} \bar{w}_1 &= e^{iz\bar{z}} w_1, \\ \bar{w}_2 &= e^{i\lambda z\bar{z}} w_2. \end{aligned} \tag{5.16}$$

We have already seen that when λ is irrational, this submanifold is almost minimal at 0, and thus not contained in any Levi-flat real-analytic subvariety of codimension 1.

When $\lambda = a/b$ is rational, M_λ is contained in a Levi-flat subvariety of codimension 1, as the meromorphic function w_1^a/w_2^b is real valued on M_λ.

By Theorem 5.4, M_λ is not a boundary of a C^2 Levi-flat hypersurface for λ irrational. We prove the following theorem to show that it cannot be a "boundary" of a real-analytic Levi-flat subvariety, even if we allow singularities.

Theorem 5.15. *Let λ be irrational and let $M_\lambda \subset \mathbb{C}^3$ be as above. Suppose H is a codimension 1 real-analytic subvariety of $D - M_\lambda$, where D is a polydisc in $\mathbb{C}^3$ centered at the origin. Suppose that there exists a point $p \in M_\lambda$, and a connected C^2 hypersurface N with boundary, such that $p \in \partial N \subset M_\lambda$, and $N^o \subset H$. Then H is not Levi-flat.*

In fact, if H is irreducible, then H is not Levi-flat at any nonsingular point of top dimension.

Proof. Assume for contradiction that H is Levi-flat. In particular this means that $N^o \cap H^*$ is Levi-flat, where H^* are the nonsingular points of hypersurface dimension. Thus N^o is Levi-flat on an open dense set. Since being Levi-flat means a certain C^1 1-form is integrable, then it is integrable on all of N^o by continuity.

Pick a point $p = (z, w_1, w_2)$ on $M = M_\lambda$, such that $p \in N$, and the local CR orbits of M are of codimension 1 in a neighbourhood $U \subset M$ of p.

If we take $q \in U$, and $\mathcal{X}_q$ is (some representative of the germ of) intrinsic complexification of the local CR orbit, then $M \cap \mathcal{X}_q$ is a hypersurface in $\mathcal{X}_q$ and hence divides $\mathcal{X}_q$ into two connected sets (we can pick a representative of $\mathcal{X}_q$ small enough). Hence we can write $\mathcal{X}_q$ as a disjoint union of three connected sets as follows:

$$\mathcal{X}_q = \mathcal{X}_q^+ \cup (M \cap \mathcal{X}_q) \cup \mathcal{X}_q^- . \tag{5.17}$$

By Lemma 5.7, we see that either $\mathcal{X}_q^+ \subset N^o$ or $\mathcal{X}_q^- \subset N^o$. Suppose $\mathcal{X}_q^+ \subset N^o \subset H$.

We will find a parametrization of $\mathcal{X}_q$ and hence of $\mathcal{X}_q^+$. We will construct this parametrization of $\mathcal{X}_q$ by the use of Segre sets. We compute the third Segre set at $q = (z^0, w_1^0, w_2^0)$, where $w_1^0 \neq 0$ and $w_2^0 \neq 0$, by the following mapping (see [BER99])

$$\varphi(t_1, t_2, t_3) := (t_3, \overline{w_1^0} e^{i(t_3 t_2 - t_2 t_1 + t_1 \overline{z^0})}, \overline{w_2^0} e^{i\lambda(t_3 t_2 - t_2 t_1 + t_1 \overline{z^0})}). \tag{5.18}$$

That is, the image of this mapping agrees with $\mathcal{X}_q$ as germs at p. We must be careful to stay within the polydisc $D \subset \mathbb{C}^3$ in the image. So let us suppose that M is only defined in D.

Let $\theta := \arg w_1^0$. On M, $\arg w_1^0 = \frac{1}{\lambda} \arg w_2^0 = \theta$. Changing variables by precomposing with $(\xi, \omega) \mapsto (0, \frac{\omega+\theta}{\xi}, \xi)$ we get the mapping:

$$\tilde{\varphi}(\xi, \omega) := (\xi, \overline{w_1^0} e^{i(\omega+\theta)}, \overline{w_2^0} e^{i\lambda(\omega+\theta)}). \tag{5.19}$$

The image of this mapping is on M when

$$\begin{aligned} \overline{w_1^0} e^{i(\omega+\theta)} &= e^{i|\xi|^2} w_1^0 e^{-i(\overline{\omega}+\theta)}, \\ \overline{w_2^0} e^{i\lambda(\omega+\theta)} &= e^{i\lambda|\xi|^2} w_2^0 e^{-i\lambda(\overline{\omega}+\theta)}. \end{aligned} \tag{5.20}$$

That is, the pullback of the CR orbit at q by $\tilde{\varphi}$ is

$$\operatorname{Re} \omega = \frac{1}{2} |\xi|^2 . \tag{5.21}$$

Let S be hypersurface in the parameter space (ξ, ω) defined by (5.21). So if, in the parameter space (ξ, ω), we stay on one or the other side of S, we are parametrizing either $\mathcal{X}_q^+$ or $\mathcal{X}_q^-$.

Let us also vary w_1^0 and w_2^0, while keeping $q = (z^0, w_1^0, w_2^0)$ within M. That is let $w_1^0 = re^{i\theta}$ and $w_2^0 = se^{i\lambda\theta}$, and now let r and s vary. We define a mapping ψ by adding the parameters r and s to $\tilde{\varphi}$

$$\psi(\xi, \omega, r, s) := (\xi, re^{-i\theta} e^{i(\omega+\theta)}, se^{-i\lambda\theta} e^{i\lambda(\omega+\theta)}) = (\xi, re^{i\omega}, se^{i\lambda\omega}). \tag{5.22}$$

As r and s vary over a small interval and ξ and ω vary over some small connected open set, such that the image of ψ never leaves D, and further, such that ξ and ω stay on one side of S, we get a parametrization of an open part of H. This is because as we vary r and s, we vary q, and then as we vary ξ and ω, we parametrize $\mathcal{X}_q^+$ (as long as ξ and ω stay on one side of S).

We will make the parametrization an immersion by restricting ω to be real. Then for a small open set $V \subset \mathbb{C} \times \mathbb{R}^3$, $\psi|_V$ is an immersion. We pick this V such that $\psi(V) \subset H$. Pick any connected open $V' \subset \mathbb{C} \times \mathbb{R}^3$, such that $V \subset V'$ and for all $(\xi, \omega, r, s) \in V'$ we have $\omega > \frac{1}{2} |\xi|^2$ (or $\omega < \frac{1}{2} |\xi|^2$) and $r, s \in (0, \epsilon)$ (where ϵ is the radius of D). It is clear that $\psi(V') \subset D \setminus M$. Further, $\psi(V') \subset H$, since H is a subvariety of $D \setminus M$ and V' is connected and if we pull back H by $\psi|_{V'}$ we must get a subvariety of V' which contains V.

Note that we can pick V' such that it contains all $\omega \in (0, \infty)$ (or in $(-\infty, 0)$). Without loss of generality suppose we can let ω go to plus infinity and still stay within H

We will show that H must be dense (in $\mathbb{C}^3$) near some point not on M, but arbitrarily close to 0. Let ξ vary in some small open set and let r and s vary in some small open interval. For a bounded interval of ω we will parametrize a 5-dimensional set. Now we can start adding 2π to ω and we add a dense set of rotations to the third component in (5.22), without changing the first two. Thus the image of ψ must be dense near some point and this contradicts H being a subvariety of codimension 1.

Now suppose that H is irreducible. We apply Lemma 1.18 and so since H^* is Levi-flat at one point, it is Levi-flat at all points. $\square$

5.7 Subanalytic hypersurfaces

If we allow subanalytic hypersurfaces (see section 1.3 or [BM88]), then we have the following result.

Theorem 5.16. *Let M be a real-analytic, codimension 2, generic submanifold that is nowhere minimal. Then there exists a subanalytic hypersurface H, which is Levi-*

flat at nonsingular points, such that $M \subset H$. Further, if H^* are the nonsingular points of top dimension of H, then $M \cap H^*$ is dense in M.

Proof. If all CR orbits of M are of codimension 2, this is trivial. Otherwise, intersect with a small ball around any point in which normal coordinates (z, w) are defined. Then take the projection π_w onto the w factor. $\pi_w(M)$ is a subanalytic hypersurface in general. Apply Lemma 3.9 to see that all the $\mathcal{X}_p$ are product sets, and $\pi_w(\mathcal{X}_p)$ is contained in $\pi_w(M)$. If $\pi_w(M)$ is of codimension 2, M had CR orbits of only codimension 2. If $\pi_w(M)$ is of codimension 0, then M must have been minimal. Hence $\pi_w(M)$ must have been a subanalytic hypersurface foliated by complex-analytic subvarieties (the projections of the CR orbits), since M is nowhere minimal. Thus $\pi_w(M)$ is the subanalytic hypersurface we are looking for. See section 3 for more details of this method. □

Note that we must intersect with a small ball first, else the image of the projection need not be subanalytic. The submanifold M_λ for λ irrational from section 5.6, when projected onto the w factor without restricting the z to be bounded, will be a dense set in $\mathbb{C}^2$ which is not subanalytic. On the other hand, if we intersect M_λ with the set $|z| \leq 1$, and we look at $\pi_w(M_\lambda)$, we get the following subanalytic hypersurface:

$$\left\{ w \in \mathbb{C}^2 \;\middle|\; \arg w_1 = t, \;\; \arg w_2 = \lambda t, \;\; -\frac{1}{2} \leq t \leq 0 \right\}. \tag{5.23}$$

Note that H^* (the nonsingular points of top dimension of H) is again a subanalytic set and hence a locally finite union of real-analytic submanifolds. Thus we have that a nowhere minimal M is contained in the closure of a locally finite union of real-analytic Levi-flat hypersurfaces. If the points where the CR foliation of M is of codimension 1 are connected, then we need only take one hypersurface. However, H need not have smooth boundary nor does the boundary need to be equal to M if it does. Thus we cannot apply Theorem 5.1.

If we allow hypersurfaces with singularities all the way up to M in the sense of [DTZ], then the above result suggests that, at least locally, any nowhere minimal submanifold could conceivably bound such a singular hypersurface.

5.8 Algebraic boundaries

Recall that a submanifold M is real-algebraic if it is contained in an algebraic subvariety of the same dimension. If we know that M is a real-algebraic codimension 2 submanifold we know by Theorem 3.13 that there exists a real-algebraic Levi-flat hypersurface. The global problem is the following: If M is real-algebraic compact codimension 2 submanifold, when does there exist a compact smooth (or singular) Levi-flat hypersurface with boundary M? While we do not have an answer to the general question, we do have the following.

Proposition 5.17. *Suppose $M \subset \mathbb{C}^N$ is a connected real-algebraic submanifold of real codimension 2, such that there exists an open set U (which intersects M) for which $U \cap M$ is a nowhere minimal CR submanifold. Then there exists a singular Levi-flat hypersurface $H \subset \mathbb{C}^N$ defined by a real polynomial ρ, such that $M \subset H$.*

Proof. Apply Theorem 3.13 to $U \cap M$ to obtain H. H contains the algebraic subvariety that contains $U \cap M$ and hence it must contain M. □

Unfortunately even if M is compact, then we do not know if $H \setminus M$ (nor $H^* \setminus M$) has a relatively compact topological component which could serve as our hypersurface with boundary M. In fact we do not a priory know that a component of $H^* \setminus M$ does not meet "both sides" of M near some point (hence H could not be the boundary).

We have the following proposition on the regularity of H however.

Proposition 5.18. *Suppose $M \subset \mathbb{C}^N$, $N \geq 3$, is a compact connected real-algebraic submanifold of codimension 2 and suppose that there exists a C^∞ Levi-flat hypersurface H with boundary M. Further, suppose that a dense set of leaves of the Levi-foliation of H meet M at a CR point. Then H is real-algebraic.*

Proof. M is, of course, nowhere minimal at CR points by Lemma 5.5. Further, we note that M cannot be flat. In fact, no CR orbits are of codimension 2. If they were, then these CR orbits would be germs of complex-analytic subvarieties of positive dimension (because $N \geq 3$) and those cannot lie in a compact algebraic submanifold, see [D'A93] section 3.2.

We construct H' by Proposition 5.17. Hence H has to agree with H' at each nonsingular point of H'. Finally, M cannot lie in the singularity of H' by Corollary 2.3. Each leaf of the Levi-foliation of H which meets a CR point, must also lie in H'. This must happen for a dense set of leaves and the CR points of $M \setminus H'_s$ are dense in M. Hence the closure of those leaves must be in H' and hence $H \subset H'$ and we are done. $\square$

6 Proper mappings between balls

6.1 General problem

Let $\mathbb{B}_n \subset \mathbb{C}^n$ be the unit ball. Let $f\colon \mathbb{B}_n \to \mathbb{B}_N$ be a proper holomorphic mapping, how can we relate the "complexity" of f to the dimensions n and N. If $n = 1$, then $z \mapsto z^d$ is of arbitrary degree, so we must first assume that $n \geq 2$. Also, $N \geq n$, since proper mappings cannot decrease dimension, as there exist no compact complex-analytic subvarieties. Forstnerič [For89] proved that if f is sufficiently smooth up to the boundary, then f is a rational mapping, and further, that there exists a bound for the degree of f in terms of n and N. D'Angelo [D'A93] made the following conjecture:

Conjecture 6.1. *Suppose $f\colon \mathbb{B}_n \to \mathbb{B}_N$ ($n \geq 2$) is a proper (holomorphic) rational mapping, then*

$$\deg(f) \leq \begin{cases} 2N-3 & \text{if } n = 2 \\ \frac{N-1}{n-1} & \text{if } n \geq 3. \end{cases} \tag{6.1}$$

There exist monomial examples (each component of the mapping is a monomial) that achieve both bounds, hence the conjecture is sharp if true. Further, D'Angelo, Kos and Riehl proved the conjecture for monomial mappings when $n = 2$ in [DKR03]. In [DLP], together with D'Angelo and Peters, we proved the conjecture for monomial mappings when d is "small" or when n is "large." Further, we proved weaker bounds in general. These results will be studied in detail the sequel. Most of the results of this chapter have appeared in some form in [DLP], except for sections 6.5, 6.6, and 6.7, which for the most part appear here for the first time.

6.2 Monomial mappings

Let $f\colon \mathbb{B}_n \to \mathbb{B}_N$ be a proper polynomial mapping. That is, each component f_k, $k = 1, \dots, N$, of the mapping is a holomorphic polynomial. Further assume that each f_k is in fact a single monomial. Using multiindex notation, such mappings can be written as

$$z \in \mathbb{C}^n \mapsto (c_1 z^{\alpha_1}, c_2 z^{\alpha_2}, \dots, c_N z^{\alpha_N}) \in \mathbb{C}^N, \tag{6.2}$$

where $c_k \in \mathbb{C}$ are constants. This is what we will mean by *monomial mappings*.

The first indication that this is an interesting case comes from the results of Faran [Far82], who proved that when $n = 2$ and $N = 3$, then, up to automorphisms of source and target, the only possible proper mappings are the following:

$$(z, w) \mapsto (z, w, 0), \tag{6.3}$$

$$(z, w) \mapsto (z, zw, w^2), \tag{6.4}$$

$$(z, w) \mapsto (z^2, \sqrt{2}zw, w^2), \tag{6.5}$$

$$(z, w) \mapsto (z^3, \sqrt{3}zw, w^3). \tag{6.6}$$

The mapping (6.6) is very special, and we will call it the *Faran mapping*.

What does it mean for a monomial (or any rational) mapping to be proper? As we noted in the introduction, it means that boundary is taken to boundary (if the mapping extends to the boundary). In particular we get the following equation:

$$\|f(z)\|^2 := \sum_{k=1}^{N} |f_k(z)|^2 = 1, \text{ when } \|z\|^2 = 1. \tag{6.7}$$

If $f_k(z) = c_k z^{\alpha_k}$, then the equation becomes:

$$\sum_{k=1}^{N} |c_k|^2 z^{\alpha_k} \bar{z}^{\alpha_k} = 1, \text{ when } \|z\|^2 = 1. \tag{6.8}$$

If we introduce new variables $x_j = |z_j|^2$ and let $a_k = |c_k|^2$, then we have the equation

$$\sum_{k=1}^{N} a_k x^{\alpha_k}, \text{ when } \sum_{j=1}^{n} x_j = 1. \tag{6.9}$$

Any nonconstant polynomial as above, where $a_k \geq 0$, induces a proper holomorphic mapping, by just reversing the above procedure. This operation is not one to one as we can multiply the c_k by $e^{i\theta}$ and this corresponds to the same a_k. For example, the Faran mapping (6.6) corresponds to the polynomial

$$p(x, y) = x^3 + 3xy + y^3. \tag{6.10}$$

Definition 6.2. Suppose $p(x)$ is a real polynomial, we will denote by $N(p)$ the number of distinct monomials in p. Further, for ease of notation we define

$$s(x) := \sum_{j=1}^{n} x_j. \tag{6.11}$$

Further, we will denote the set of polynomials $p \colon \mathbb{R}^n \to \mathbb{R}$ with nonnegative coefficients, such that $p = 1$ when $s = 1$ and are of degree d exactly, by $\mathcal{H}(n, d)$. We will say that p is *optimal* in $\mathcal{H}(n, d)$, if it minimizes $N(p)$.

We will also frequently decompose $p \in \mathcal{H}(n, d)$ into its homogeneous parts. That is we will write

$$p = p_d + p_{d-1} + \cdots + p_0, \tag{6.12}$$

where p_k are the monomials of p which are of homogeneous degree k.

In this new formulation, the D'Angelo conjecture is stated as follows.

Conjecture 6.3. *Suppose $p \in \mathcal{H}(n, d)$, then*

$$d \leq \begin{cases} 2N(p) - 3 & \text{if } n = 2 \\ \frac{N(p)-1}{n-1} & \text{if } n \geq 3. \end{cases} \tag{6.13}$$

As stated before, the above is proved for $n = 2$ in [DKR03]. We will use this result repeatedly, so we restate it as a theorem.

Theorem 6.4 (D'Angelo-Kos-Riehl)**.** *Suppose $p \in \mathcal{H}(2, d)$, then*

$$d \leq 2N(p) - 3, \tag{6.14}$$

and this result is sharp.

By sharp we mean that there exist $p \in \mathcal{H}(2,d)$ such that $d = 2N(p) - 3$. One such p is for example (when d is odd),

$$p(x,y) = y^d + \left(\frac{x + \sqrt{x^2+4y}}{2} \right)^d + \left(\frac{x - \sqrt{x^2+4y}}{2} \right)^d . \tag{6.15}$$

It is not hard to see that

$$p(x,y) = (x^d, y^d, c_1 x^{d-2} y, c_2 x^{d-4} y^2, \ldots, c_k x y^k), \tag{6.16}$$

where $k = \frac{d-1}{2}$, and $c_1, \ldots, c_k$ are integer constants. For $d = 3$ this is the polynomial induced by the Faran mapping (6.10). Hence we will call the above polynomials the *generalized Faran mappings*. We will identify polynomials and monomial mappings in the sequel.

There exist other optimal polynomials for some degrees. For example, the polynomial

$$x^7 + y^7 + \tfrac{7}{2} x^5 y + \tfrac{7}{2} x y^5 + \tfrac{7}{2} x y. \tag{6.17}$$

As we can see an optimal polynomial can have rational coefficients, but as we will see later in Proposition 6.21, optimal polynomials never have irrational coefficients.

It is sometimes easier, in this context, to think of the estimate (6.13) as

$$N(p) \geq \begin{cases} \frac{d+3}{2} & \text{if } n = 2 \\ d(n-1)+1 & \text{if } n \geq 3. \end{cases} \tag{6.18}$$

One way to construct new polynomials in $\mathcal{H}(n,d)$ is to multiply or divide by s. That is, $p = p' + p'' = 1$ on $s = 1$ if and only if $p' + p''s = 1$ on $s = 1$.

Definition 6.5. If $p = p' + p''$, we define

$$X_{p''}(p) = p' + p''s, \tag{6.19}$$

and we will generally drop the subscript and call this *operation* X.

One mapping constructed by successive application of operation X is $s^d \in \mathcal{H}(n,d)$. We note that this is the only homogeneous mapping in $\mathcal{H}(n,d)$.

Proposition 6.6. *Suppose $p \in \mathcal{H}(n,d)$ is homogeneous, then $p = s^d$.*

Proof. First we note that $p(x) = s(x)^d$ when $s(x) = 1$. By homogeneity, $p(tx) = t^d p(x) = t^d s(x)^d = s(tx)$ for all t. Hence $p = s^d$. □

Definition 6.7. We let $\mathcal{W}(n, d) \subset \mathcal{H}(n, d)$ be the set of polynomials gotten by

$$X_{p_1}(X_{p_2}(\cdots X_{p_k}(1)\cdots)), \tag{6.20}$$

that is, by finitely many applications of operation X to the polynomial 1.

Unfortunately there do exist polynomials in $\mathcal{H}$, which are not in $\mathcal{W}$, for example the Faran mapping (6.10). This fact can be easily checked directly. For polynomials in $\mathcal{W}$ we have the following proposition.

Proposition 6.8. *If $p \in \mathcal{W}(n, d)$, then $d \leq \frac{N(p)-1}{n-1}$.*

Proof. Each application of operation X cannot decrease the number of terms. If we apply operation X to a top degree monomial, we must gain $n-1$ new monomials. Hence by induction on degree we can see that $N(p) \geq d(n-1)+1$. □

We show that we can construct all $p \in \mathcal{H}(n, d)$ by a procedure of dividing out s from s^d, i.e., by reversing (undoing) an operation X. I.e., we note that if p is in the form (6.12), then by Proposition 6.6

$$p_d + p_{d-1}s + p_{d-2}s^2 + \cdots + p_1 s^{d-1} + p_0 s^d = s^d. \tag{6.21}$$

This is because the left hand side is homogeneous. Hence, we can start with s^d, and by division as above arrive at p.

Using Theorem 6.4, we can give a crude bound for certain polynomials and demonstrate the method that we will apply in Section 6.3.

Lemma 6.9. *Suppose $n \geq 2$ and $p \in \mathcal{H}(n, d)$. If p contains a monomial in one or two variables of degree d, then*

$$d \leq \frac{2N(p)-3}{2n-3}. \tag{6.22}$$

Proof. After renumbering we may assume that p contains either x_1^d or $x_1^a x_2^b$ where $a + b = d$. We "pull back" by the generalized Faran mapping (6.15) of degree

$2n-3$. Call this φ. Note that φ contains a term x^{2n-3}. And further $N(\varphi) = n$. To "pull back" we just think of φ and p as a monomial mapping and then get a polynomial $\varphi^*(p)$ in two variables of degree $(2n-3)(d)$. Since all coefficients are positive, we can see that $N(p) \geq N(\varphi^*(p))$. Hence applying Theorem 6.4

$$d = \frac{\deg(\varphi^*(p))}{2n-3} \leq \frac{2N(\varphi^*(p)) - 3}{D} \leq \frac{2N(p) - 3}{D} = \frac{2N-3}{2n-3} \tag{6.23}$$

□

We can now give an elementary combinatorial proof of the conjecture for small degree or codimension, for $n \geq 3$.

Theorem 6.10. *Suppose $n \geq 3$ and $p \in \mathcal{H}(n,d)$. If $d \leq 4$ or $N(p) < 4n-3$, then*

$$d \leq \frac{N(p)-1}{n-1}. \tag{6.24}$$

Before we prove this proposition we will need the following useful lemma.

Lemma 6.11. *Suppose $p \in \mathcal{H}(n,d)$, then there are at least n unique monomials of degree d.*

Proof. First write $p - 1 = q(s-1)$ for some polynomial q. Now identify the terms of degree d, then $p_d = q_d s$. It is an easy matter to see that such a p_d must have at least n terms. □

We observe that if $p(x,y)$ is such that $p \in \mathcal{H}(2,d)$ and $d \geq 2$, then there exists a mixed term (monomial depending on both x and y). This can be seen by just plugging in $x = t$ and $y = (1-t)$ and expanding. We also note that if we set k variables to zero in $p \in \mathcal{H}(n,d)$, then we get a polynomial in $\mathcal{H}(n-k,d)$.

Proof of Theorem 6.10. What we will prove are the following four statements. Suppose that $p \in \mathcal{H}(n,d)$.

(i) If $N(p) < n$, then $d = 0$.

(ii) If $N(p) < 2n-1$, then $d \leq 1$.

(iii) If $N(p) < 3n-2$, then $d \leq 2$.

(iv) If $N(p) < 4n-3$, then $d \leq 3$.

The contrapositive of Statement (i) is easy. When $d \geq 1$ there must be at least n distinct monomials of degree d, by Lemma 6.11.

We call terms of the form x_i^k *pure* terms, and we call monomials depending on at least 2 variables *mixed* terms. By pulling back to the one-dimensional case in n ways (by setting $n-1$ of the variables equal to zero), we note that there must be at least n distinct pure terms. If $d = 1$ then all the terms are pure terms and $p = s$. We may therefore assume that $d \geq 2$ in proving the rest of the statements.

If no pure term is of degree at least 2, then $p = s$ as above. We may thus assume that the monomial x_1^a occurs for some $a \geq 2$. By setting all variables except x_1 and x_j equal to 0, we see that a monomial $x_1^k x_j^l$ must occur for $2 \leq j \leq n$. Hence we have at least $n-1$ mixed terms. Counting also the n pure terms shows that $N(p) \geq (n-1) + n$ and (ii) follows.

Now assume that $d \geq 3$. To prove (iii) we must show that $N \geq 3n-2$. There are two cases. If x_1^a is the only pure term of degree greater than 1, then p must be equal to $x_1 r(x) + s - x_1$, for some $r(x) \in \mathcal{H}(n,d)$. The polynomial r has $n-1$ fewer terms than p does and it must have degree at least 2. Applying (ii) shows that $N(r) \geq 2n-1$ and hence $N(p) \geq (2n-1) + (n-1) = 3n-2$. Thus (iii) holds in this case.

The remaining case of (iii) is when at least two pure terms of degree at least 2 occur. Hence we assume that x_2^b occurs as well, with $b \geq 2$. We then have at least $2(n-2)+1$ mixed terms and n pure terms for a total of $3n-3$. We want $N \geq 3n-2$. Let us therefore assume for the purpose of contradiction that there are no other terms. For $d \geq 3$ the only element of $\mathcal{H}(2,d)$ that has at most 3 distinct monomials is $u^3 + 3uv + v^3$. Hence all pure terms must be of degree 3 and we obtain

$$p(x) = \sum_{j=1}^{n} x_j^3 + 3 \sum_{i \neq j} x_j x_i. \tag{6.25}$$

We claim that the polynomial in (6.25) is not in $\mathcal{H}(n,3)$ unless $n = 2$. One way to verify the claim is to note that $p\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) > 1$ when $n \geq 3$. Thus (iii) holds in this case, and hence in general.

To prove (iv) assume that $N < 4n-3$ and $d \geq 4$. Suppose first that Lemma 6.9 does not apply; that is, there is no term of degree d involving at most two of the variables. Then we must have at least n terms of top degree, n additional pure terms, and (as above) at least $2n-3$ additional mixed terms involving two variables. The total is $4n-3$. Thus the Lemma applies when $N \leq 4n-4$.

If $N \leq 4n-5$ then Lemma 6.9 applies and we obtain a contradiction as follows: By Lemma 6.9,

$$d(2n-3)+3 \leq 2N. \tag{6.26}$$

Including the information on N and d yields

$$4(2n-3)+3 \leq d(2n-3)+3 \leq 2N \leq 2(4n-5) \tag{6.27}$$

from which we obtain the contradiction $-9 \leq -10$. Thus, for $N \leq 4n-5$ we have $d \leq 3$.

The remaining case is when $N(p) = 4n-4$ and $d \leq 4$, and it has two subcases. First suppose that $n \geq 4$. Setting in turn $x_1 = 0$ and $x_2 = 0$ we get polynomials in $n-1$ variables with at least n fewer terms. Thus these polynomials must have degree at most 3. Therefore if $d = 4$ then the top degree terms must be divisible by $x_1 x_2$, and thus $p_4 = s(x)x_1x_2q(x)$ where q is homogeneous of degree 1. We can easily check that q must have all positive coefficients, and we can undo an operation X to reduce to a previous case.

The other subcase is when $n = 3$, $N(p) = 4n-4 = 8$ and $d \leq 4$. We claim that no polynomial in $\mathcal{H}(3,4)$ has exactly 8 distinct monomials. There are only finitely many possibilities that need to be checked and we outline how to do this by hand.

If all terms of degree 4 depend on 3 variables, we undo and reduce to a previous case to get a contradiction. By a simple counting argument, up to renaming of variables, we show that $p(x_1, x_2, 0)$ and $p(x_1, 0, x_2)$ must have exactly 4 terms and be of degree 4 while $p(0, x_2, x_3)$ must have 3 terms and must be of degree 3 or less. By a study of the 2-dimensional case we see that x_1^4 must appear. One can then check by hand that the only possible configuration of degree 4 terms is $x_1^3(x_1 + x_2 + x_3)$ and reducing to a previous case produces a contradiction. $\square$

6.3 Bound for all monomial mappings

In this section we prove a general bound which is probably not sharp, but it is of the correct order. The method of proof is similar to Lemma 6.9, but will now hold for all polynomials in $\mathcal{H}(n,d)$.

Theorem 6.12. *Suppose $p \in \mathcal{H}(n,d)$. Then*

$$d \leq \frac{2n(2N(p)-3)}{3n^2-3n-2} \leq \frac{4}{3}\frac{2N(p)-3}{2n-3}. \tag{6.28}$$

Proof. Suppose we have a monomial of degree d of the form $m = x_1^{a_1} \cdots x_k^{a_k}$, i.e. $a_1 + \cdots + a_k = d$. First we prove the estimate

$$d \leq \frac{2N(p)-3+\sum_{j=3}^{k}(j-2)a_j}{2n-3}. \tag{6.29}$$

As in the proof of Lemma 6.9 we let φ be the generalized Faran mapping (6.15) of degree $D := 2n-3$. We then reorder the variables such that after pulling back we have

$$(x_1, x_2, x_3, x_4, \ldots) = (u^D, v^D, c_1 u^{D-2} v, c_2 u^{D-4} v^2, \ldots) = \varphi(u,v). \tag{6.30}$$

Where c_j are the constants in φ. That φ is of the above form can be seen by direct computation. Pulling back the monomial m guarantees a term of degree

$$a_1 D + a_2 D + a_3(D-1) + \cdots + a_k(D-k+2) = D\sum_{j=1}^{k} a_j - \sum_{j=3}^{k}(j-2)a_j \tag{6.31}$$

in $\varphi^*(p)$. As the sum of the a_j is d, we obtain

$$dD - \sum_{j=3}^{k}(j-2)a_j \leq d(\phi^*(p)) \leq 2N(\phi^*(p)) - 3 \leq 2N(p)-3, \tag{6.32}$$

and hence

$$d \leq \frac{2N(p)-3+\sum_{j=3}^{k}(j-2)a_j}{D} = \frac{2N(p)-3+\sum_{j=3}^{k}(j-2)a_j}{2n-3}. \tag{6.33}$$

If we let $A := \frac{2N(p)-3}{2n-3}$ and $B := \frac{1}{2n-3}\sum_{j=3}^{k}(j-2)a_j$, then we have

$$d \leq A + B. \tag{6.34}$$

We may assume $k \geq 2$, and that $a_1 \geq a_2 \geq \cdots \geq a_k$. We estimate B as

$$\begin{aligned} B = \frac{1}{2n-3} \sum_{j=3}^{k} (j-2) a_j &\leq \frac{d}{k(2n-3)} \sum_{j=3}^{k} (j-2) \\ &= \frac{d}{k(2n-3)} \binom{k-1}{2} \\ &\leq \frac{d}{n(2n-3)} \binom{n-1}{2} \end{aligned} \tag{6.35}$$

Let $c(n) = \frac{1}{n(2n-3)}\binom{n-1}{2}$, hence $B \leq c(n)d$. It is not hard to see that $c(n) < 1$. Thus

$$d \leq A + B \leq A + c(n)d, \tag{6.36}$$

and so

$$d \leq \frac{1}{1-c(n)} A = \frac{1}{1-c(n)} \frac{2N(p)-3}{2n-3} = \frac{2n(2N(p)-3)}{3n^2-3n-2}. \tag{6.37}$$

To finish the proof we note that for $n \geq 2$ we have

$$\frac{2n}{3n^2-3n-2} \leq \frac{4}{3(2n-3)}, \tag{6.38}$$

and therefore the inequality on the far right-hand side of (6.28) holds. □

6.4 Bound for large source dimension

If we assume that n is sufficiently large, we can in fact prove that if $p \in \mathcal{H}(n,d)$ and p is optimal, then $p \in \mathcal{W}$, hence the Conjecture 6.3 holds by Proposition 6.8.

Theorem 6.13. *Fix d and assume $n \geq 2d^2 + 2d$. If $p \in \mathcal{H}(n,d)$ then $N(p) \geq d(n-1)+1$. Furthermore, if equality holds then $p \in \mathcal{W}$.*

Remark 6.14. If we invoke Theorem 6.12 we can change the assumption in Theorem 6.13 to $2N - 3 < \frac{2n-3}{2}\sqrt{n}$, as this will force $n \geq 2d^2 + 2d$. Better and far more complicated conditions can be gotten by combining the previous results and applying Theorem 6.13, but the utility of such improvements does not seem to be large.

Before we prove Theorem 6.13 we give a simple condition guaranteeing that $p \in \mathcal{W}$. Let $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and define $s'(x') := \sum_{j=1}^{n-1} x_j$. We will say that p is *affine in* x_n if we can write $p(x', x_n) = a(x') + x_n b(x')$ for some polynomials a and b.

Lemma 6.15. *If $p \in \mathcal{H}(n,d)$ and suppose p is affine in x_n, then $p \in \mathcal{W}$.*

Proof. We induct on the degree d. When $d = 1$ the result is obvious. Suppose $d \geq 2$ and that the result is known for such affine polynomials of degree $d-1$. Assume $p(x', x_n) = a(x') + x_n b(x')$. Write $p - 1 = q(s-1)$, then equating the highest degree parts we get $p_d = s q_{d-1}$, that is

$$\begin{aligned} a_d(x') + x_n b_{d-1}(x') &= \left(\sum_{j=1}^{n-1} x_j + x_n \right) q_{d-1}(x') \\ &= s'(x') q_{d-1}(x') + x_n q_{d-1}(x'). \end{aligned} \tag{6.39}$$

Hence $q_{d-1} = b_{d-1}$ and $a_d = s' q_{d-1}$. Therefore

$$p = p - p_d + s b_{d-1} = X(p - p_d + b_{d-1}), \tag{6.40}$$

and $p - p_d + b_{d-1} \in \mathcal{H}(n, d-1)$. It is also affine in x_n and hence lies in $\mathcal{W}$ by the induction hypothesis. Thus $p \in \mathcal{W}$ as well. □

We now prove two simple technical results that we use in the proof of Theorem 6.13.

Lemma 6.16. *Let $p \in \mathcal{H}(2,d)$ and suppose that $p(x,y) = a(x) + y b(x)$. Then $N(p) \geq d+1$. The monomial x^d must appear and $x^j y$ must appear for each j with $0 \leq j \leq d-1$. Furthermore, p has exactly $d+1$ distinct monomials if and only if*

$$p(x,y) = x^d + y(x^{d-1} + \cdots + x + 1). \tag{6.41}$$

Proof. By Lemma 6.15 we know $p \in \mathcal{W}$, and the statement follows by induction on d. □

For two monomials $m_1 = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $m_2 = x_1^{\beta_1} \cdots x_n^{\beta_n}$ we define the distance between them by

$$\delta(m_1, m_2) := \sum_j |\alpha_j - \beta_j|. \tag{6.42}$$

Note that for monomials of the same degree $\delta(m_1, m_2)$ must be even.

Lemma 6.17. *Let $p \in \mathcal{H}(3,d)$, and suppose that $p(x_1,x_2,x_3) = a(x_1,x_2) + x_3 b(x_1,x_2)$. If two monomials $m_1(x_1,x_2), m_2(x_1,x_2)$ of degree d occur in $p(x)$ with $\delta(m_1,m_2) \geq 4$, then p has at least $d+1$ distinct monomials that depend on x_3.*

Proof. It follows from Lemma 6.15 that $p \in \mathcal{W}$, and from Lemma 6.16 that p must have at least one monomial of every degree that depends on x_3. Since $\delta(m_1,m_2) \geq 4$ there must be at least 2 monomials of maximal degree that depend on x_3, which gives at least $d+1$ monomials. □

For the rest of this section we assume $n \geq 2d^2 + 2d$. In particular $n \geq 3$. Let $p \in \mathcal{H}(n,d)$ and let $N = N(p)$. We assume both that $N \leq d(n-1)+1$ and that p is optimal. We will show that p must be a generalized Whitney mapping, thereby $N = d(n-1)+1$ and thus Theorem 6.13 is proved.

Fix $k \leq 2d$, and after renaming the variables if necessary, we may assume that p has at least one monomial of degree d that involves only the variables x_1 through x_k. We define new polynomials in $\mathcal{H}(2,d)$ and $\mathcal{H}(3,d)$

$$P_j(\xi, x_j) := p\Big(\underbrace{\frac{\xi}{k},\ldots,\frac{\xi}{k}}_{k \text{ times}},0,\ldots,0,x_j,0,\ldots\Big), \tag{6.43}$$

$$P_{ij}(\xi, x_i, x_j) := p\Big(\underbrace{\frac{\xi}{k},\ldots,\frac{\xi}{k}}_{k \text{ times}},0,\ldots,0,x_i,0,\ldots,0,x_j,0,\ldots\Big). \tag{6.44}$$

Claim 6.18. *The polynomial P_j is affine in x_j for each $j \in \{k+1,\ldots,n\}$.*

Proof. Seeking a contradiction we assume $k+1 \leq l \leq n$, that P_j is not affine for $k+1 \leq j \leq l$, and that P_j is affine for $l+1 \leq j \leq n$.

If P_j is affine in x_j then by Lemma 6.15 we have

$$P_j(\xi, x_j) = c_1 \xi^d + c_2 \xi^{d-1} x_j + \cdots + c_d \xi x_j + c_{d+1} x_j + q(\xi), \tag{6.45}$$

where q is a possibly zero polynomial in ξ of degree $d-1$ or less. If P_j is not affine in x_j then there must be at least $\lceil \frac{d-3}{2} \rceil$ terms by Theorem 6.4.

We will proceed to find a lower estimate for the number of monomials of p, and we must take care not to count the same monomial twice. We first count the monomial m. For each P_j where $k+1 \leq j \leq l$ we have at least $\lceil \frac{d+3}{2} \rceil - 1$ extra monomials and for each P_j for $j > k$ we get at least d extra monomials.

For P_{ij} where $k+1 \leq i < j \leq l$ we know that there must be at least one monomial that depends on x_i as well as x_j (keep ξ constant to see this), and thus we get least $(l-k)(l-k-1)/2$ more monomials that we have not counted yet.

For the same reason we can count one extra monomial depending on both x_i and x_j for each possible choice $k+1 \leq i \leq l < j \leq n$ so we get $(l-k)(n-l)$ more monomials.

When we add the number of all these monomials we obtain

$$N \geq 1 + (l-k)\left(\left\lceil \frac{d+3}{2} \right\rceil - 1 + \frac{l-k-1}{2} + (n-l)\right) + (n-l)d. \tag{6.46}$$

By our assumption $l \geq k+1$. If

$$(l-k)\left(\left\lceil \frac{d+3}{2} \right\rceil - 1 + \frac{l-k-1}{2} + (n-l)\right) > (l-1)d, \tag{6.47}$$

then p cannot be optimal. This happens when

$$(l-k)(d-l-k+2n) - 2(l-1)d > 0. \tag{6.48}$$

Fixing k, d and n the expression in (6.48) is concave down in l and thus must achieve a minimum if $l = k+1$ or $l = n$. We know $1 \leq k \leq 2d$ and so get two bounds for n:

$$n > \frac{4d^2 + 3d + 1}{2}, \tag{6.49}$$

$$n^2 > 2d^2 + (5n-2)d - 1. \tag{6.50}$$

Our assumption that $n \geq 2d^2 + 2d$ implies both bounds (noting that $d \geq 2$ and $n \geq 3$). $\square$

We now know that P_j must be affine in x_j for every variable j that does not appear in the fixed monomial of highest degree, and we can complete the proof of Theorem 6.13.

Claim 6.19. *If m_1 and m_2 are two distinct monomials of maximal degree that occur in p then $\delta(m_1, m_2)$ must be 2.*

Proof. Suppose for the sake of contradiction that p has two highest degree terms m_1, m_2 whose distance is at least 4. We rename the variables $x_1, \ldots, x_n$ such that the monomials m_1 and m_2 only depend on $x_1, \ldots, x_k$, with $k \leq 2d$. Write $m_1 = \prod_{i=1}^k x_i^{r_i}$ and $m_2 = \prod_{i=1}^k x_i^{s_i}$. We may further assume that there exists an integer t such that for $i = 1, \ldots, t$ we have that $r_i \geq s_i$, and for $i = t+1, \ldots, k$ we have $r_i \leq s_i$. We define P_j and P_{ij} as in (6.43) and (6.44) and apply Claim 6.18 to see that for every $j = k+1, \ldots, n$, the polynomial P_j must be affine in x_j.

Let

$$P(y, z, x_{k+1}, \ldots, x_n) := p\Big(\underbrace{\frac{y}{t}, \ldots, \frac{y}{t}}_{t \text{ times}}, \underbrace{\frac{z}{k-t}, \ldots, \frac{z}{k-t}}_{k-t \text{ times}}, x_{k+1}, \ldots, x_n\Big). \tag{6.51}$$

It follows that P has two terms of highest degree $y^{r_1} z^{r_2}$ and $y^{s_1} z^{s_2}$ with $r_1 > s_1 + 1$ and $r_2 < s_2 - 1$.

For every $j = k+1, \ldots, n$, the polynomial $P(y, z, 0, \ldots, 0, x_j, 0, \ldots, 0)$ is a polynomial in three variables that satisfies the conditions of Lemma 6.17, and therefore has at least $d+1$ terms that depend on x_j. Therefore, P (and thus also p) has at least $(d+1)(n-2d) = dn + n - 2d^2 - 2d$ distinct monomials. We assumed that $n \geq 2d^2 + 2d$, so the polynomial cannot be optimal. $\square$

Proof of Theorem 6.13. By Claim 6.19 we know that all highest degree terms have distance 2 from each other. By Lemma 6.11, there exist at least n terms of highest degree. It follows that the terms of highest degree must be equal to $cs \cdot m$ for some constant c and some monomial m of degree $d-1$.

Thus we can undo the operation X to obtain a new polynomial of degree $d-1$, with exactly $n-1$ terms fewer than p. The reason is that p is optimal; undoing the operation X must create a new term of degree $d-1$ (otherwise multiplying that term by s would get a polynomial with fewer terms than p). This new polynomial of degree $d-1$ must again be optimal, because if there existed a polynomial of degree $d-1$ with fewer terms, we could apply operation X to it and again and invalidate the optimality of p.

An inductive argument with respect to the degree shows that $p \in \mathcal{W}$. By Proposition 6.8 we are done. $\square$

6.5 Optimal polynomials

Recall that $p \in \mathcal{H}(n,d)$ is optimal if $N(p)$ is minimal in $\mathcal{H}(n,d)$. Our main result is Theorem 6.20 below, which will enable us to prove some interesting properties of optimal polynomials, which are of independent interest, but will also make it possible to prove a bound for all polynomials in $\mathcal{H}(n,d)$, albeit not as good as Theorem 6.12. Let us denote the set of all polynomials (not necessarily with non-negative coefficients) that are 1 on $s = 1$ by $\mathcal{I}(n,d)$. If we want to denote such polynomials of degree d or less, we will use the notation $\mathcal{I}(n, \leq d)$.

We will treat the coefficients of a generic polynomial of degree d as unknowns. Since for $p \in \mathcal{I}(n,d)$, $p = 1$ on $s = 1$, we can solve $s = 1$ for x_n and look at,

$$p\left(x_1, \ldots, x_{n-1}, 1 - \sum_{k=1}^{n-1} x_k\right) = 1. \tag{6.52}$$

As this is true for all $x \in \mathbb{R}^n$, we obtain a linear equation in the coordinates whose solutions are polynomials in $\mathcal{I}(n, \leq d)$. In the following when we endow the set of polynomials of degree d or less with a topology, it is the euclidean topology on the space of coefficients, $\mathbb{R}^K$ for some large K.

Theorem 6.20. *If there exists a continuous 1 parameter family $t \mapsto p(t) \in \mathcal{H}(n,d)$ and further assume that $N(p(t))$ is constant for t in some open interval, no such $p(t)$ is optimal.*

Proof. If there were a family, then since the polynomials come from a linear system, there is a "line" in the parameter space. That is, we can pick two polynomials p and q such that $\varphi_t := tp + (1-t)q \in \mathcal{H}(n,d)$ and the number of terms in φ_t is constant for t in some small interval. We can however extend this line and it is obvious that for any t the polynomials must be in $\mathcal{I}(n, \leq d)$. If we travel on this line in the correct direction we will hit a place where at least one term becomes zero and hence $N(\varphi_t)$ becomes smaller. The only problem could arise if all the terms of degree d become zero at once. Suppose that this place is $t = a$, and here all coefficients are nonnegative, and further all coefficients of degree d vanish. Note that $\varphi_a \in \mathcal{H}$ (of lower degree than d) and also any monomial that appears in φ_a

appears in p as well. Hence, for small $\epsilon > 0$

$$\psi_\epsilon := \frac{1}{1-\epsilon}(p - \epsilon(\varphi_a)) \in \mathcal{H}(n,d) \tag{6.53}$$

Note that $p - \varphi_a$ must have a negative coefficient since it must be a nonzero polynomial and is zero on $s = 1$. Hence ψ_ϵ is not in $\mathcal{H}(n,d)$ for some $\epsilon < 1$. Take

$$\epsilon_0 := \sup\{\epsilon \in (0,1) \mid \psi_\epsilon \in \mathcal{H}(n,d)\} \tag{6.54}$$

Then $\psi_{\epsilon_0} \in \mathcal{H}(n,d)$ and obviously one coefficient (of degree less than d) must have vanished and hence $N(\psi_{\epsilon_0}) < N(p)$. Hence contradicting the optimality of p. □

Let a polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha$, then let the *signature* of p, $\mathrm{sig}(p)$, be the set of multiindices α for which c_α is not zero. We will not use the following properties, but they are of independent interest.

Proposition 6.21. *Fix n and d.*

(i) If $p \in \mathcal{H}(n,d)$, p is optimal, $q \in \mathcal{H}(n,d')$ for $d' \le d$, then $\mathrm{sig}(q) \subset \mathrm{sig}(p)$ *if and only if $p = q$.*

(ii) If $p, q \in \mathcal{H}(n,d)$ are optimal, then $\mathrm{sig}(p) \neq \mathrm{sig}(q)$.

(iii) The set of optimal polynomials in $\mathcal{H}(n,d)$ is finite.

(iv) If $p \in \mathcal{H}(n,d)$ is optimal, then all coefficients of p are rational numbers.

(v) Every polynomial in $\mathcal{I}(n,d)$ is a linear combination of optimal polynomials in $\mathcal{W}$ of degree less than or equal d (optimal in $\mathcal{W}$).

By p optimal in $\mathcal{W}$ we really mean that $N(p) = \deg(p)(n-1)+1$, since that is the minimum number of terms for a polynomial of degree $\deg(p)$ in $\mathcal{W}$.

Proof. (i) is proved in the same manner as Theorem 6.20 using the convex combination $tp + (1-t)q$. (ii) now follows easily. (iii) follows from (ii), since there are only finitely many possible signatures for polynomials in $\mathcal{H}(n,d)$. (iv) follows from the fact that, as noted earlier, coefficients of polynomials in $\mathcal{I}(n, \le d)$ are solutions to a linear problem coming from (6.52). Similarly polynomials that are constant on $s = 1$ are vectors in the nullspace of a certain matrix with integer coefficients (just changing the 1 on the right hand side of (6.52) to a variable and moving it to the left hand side). If one considers only those columns corresponding to multiindices in $\mathrm{sig}(p)$, then the nullspace of this smaller matrix must have a basis of vectors with only rational elements (as the matrix consists of only integers) and the claim follows by another application of Theorem 6.20 by noting that this nullspace must then be only one-dimensional. This one dimension corresponds to the extra variable coming from the right hand side of (6.52). For claim (v) we just notice that the polynomial $s - sx^\alpha$, corresponding to the simplest operation X, (for any multiindex α) is the difference of two optimal polynomials in $\mathcal{W}$ (one of degree $|\alpha|$ and one of degree $|\alpha| + 1$). $\square$

We prove the following simple induction lemma.

Lemma 6.22. *If $p \in \mathcal{H}(n, d)$ and $p'_\epsilon(x_1, \ldots, x_{n-1}) = p(x_1, \ldots, \epsilon x_{n-1}, (1 - \epsilon)x_{n-1})$ (where $0 < \epsilon < 1$), then $p' = p'_\epsilon \in \mathcal{H}(n - 1, d)$ and $N(p') + 1 \leq N(p)$.*

Proof. That $p' \in \mathcal{H}(n - 1, d)$ is obvious. What we want to show is that p' has at least one less term than p. We look at the degree d homogeneous part of p, let us call it p_d, and show that at least two terms have to collapse when we construct p'. Since $p = 1$ when $s = \sum_i x_i = 1$, then $(p - 1) = q(s - 1)$ for some polynomial q. q is of degree $d - 1$ and thus $p_d = q_{d-1}s$. There is a term in q_{d-1}, $c_\alpha x^\alpha$, where $\sum_{i=1}^{n-2} \alpha_i$ is minimal. There may be more than one such term, so let α be such multiindex where α_{n-1} is maximal and β be such multiindex where β_n is maximal. Then once we multiply by s, we see we have the following two nonzero terms of p_d

$$c_\alpha x_{n-1} x^\alpha \text{ and } c_\beta x_n x^\beta. \tag{6.55}$$

Further, it is obvious that these two terms collapse in p'. $\square$

We will be able to improve our induction by one more term using the following lemma.

Lemma 6.23. *If $p \in \mathcal{H}(n, d)$, $p'_\epsilon(x_1, \ldots, x_{n-1}) = p(x_1, \ldots, \epsilon x_{n-1}, (1 - \epsilon)x_{n-1})$ (where $0 < \epsilon < 1$), and p'_ϵ is optimal in $\mathcal{H}(n - 1, d)$ for ϵ in a small interval, then*

p'_ϵ *does not depend on* ϵ *and*

$$p(x_1, \dots, x_n) = p'_\epsilon(x_1, \dots, x_{n-1} + x_n). \tag{6.56}$$

Proof. By Theorem 6.20, it is clear that p'_ϵ cannot be optimal unless it does not depend on ϵ. Assume it does not and call it p'. Now suppose that for an n-multiindex γ, c_γ is the coefficient of p corresponding to the monomial x^γ. Fix certain $(n-2)$-multiindex α and we look at the coefficient of p' corresponding to (α, m), the $n-1$-multiindex we get by concatenating m onto α. This coefficient of p' has to be constant as ϵ changes and is in fact equal to

$$\sum_{k=0}^{m} c_{(\alpha,k,m-k)} \epsilon^k (1-\epsilon)^{m-k}. \tag{6.57}$$

Substitute $y = \epsilon$ and $z = (1-\epsilon)$ above and note that we have a polynomial $q(y,z)$ which is constant on $y + z = 1$. Unless all coefficients are zero we can scale q to be in $\mathcal{H}(2, m)$. We also note that q is homogeneous of degree m, and thus by Proposition 6.6 it has to be $c(y+z)^m$ for some positive constant c. Thus for fixed α, $c_{(\alpha,k,m-k)}$ are just coefficients of $c(y+z)^m$, and hence the lemma follows. $\square$

Example 6.24. An example in $\mathcal{H}(3,3)$ to keep in mind for this is the following, where variables are x, y, z.

$$p(x,y,z) = x^3 + y^3 + z^3 + 3(z^2y + y^2z) + 3(xy + xz). \tag{6.58}$$

p is optimal, and further,

$$p'(x,y) = p(x, \frac{y}{2}, \frac{y}{2}) = x^3 + y^3 + 3xy \tag{6.59}$$

is also optimal. Is is easy to see that $p(x,y,z) = p'(x, y+z)$. Of course, if you collapse x and y (or x and z) instead of y and z you no longer get an optimal polynomial.

Lemma 6.25. *Let* $d > 1$, $q \in \mathcal{H}(n-1, d)$ *be optimal and* $p \in \mathcal{H}(n, d)$, *then* $N(q) + 2 \leq N(p)$.

Proof. Construct p'_ϵ as above. If p'_ϵ is not optimal then we know at least $N(q)+1 \leq N(p'_\epsilon)$, and by Lemma 6.22 $N(q)+2 \leq N(p)$. So assume that p'_ϵ is optimal for all permutations of the variables x. By Lemma 6.23, we find that there exists a polynomial r in one variable such that $r(x_1+\cdots+x_n) = p(x_1,\ldots,x_n)$. It is plain to see that p cannot be optimal; it has at least as many terms as s^d. Further it is clear that p'_ϵ cannot possibly be optimal for the same reason, hence a contradiction. $\square$

If we now apply Theorem 6.4 to the result of Lemma 6.25, we get the following result. Of course, this is not as good as Theorem 6.12, but it illustrates the techniques of this section, which may generalize to the case of polynomial or rational mappings.

Theorem 6.26. *If $p \in \mathcal{H}(n,d)$, then $N(p) \geq \lceil \frac{d+3}{2} \rceil + 2(n-2)$ or in other words $d \leq 2(N(p)-2n)+5$.*

6.6 Rational and polynomial proper mappings

In this section we will look at how the ideas of this chapter could be generalized to polynomial or rational proper mappings. First, we will generalize the idea of pulling back the mappings to source dimensions for which the result is known. We can prove the following result generalizing Lemma 6.9.

Proposition 6.27. *Suppose that Conjecture 6.1 is true for $n = 2$ for rational (resp. polynomial) mappings. If $f\colon \mathbb{B}_n \to \mathbb{B}_N$ $(n \geq 2)$ is a rational (resp. polynomial) proper mapping of degree d, then*

$$d \leq \frac{2N-3}{2n-3}. \tag{6.60}$$

Proof. The proof is similar as that of Lemma 6.9. However, we guarantee that the composition of f with a generalized Faran mapping φ is of the proper degree by applying a unitary matrix U to $\mathbb{C}^n$ and looking at the mapping $f \circ U \circ \varphi$. By choosing the right U, we can ensure that f has a pure monomial of top degree. $\square$

Similarly by composing with generalized Whitney mappings we can prove the following proposition.

Proposition 6.28. *Suppose that Conjecture 6.1 is true for rational (resp. polynomial) mappings, for all $n \geq 3$ such that $n-1$ is prime. Then Conjecture 6.1 is true for all $n \geq 3$ for rational (resp. polynomial) mappings.*

Proof. Suppose $f\colon \mathbb{B}_{k(n-1)+1} \to \mathbb{B}_N$ is a proper rational mapping then by composing with a generalized Whitney mapping of degree k, i.e. $\varphi\colon \mathbb{B}_n \to \mathbb{B}_{k(n-1)+1}$ (and a unitary matrix if needed as before), we can see that the conjecture must hold for the mapping f. □

Note in the proof that if the conjecture is true for $n = 3$, then it is true for all n such that $n = 2k + 1$. However, supposing that $f\colon \mathbb{B}_{2k+2} \to \mathbb{B}_N$, we can still use the generalized Whitney mapping of degree k, with an added zero component to make it map into $\mathbb{B}_{2k+2}$. The calculation of the degree then becomes

$$d = \frac{\deg(\varphi^*(f))}{k} \leq \left\lfloor \frac{N-1}{2} \right\rfloor \frac{1}{k} = \left\lfloor \frac{N-1}{2} \right\rfloor \frac{2}{n-2} \leq \frac{N-1}{n-2}. \tag{6.61}$$

So given the conjecture for $n = 3$, then we get the sharp result for n odd and (6.61) for $n \geq 3$ even. Note that if the conjecture is true for $n = 2$, then (6.60) is better than (6.61) for n even.

Even if we do not have the sharp result for $n = 2$, we can use the idea of composing with a generalized Faran mapping to get a bound for $n \geq 3$. For example, Meylan [Mey06] recently proved the bound

$$d \leq \frac{N(N-1)}{2}, \tag{6.62}$$

for all rational proper mappings. Hence, by applying the same logic as in Proposition 6.27, we get the following theorem.

Theorem 6.29. *Suppose that $f\colon \mathbb{B}_n \to \mathbb{B}_N$ ($n \geq 2$) is a rational proper mapping of degree d, then*

$$d \leq \frac{N(N-1)}{2(2n-3)}. \tag{6.63}$$

Proof. We compose with a generalized Faran mapping φ (and a unitary if necessary) and compute.

$$d = \frac{\deg(\varphi^*(f))}{2n-3} \leq \frac{(N-1)N/2}{2n-3}. \tag{6.64}$$

□

We can also generalize the linear system arising from (6.52). For simplicity, let us for a moment assume $n = 2$, and let $p\colon \mathbb{B}_2 \to \mathbb{B}_N$ be a proper polynomial mapping, and let (z, w) be the coordinates of $\mathbb{C}^2$. A general polynomial mapping then looks like

$$p(z, w) = \big(p_1(z, w), \dots, p_N(z, w)\big), \tag{6.65}$$

where

$$p_\ell(z, w) = \sum_{\substack{0 \leq j,k \\ j+k=d}} c_{\ell jk} z^j w^k, \tag{6.66}$$

for $c_{\ell jk} \in \mathbb{C}$. This is a proper mapping only if

$$\sum_{\ell=1}^{N} p_\ell(z, w)\overline{p_\ell(z, w)} = 1 \quad \text{whenever} \quad z\bar{z} + w\bar{w} = 1. \tag{6.67}$$

We can solve $z\bar{z} + w\bar{w} = 1$ for $\bar{w}$ to get $\bar{w} = \frac{1-z\bar{z}}{w}$ and plugging in to get

$$\sum_{\ell=1}^{N} p_\ell(z, w)\overline{p_\ell}\left(\bar{z}, \frac{1 - z\bar{z}}{w}\right) = 1. \tag{6.68}$$

Treating $\bar{z}$ as a separate variable we have a polynomial that is constant everywhere and hence all the coefficients except the constant coefficient must be zero. This will give us quadratic equations in $c_{\ell jk}$. Let us denote by c_{jk} the vector $(c_{1jk}, \dots, c_{Njk})$. Then the equations become

$$\sum_{k=0}^{d} \left\| c_{0k} \right\|^2 = 1, \tag{6.69}$$

and a number of equations of the form

$$\sum_{j,k,s,t} a_{jkst} \langle c_{jk}, c_{st} \rangle = 0, \tag{6.70}$$

where a_{jkst} are integers, and $\langle \cdot, \cdot \rangle$ is the standard hermitian inner product.

For rational mappings we need to write

$$f = (p_1/q, \dots, p_N/q), \tag{6.71}$$

for p_k and q polynomials. The mapping f is proper if and only if

$$\sum_{\ell=1}^{N} \left| p_\ell \right|^2 = \left| q \right|^2 \quad \text{whenever} \quad z\bar{z} + w\bar{w} = 1. \tag{6.72}$$

Again we will end up with a set of quadratic equations in the coefficients of p_ℓ and q. Satisfying these equations is sufficient for being a rational proper mapping. Hence the general problem is really a problem about quadratic equations. We will look at another way to generalize this system in section 6.7.

Another way to measure the complexity of a polynomial mapping is to count the distinct monomials that appear. If $f \colon \mathbb{B}_n \to \mathbb{B}_N$ is a proper polynomial mapping, we can write it as (using the multiindex notation)

$$\sum_{|\alpha| \leq d} c_\alpha z^\alpha. \tag{6.73}$$

Here, c_α are n-dimensional vectors.

Definition 6.30. We will denote by $\#(f)$ the number of distinct monomials α such that $c_\alpha \neq 0$.

Of course this measure of complexity is not invariant even under unitary transformations of the source. If we wanted to make this invariant we would need to take the minimum over pre and postcomposing by all the automorphisms of the source and target which leave the mapping a polynomial. It is also not hard to see that by choosing the right unitary matrix U on the source such that $\#(f \circ U) = \binom{n+d}{n}$, i.e. all the monomials appear.

A previously known elementary result that relates monomial and polynomial mappings, that has thus far not appeared in the literature as such, is the following proposition. D'Angelo [D'A88] has used the technique of its proof to prove a factorization theorem for polynomials similar to the factorization for monomial mappings.

Proposition 6.31. *Let $f \colon \mathbb{B}_n \to \mathbb{B}_N$ be a proper polynomial mapping. Then there exists a proper monomial mapping $p \colon \mathbb{B}_n \to \mathbb{B}_K$, where $K = \#(f)$, and an $N \times K$ matrix A such that $f = A \circ p$.*

Proof. The crux of this argument is to show that if f has the form (6.73), then the real polynomial

$$\sum_{|\alpha| \leq d} |c_\alpha|^2 \, |z^\alpha|^2 \tag{6.74}$$

corresponds to a monomial mapping in the sense of section 6.2. The matrix A is then constructed by taking as its column vectors the vectors $c_\alpha / |c_\alpha|$.

To see the claim, note that we can fix a z on the sphere, and multiply each element by $e^{i\theta_k}$. Then we look at the squared norm of p at $e^{i\theta} z$, for an n-vector θ, where by this notation we mean $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$. Also by the notation $\alpha\theta$ for a multiindex α we will mean $\alpha_1\theta_1 + \cdots + \alpha_n\theta_n$.

$$\left(\sum_{|\alpha| \leq d} c_\alpha (e^{i\theta} z)^\alpha \right) \overline{\left(\sum_{|\alpha| \leq d} c_\alpha (e^{i\theta} z)^\alpha \right)} = \sum_{\substack{|\alpha| \leq d \\ |\beta| \leq d}} c_\alpha \overline{c_\beta} e^{i(\alpha - \beta)\theta} z^\alpha \bar{z}^\beta. \tag{6.75}$$

As a trigonometric polynomial in θ this has to be identically 1 when z is on the sphere. Hence, each coefficient of the trigonometric polynomial has to be 0, except for the constant coefficient which must be 1. The constant coefficient is precisely the coefficient we get when $\alpha = \beta$. Hence $\sum |c_\alpha|^2 |z^\alpha|^2 = 1$ when z on the sphere and we are done. $\square$

If f is the polynomial mapping we will denote the induced monomial mapping by $M(f)$. As there is ambiguity in what monomial mapping we can take, we will take the unique one with the positive real coefficients.

The corollaries of this result are that we can rewrite all the results about monomial mappings by replacing $N(p)$ with $\#(p)$ as follows.

Corollary 6.32. *Suppose $n \geq 3$ and $p \colon \mathbb{B}_n \to \mathbb{B}_N$ is a proper polynomial mapping. If $d \leq 4$ or $\#(p) < 4n - 3$, then*

$$d \leq \frac{\#(p) - 1}{n - 1}. \tag{6.76}$$

Corollary 6.33. *Suppose $p \colon \mathbb{B}_n \to \mathbb{B}_N$ is a proper polynomial mapping. Then*

$$d \leq \frac{2n(2\#(p) - 3)}{3n^2 - 3n - 2} \leq \frac{4}{3} \frac{2\#(p) - 3}{2n - 3}. \tag{6.77}$$

Corollary 6.34. *Fix d and assume $n \geq 2d^2 + 2d$. If $p \colon \mathbb{B}_n \to \mathbb{B}_N$ is a proper polynomial mapping, then $\#(p) \geq d(n-1) + 1$. Furthermore, if equality holds then $M(p) \in \mathcal{W}$.*

Of course we also get the corollary for the $n = 2$ result of [DKR03].

Corollary 6.35. *Suppose $p\colon \mathbb{B}_2 \to \mathbb{B}_N$ is a proper polynomial mapping, then*

$$d \leq 2\#(p) - 3, \tag{6.78}$$

and this result is sharp.

6.7 Relating monomial and polynomial proper mappings

Let us look at how the results of section 6.5 generalize to polynomial proper mappings. We will relate polynomials in $\mathcal{H}(n,d)$ and $\mathcal{I}(n,d)$ to proper polynomial mappings. We do this by generalizing the procedure of constructing elements of $\mathcal{H}(n,d)$ from proper monomial mappings. Further, we will outline a possible approach to solving Conjecture 6.1 in the case of polynomials.

Let us for the moment work in $\mathbb{C}^2$, with variables (z,w). All of these ideas generalize readily to higher dimensions. Let us first fix an integer $d \geq 1$ and call $\mathcal{Z}$ the monomial mapping

$$(z,w) \mapsto (1, z, z^2, \ldots, z^d, w, zw, z^2w, \ldots, z^{d-1}w^d, z^dw^d). \tag{6.79}$$

A polynomial mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^N$ of degree d can be represented as an N by $(d+1)(d+1)$ matrix C composed with $\mathcal{Z}$, that is $C \circ \mathcal{Z}$ (we will just write $C\mathcal{Z}$). Since f is of degree d, many columns of C will of course be zero. If f is a proper mapping between balls, then we know that

$$\|C\mathcal{Z}\|^2 = 1 \quad \text{whenever} \quad |z|^2 + |w|^2 = 1. \tag{6.80}$$

Let $\langle \cdot, \cdot \rangle$ be the normal euclidean (hermitian) inner product. Then $\|C\mathcal{Z}\|^2 = \langle C^*C\mathcal{Z}, \mathcal{Z}\rangle$. It is not hard to see that any real polynomial of degree less then or equal to d can be written as $\langle M\mathcal{Z}, \mathcal{Z}\rangle$ for a hermitian matrix M. We can therefore write

$$\langle C^*C\mathcal{Z}, \mathcal{Z}\rangle - 1 = \langle Q\mathcal{Z}, \mathcal{Z}\rangle(|z|^2 + |w|^2 - 1). \tag{6.81}$$

Q will furthermore have all the rows and columns corresponding to monomials of degrees d or larger being zero. Let S be the shift matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6.82}$$

It is then not hard to see that due to the ordering of the monomials in $\mathcal{Z}$ that

$$\langle Q\mathcal{Z}, \mathcal{Z}\rangle \, |z|^2 = \langle S^t Q S \mathcal{Z}, \mathcal{Z}\rangle \quad \text{and} \quad \langle Q\mathcal{Z}, \mathcal{Z}\rangle \, |w|^2 = \langle (S^{d+1})^t Q S^{d+1} \mathcal{Z}, \mathcal{Z}\rangle. \tag{6.83}$$

If we let O represent the matrix such that $O_{11} = 1$ and $O_{jk} = 0$ for all other j and k. Then we rewrite (6.81)

$$\langle C^* C \mathcal{Z}, \mathcal{Z}\rangle - \langle O\mathcal{Z}, \mathcal{Z}\rangle = \langle S^t Q S \mathcal{Z}, \mathcal{Z}\rangle + \langle (S^{d+1})^t Q S^{d+1} \mathcal{Z}, \mathcal{Z}\rangle - \langle Q\mathcal{Z}, \mathcal{Z}\rangle \tag{6.84}$$

$$\langle (C^* C - O)\mathcal{Z}, \mathcal{Z}\rangle = \langle (S^t Q S + (S^{d+1})^t Q S^{d+1} - Q) \ \mathcal{Z}, \mathcal{Z}\rangle \tag{6.85}$$

$$0 = \langle (S^t Q S + (S^{d+1})^t Q S^{d+1} - Q - C^* C + O) \ \mathcal{Z}, \mathcal{Z}\rangle. \tag{6.86}$$

This is true for all z and w. By complexifying this equation we have a polynomial in $z, w, \bar{z}$ and $\bar{w}$ that is identically zero and hence all the coefficients are zero. This means that as matrices

$$C^* C = S^t Q S + (S^{d+1})^t Q S^{d+1} - Q + O. \tag{6.87}$$

Hence we only need to search for matrices of the form $S^t Q S + (S^{d+1})^t Q S^{d+1} - Q + O$ which are positive semidefinite. Given a hermitian matrix Q, let us define the mapping

$$A(Q) := S^t Q S + (S^{d+1})^t Q S^{d+1} - Q + O. \tag{6.88}$$

For example, if $d = 2$ and $(Q)_{jk} = q_{jk}$, then $A(Q) = C^*C$ can be written as

$$A(Q) = \begin{pmatrix} q_{11}+1 & q_{12} & 0 & q_{14} & 0 & 0 & 0 & 0 & 0 \\ q_{21} & q_{11}-q_{22} & q_{12} & q_{24} & q_{14} & 0 & 0 & 0 & 0 \\ 0 & q_{21} & q_{22} & 0 & q_{24} & 0 & 0 & 0 & 0 \\ q_{41} & q_{42} & 0 & q_{11}-q_{44} & q_{12} & 0 & q_{14} & 0 & 0 \\ 0 & q_{41} & q_{42} & q_{21} & q_{22}+q_{44} & 0 & q_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{41} & q_{42} & 0 & q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.89}$$

If Q corresponds to a proper ball mapping of $\mathbb{B}_2 \to \mathbb{B}_N$, then $N \geq \operatorname{rank}(A(Q))$. The image of A is an affine real linear subspace of the space of hermitian matrices. That is, the matrices of the form $A(Q) - O$ give a linear subspace. Furthermore, the set of matrices of the form $A(Q)$ that lie in the cone of positive semidefinite matrices is a compact convex set. Let us call this set $\mathcal{K}$.

Hence the matrices that minimize N are those that are the extremal points of this convex set. A priory, these extreme points could correspond to proper polynomial mappings of degree less then d. We have, however, the following proposition. We will say that a proper polynomial mapping $f\colon \mathbb{B}_n \to \mathbb{B}_N$ of degree d is *optimal*, if it minimizes N for this fixed d. Let us write $N(n, d)$ for this minimal N.

Proposition 6.36. *For a fixed n and d, the set of optimal proper polynomial mappings is a subset of the set of extreme points of $\mathcal{K}$.*

Proof. What we show is that there is no linear one parameter family of optimal proper polynomial mappings in the following sense. Fix $d \geq 1$, and let $N = N(n, d)$. Then there do not exist two positive semidefinite matrices E and F such that for $t \in [0, 1]$, $tE+(1-t)F$ is of rank N and corresponds to a proper polynomial mapping of degree d.

As we said before, if we are to minimize the rank of $A(Q)$ we must be at an extreme point. We must however deal with the case where the degree drops at the extreme point. Suppose that $tE + (1 - t)F$ corresponds to a degree d proper mapping for $t \in (0,1]$ and suppose that E corresponds to a lower degree proper polynomial mapping. We notice that for $t \geq 1$, $tE + (1-t)F$ still corresponds to a proper polynomial mapping and furthermore there exists a $t_0 \geq 1$ such that $t_0 E + (1+t_0)F$ is still positive semidefinite, and of lower rank than F. As the diagonal elements of E that correspond to degree d monomials are all zero, it is clear that these cannot all be zero in $t_0 E + (1+t_0)F$ and hence $t_0 E + (1+t_0)F$ corresponds to a proper polynomial mapping with lower target dimension and we are done. $\square$

Notice that any diagonal of $A(Q)$ depends only on the corresponding diagonal of Q. We make the following observation. Suppose that $A(Q)$ has the following form

$$A(Q) = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & \ldots \\ \bar{b}_1 & a_2 & b_2 & c_2 & \ldots \\ \bar{c}_1 & \bar{b}_2 & a_3 & b_3 & \ldots \\ \bar{d}_1 & \bar{c}_2 & \bar{b}_3 & a_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.90}$$

Then notice that the vector $a = (a_1, a_2, \ldots)$ corresponds to a real polynomial in $\mathcal{H}(n,d)$ as each a_j corresponds to a monomial in $\mathcal{Z}$. To see that this polynomial is in $\mathcal{H}(n,d)$, we notice that first since $A(Q)$ must be positive semidefinite, then $a_j \geq 0$. Second if we zero out all the off diagonal elements of Q we precisely get the corresponding proper monomial mapping. That is, if Q_d is the matrix of diagonal elements of Q_d, we notice that the following matrix also corresponds to a monomial mapping.

$$A(Q_d) = \begin{pmatrix} a_1 & 0 & 0 & 0 & \ldots \\ 0 & a_2 & 0 & 0 & \ldots \\ 0 & 0 & a_3 & 0 & \ldots \\ 0 & 0 & 0 & a_4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.91}$$

This gives a completely algebraic and elementary proof Proposition 6.31 from section 6.6.

However, we get something even more striking. We notice that the vectors

$\operatorname{Re} b = (\operatorname{Re} b_1, \operatorname{Re} b_2, \dots)$ and $\operatorname{Im} b = (\operatorname{Im} b_1, \operatorname{Im} b_2, \dots)$ each correspond to two different polynomials in $\mathcal{I}(n, \leq d)$. For example, for $\operatorname{Re} b$, we can easily find two different Q_1 and Q_2 such that

$$S^t Q_1 S + (S^{d+1})^t Q_1 S^{d+1} - Q_1 = \begin{pmatrix} \operatorname{Re} b_1 & 0 & 0 & 0 & \dots \\ 0 & \operatorname{Re} b_2 & 0 & 0 & \dots \\ 0 & 0 & \operatorname{Re} b_3 & 0 & \dots \\ 0 & 0 & 0 & \operatorname{Re} b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6.92}$$

and

$$S^t Q_2 S + (S^{d+1})^t Q_2 S^{d+1} - Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & \operatorname{Re} b_1 & 0 & 0 & \dots \\ 0 & 0 & \operatorname{Re} b_2 & 0 & \dots \\ 0 & 0 & 0 & \operatorname{Re} b_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.93}$$

Just take the superdiagonal in Q, take the real part of it, and shift it to the diagonal in two different ways. We can do similarly with $\operatorname{Im} b$, and also with c, d, and all other diagonals. Since $A(Q)$ must be positive semidefinite we find that $\operatorname{Re} b_k$ is nonzero only if both a_k and a_{k+1} are nonzero.

This means that $\frac{1}{1-t} a + t \operatorname{Re} b$ for example is a one parameter family of polynomials in $\mathcal{H}(n, d)$ for small t. To see that this is a nontrivial family notice again that $A(Q)$ being positive semidefinite implies that $\operatorname{Re} b$ must have at least one less nonzero element then a.

Denote by $\#(f)$ again the number of distinct monomials in f, and by $M(f)$ the monomial mapping induced by f (that is the mapping induced by the vector a above). We then have the following proposition by the above argument and Theorem 6.20. As we noticed before, $\mathcal{H}(n, d)$ is a convex set. We will say that a monomial mapping is *isolated* if it is an extreme point of $\mathcal{H}(n, d)$. Then Theorem 6.20 just says that an optimal monomial mapping is an extreme point of $\mathcal{H}(n, d)$. What we proved above is that if the vector a represents an extreme point of $\mathcal{H}(n, d)$, then all the off diagonal entries must be zero.

Proposition 6.37. *Let $f: \mathbb{B}_n \to \mathbb{B}_N$ be a proper polynomial mapping. Further, suppose that $M(f)$ is either optimal (i.e. $\#(f)$ is minimal) or isolated, then f is a monomial mapping.*

The following is a reasonable conjecture to make.

Conjecture 6.38. *Let $f: \mathbb{B}_n \to \mathbb{B}_N$ be a proper polynomial mapping of degree d. Suppose that $M(f)$ is not isolated, then there exists a proper polynomial mapping $g: \mathbb{B}_n \to \mathbb{B}_N$ of degree d, such that $\#(g) < \#(f)$.*

The intuition is that if there is a nontrivial one parameter family $p(t) \in \mathcal{H}(n, d)$, such that $p(0)$ corresponds to $M(f)$, then it is reasonable to expect there to be a one parameter family of proper mappings $f_t: \mathbb{B}_n \to \mathbb{B}_N$ such that $p(t)$ corresponds to $M(f_t)$. This family f_t need not be affine, and will never be affine if f was optimal for example. If f_t exists as above, then this implies Conjecture 6.38, since we can travel along this one parameter family in the same manner as in the proof of Theorem 6.20 to produce the mapping g.

Supposing that Conjecture 6.38 is true, we take the f that minimizes $\#(f)$ for a fixed N. Then we know that this f is a monomial mapping by Proposition 6.37. By an argument similar to that in Proposition 6.27, we get the following.

Proposition 6.39. *Let $f: \mathbb{B}_n \to \mathbb{B}_N$ be a proper polynomial mapping of degree d. If Conjecture 6.38 is true, then*

$$d \leq \frac{2N - 3}{2n - 3}, \tag{6.94}$$

and for $n \geq 3$, if $d \leq 4$, or $N < 4n - 3$, or $n \geq 2d^2 + 2d$, then

$$d \leq \frac{N - 1}{n - 1}. \tag{6.95}$$

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