QUASICONFORMAL EXTENSIONS OF QUASISYMMETRIC MAPPINGS

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CHAPTER 1

INTRODUCTION

When looking at mappings of the plane, the nicest mappings one can imagine are conformal mappings. That is mappings that preserve angles between lines. These mappings are really the complex analytic functions that have many nice properties. However they also have many restrictions. For example we cannot conformally map a rectangle onto another rectangle with a different aspect ratio while mapping corners to corners. So the question is how nice can mappings be and still preserve some of the nice properties. If we relax the restriction that the mapping must preserve angles and say that the mapping must not change angles *too much*, then we have what is called a *quasiconformal mapping.* In 1928, H. Grötzsch introduced this problem and also the first definition of such a mapping [1]. What is of particular interest is the behavior of these mappings close to the boundary. The primary question that we will deal with is the question of which boundary correspondence maps have a quasiconformal extension to the whole domain. Since there can be many such extensions with the same boundary condition, we study two explicit constructions, the first done by Beurling and Ahlfors [2] and the second by Douady and Earle [4]. We will then present applications of these ideas to related theory.

1.1 Quasiconformality

We wish to now rigorously define what is a quasiconformal mapping. Suppose f is a differentiable topological mapping $f : \mathbb{C} \to \mathbb{C}$. And we define the partial derivatives in terms of the real and complex parts where $f(x, y) = u(x, y) + i\,v(x, y)$ as

$$\begin{aligned} f_z &= 1/2(u_x + v_y) + i/2(v_x - u_y), \\ f_{\overline{z}} &= 1/2(u_x - v_y) + i/2(v_x + u_y). \end{aligned} \tag{1.1}$$

We will only consider sense preserving mappings, that is mappings that preserve orientation of Jordan curves. For such mappings we have that $|f_{\overline{z}}| < |f_z|$ because only that way do we have a positive Jacobian which is defined [1] as $J = |f_z| - |f_{\overline{z}}|$.

So let us now define the dilatation of a function at a point [1].

Definition 1.1 *Dilatation of a function at a particular point is defined as*

$$D_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1.$$

A mapping is conformal if and only if $D_f(z)$ is 1 for all z, since then $|f_{\overline{z}}| = 0$. Geometrically the dilatation is the ratio of the major and minor axis of the infinitesimal ellipse that is the image of the infinitesimal circle under the mapping f. Let us look at a simple example of a linear function, which takes all circles to ellipses. Let's consider (using two dimensional notation rather than complex notation for simplicity)

$$f(x,y) = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{1.2}$$

This function will take all circles to ellipses. Figure 1.1 shows for example the result of applying this map to the unit circle.

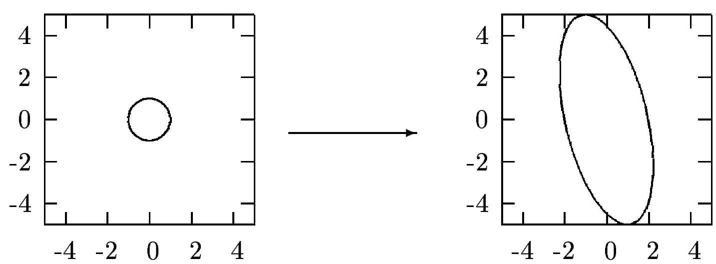


Figure 1.1: The Graph of Where the Linear Map $f(x,y)$ Takes the Unit Circle

To calculate the dilatation of this function we first calculate the partial derivatives with respect to z and $\overline{z}$

$$|f_z| = |1/2(u_x + v_y) + i/2(v_x - u_y)| = |1/2(1+4) + i/2(3+2)| = \sqrt{25/2} = 5/\sqrt{2},$$

$$|f_{\overline{z}}| = |1/2(u_x - v_y) + i/2(v_x + u_y)| = |1/2(1-4) + i/2(3-2)| = \sqrt{5/2} = \sqrt{5}/\sqrt{2}.$$

Now we can calculate the dilatation itself. We note that the dilatation of this function is the same at every point and this is

$$D_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} = \frac{5/\sqrt{2} + \sqrt{5}/\sqrt{2}}{5/\sqrt{2} - \sqrt{5}/\sqrt{2}} = \frac{5+\sqrt{5}}{5-\sqrt{5}} \approx 2.62\,.$$

We are really interested in the uniform bound on the dilatation over the entire domain, if in fact it is actually uniformly bounded. So we can now define what it means for a mapping to be *quasiconformal* [1] [2].

Definition 1.2 *A mapping is* quasiconformal *if the dilatation is uniformly bounded over the entire domain of the mapping. We call*

$$K_f = \sup_z D_f(z)$$

*the maximal dilatation of f. A mapping is K-*quasiconformal *if the maximal dilatation of the function is K.*

From this definition we can see that the linear function we defined above is quasiconformal and that $K_f \approx 2.62$, since the dilatation of that function is constant. Also any conformal mapping is 1-quasiconformal as its dilatation is 1 everywhere and therefore constant.

There are some related quantities to dilatation. The first is called the *small dilatation* and is defined in [1] as:

Definition 1.3 *The small dilatation of a sense-preserving quasiconformal map at a particular point is defined as*

$$d_f(z) = \frac{|f_{\bar{z}}|}{|f_z|} < 1.$$

Note that this quantity is less than 1 only for sense preserving maps. This can be seen by looking at the Jacobian $J = |f_z| - |f_{\bar{z}}|$ and noting that it must be then positive for sense preserving maps. If it would equal to 1, then the dilatation would not be bounded at that point. Also to note that some books, such as [13] use the notation κ-quasiconformal to mean that the $d_f \leq \kappa$ for all z. This quantity is related to D_f (the dilatation) by

$$D_f = \frac{1 + d_f}{1 - d_f}, \qquad d_f = \frac{D_f - 1}{D_f + 1}.$$

If the mapping is conformal if $D_f = 1$ and $d_f = 0$. This quantity may be more convenient sometimes because of the simpler definition, although it doesn't have such a nice geometric description. As an example we look at (1.2) and calculate the small dilatation

$$d_f = \frac{|f_{\overline{z}}|}{|f_z|} = \frac{\sqrt{5}/\sqrt{2}}{5/\sqrt{2}} = \frac{\sqrt{5}}{5} \approx 0.447.$$

Another related quantity is called the *complex dilatation* and is defined as follows.

Definition 1.4 *The complex dilatation of a function at a particular point is defined as*

$$\mu_f(z) = \frac{f_{\overline{z}}(z)}{f_z(z)}.$$

Now obviously we have that $|\mu_f| = d_f$. This complex dilatation is seen in the *Beltrami differential equation*

$$f_{\overline{z}}(z) = \mu_f(z) f_z(z).$$

And so a quasiconformal mapping is a solution to the Beltrami equation where the μ_f is uniformly bounded by some $k < 1$. Obviously it can be seen that for a conformal map, μ_f must equal to 0 for all z.

1.2 The M-condition

It is known that a quasiconformal mapping from the unit disc onto itself is continuous on the boundary [2]. This means that a quasiconformal mapping induces a topological mapping between the boundaries. It is also known that composition with conformal mappings does not change the maximal dilatation [1], so this means that looking at the mappings of the unit disc to itself is the same as looking at the mappings from the upper halfplane to itself. These mappings have the advantage of having its boundary on the real line and thus the function that defines boundary correspondence is a real function onto the real line, which is also one to one. So let's define a so called *M-condition* [1] [2] on any such function $\mu(x)$ that fixes ∞.

Definition 1.5 *A function $\mu(x)$ satisfies the M-condition if there is some M such that for any x and t,*

$$\frac{1}{M} \leq \frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)} \leq M.$$

Geometrically this means that the ratio of the length of the intervals $\mu[(x-t, x)]$ and $\mu[(x, x+t)]$ is bounded. This also implies that the function must be one to one, since if $\mu(x) = \mu(y)$, then if we let $t = x - y$, then the M-condition would not be satisfied since then $\mu(x) - \mu(x-t)$ would be zero and thus the ratio would not be defined.

Definition 1.6 *A function that satisfies this condition is also called* quasisymmetric. *In particular if a function satisfies the M-condition for a particular M, then the function is called* M-quasisymmetric.

Let us look at some examples of this. It is fairly trivial to see that all linear maps satisfy the M-condition with $M = 1$. So let's look at a little less trivial example of $\mu(x) = x^3$. We note that we have that $\mu(kx) = k^3\mu(x)$ and so we can really factor any multiple out of the M-condition. This means that we really only need to look at the case where $x = 1$ and just let t vary. So we have (noting that $t \neq 0$)

$$\frac{\mu(1+t) - \mu(1)}{\mu(1) - \mu(1-t)} = \frac{t^3 + 3t^2 + 3t}{t^3 - 3t^2 + 3t} = \frac{t^2 + 3t + 3}{t^2 - 3t + 3}.$$

The denominator is always positive and never zero, and so this is a continuous function. In fact it is easy to see that it approaches 1 as we approach $t = 0$. The limit as t approaches ∞ (in both positive and negative directions) goes to 1, and so this function achieves both a maximum and a minimum, both of which are positive. In fact with a bit of calculus we can see that these are achieved at $\sqrt{3}$ and $-\sqrt{3}$. So we in fact have that the M-condition is satisfied for $M = \frac{6+3\sqrt{3}}{6-3\sqrt{3}}$.

It is also useful to see examples of functions which do not satisfy the M-condition. Consider $\mu(x) = e^x - e^{-x}$. This is a one-to-one and onto function of the real line onto itself. Let us now look at the M-condition in the case where $t = x$,

$$\begin{aligned}\frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)} &= \frac{(e^{2x} - e^{-2x}) - (e^x - e^{-x})}{(e^x - e^{-x}) - 0} \\ &= \frac{(e^x - e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})}{e^x - e^{-x}} \\ &= e^x + e^{-x} - 1.\end{aligned}$$

We can see that this is not bounded as x goes to ∞, and thus this function does not satisfy the M-condition and is therefore not quasisymmetric.

We will need a few definitions and lemmas that arise from the M-condition. First it is useful to consider a so called *normalized* quasisymmetric function.

Definition 1.7 *We will call an M-quasisymmetric function μ* normalized *if $\mu(0) = 0$ and $\mu(1) = 1$.*

Lemma 1.1 *For a normalized M-quasisymmetric μ we have*

$$\left(1+\frac{1}{M}\right)^n \leq \mu(2^n) \leq (M+1)^n \quad \textit{(for all integers } n > 0\textit{)}.$$

Proof: We know μ satisfies the M-condition and is normalized so we plug $x = 2^n$ and $t = 2^n$ into the M-condition to get (noting that $\mu(0) = 0$ and $\mu(2^n)$ must be positive)

$$\begin{aligned} \frac{1}{M} \leq \frac{\mu(2^{n+1}) - \mu(2^n)}{\mu(2^n) - \mu(0)} &\leq M, \\ \frac{\mu(2^n)}{M} + \mu(2^n) \leq \mu(2^{n+1}) &\leq \mu(2^n)M + \mu(2^n), \\ \mu(2^n)\left(1+\frac{1}{M}\right) \leq \mu(2^{n+1}) &\leq \mu(2^n)(M+1). \end{aligned} \tag{1.3}$$

With $n = 0$ we get (noting that $\mu(1) = 1$)

$$\left(1+\frac{1}{M}\right) \leq \mu(2) \leq (M+1).$$

And now using induction on n, we suppose that $\left(1+\frac{1}{M}\right)^n \leq \mu(2^n) \leq (M+1)^n$ holds then by (1.3) we have that

$$\begin{aligned} \left(1+\frac{1}{M}\right)^{n+1} &\leq \left(1+\frac{1}{M}\right)^n \left(1+\frac{1}{M}\right) \\ &\leq \mu(2^n)\left(1+\frac{1}{M}\right) \\ &\leq \mu(2^{n+1}) \\ &\leq \mu(2^n)(M+1) \\ &\leq (M+1)(M+1)^n \\ &\leq (M+1)^{n+1}. \end{aligned}$$

QED!

Lemma 1.2 *For a normalized M-quasisymmetric μ we have*

$$-M(M+1)^n \leq \mu(-2^n) \leq -\frac{1}{M}\left(1+\frac{1}{M}\right)^n \quad \textit{(for all integers } n > 0\textit{)}.$$

Proof: We plug $x = 0$ and $t = 2^n$ into the M-condition getting

$$\begin{aligned} \frac{1}{M} &\le \frac{\mu(2^n) - \mu(0)}{\mu(0) - \mu(-2^n)} \le M, \\ \frac{1}{M} &\le \frac{\mu(2^n)}{-\mu(-2^n)} \le M, \\ \frac{1}{M}\mu(2^n) &\le -\mu(-2^n) \le M\mu(2^n). \end{aligned} \tag{1.4}$$

Using Lemma 1.1 on the page before and (1.4) we then obtain

$$\frac{1}{M}\left(1 + \frac{1}{M}\right)^n \le \frac{1}{M}\mu(2^n) \le -\mu(-2^n) \le M\mu(2^n) \le M(M+1)^n.$$

QED!

CHAPTER 2

THEOREM OF BEURLING AND AHLFORS

2.1 Boundary Condition

The main result of the Beurling and Ahlfors paper [2] was that a mapping μ of the real line to itself satisfying the M-condition is a sufficient and necessary condition for the existence of a quasiconformal mapping from the upper halfplane to itself with μ as the boundary correspondence.

Theorem 2.1 (Beurling and Ahlfors) *There exists a quasiconformal extension of the upper halfplane to itself if and only if the boundary correspondence mapping $\mu(x)$ satisfies the M-condition. Furthermore there exists an extension of μ to a quasiconformal map of the upper halfplanes such that the maximal dilatation of the extension depends only on M, and not on μ.*

Beurling and Ahlfors proved that the dilatation must be less than M^2, however there are better bounds in terms of M and they are given in Section 4.1 on page 45. They also proved that a quasiconformal mapping must have a maximal dilatation greater than or equal to $1 + A \log M$ where A is a constant, about 0.2284. We will prove the sufficiency condition by constructing the extension as given by Ahlfors [1].

We will not prove all of Theorem 2.1, but just the sufficiency condition. This is because we are really interested in the explicit construction of the extension and the proof of the necessity is rather involved. Proof can be seen in either [2] or [1]. The sufficiency part, including a bound on the maximal dilatation, will be proved in Section 2.3.

As an example take $\mu(x) = 2x$. We know that this function satisfies the M-condition with $M = 1$, which is obvious from the M-condition. So according to Beurling and Ahlfors there exists a quasiconformal mapping with a maximal dilatation of 1 (since it must be greater than or equal to 1 and less than or equal to 1). Since a quasiconformal mapping with a maximal dilatation of 1 is conformal, this mapping will in fact also be conformal. It is easy to see that such a mapping is just $f(z) = 2z$.

In fact we can expand on this example. If μ satisfies the M-condition with $M = 1$, then

$$1 \leq \frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)} \leq 1,$$

which implies that

$$\mu(x+t) - \mu(x) = \mu(x) - \mu(x-t).$$

So μ must be a linear mapping of the form $\mu(x) = ax+b$ where $a > 0$ since μ is increasing. We know that there exists a quasiconformal mapping of a dilatation of at most $M^2 = 1$, that is a conformal mapping, which in fact is the mapping $\phi(z) = az + b$. It is also easy to see that this must be the only conformal mapping of the upper halfplane to itself that fixes ∞.

2.2 The M-condition as a Compactness Condition

It is also possible to give a more qualitative meaning to the M-condition. First let's define a mapping $\mu(x)$ of the real line onto itself to be *normalized* if and only if $\mu(0) = 0$ and $\mu(1) = 1$. Now let's consider the following compactness condition [2]:

Definition 2.1 *A family $\mathcal{F}$ of mappings μ which is closed under composition with affine transformations (of the form $ax + b$, where $a > 0$) satisfies the compactness condition if every infinite set of normalized mappings $\mu \in \mathcal{F}$ contains a sequence $\{\mu_n\}_1^\infty$ which converges to a strictly increasing limit function.*

The second result of Beurling and Ahlfors in their paper is that the above compactness condition is equivalent to the M-condition.

Theorem 2.2 *The mappings μ in a family $\mathcal{F}$, which is closed under affine transformations, satisfy the M-condition, all for the same M, if and only if the compactness condition (Definition 2.1) is satisfied. In fact every infinite sequence of normalized mappings has a subsequence that converges to a mapping satisfying the M-condition.*

This will in fact mean that the set of all normalized mappings that satisfy the M-condition for some M is a compact subset of all the mappings that satisfy the M-condition for that same M. Before we can prove this result we must prove a small but useful lemma.

Lemma 2.1 *Suppose μ is a mapping which satisfies the M-condition, then composing it with two affine mappings does not modify the M-condition. That is $A\mu(ax+b)+B$ where $A, a > 0$ satisfies the same M-condition as μ.*

Proof: From

$$\frac{1}{M} \leq \frac{\mu((ax+b)+at)-\mu(ax+b)}{\mu(ax+b)-\mu((ax+b)-at)} \leq M,$$
$$\frac{1}{M} \leq \frac{\mu(a(x+t)+b)-\mu(ax+b)}{\mu(ax+b)-\mu(a(x-t)+b)} \leq M,$$
$$\frac{1}{M} \leq \frac{A\mu(a(x+t)+b)+B-(A\mu(ax+b)+B)}{A(\mu(ax+b)+B)-(A\mu(a(x-t)+b)+B)} \leq M,$$

we can see that since x and t are arbitrary, we could just as well take $(ax+b)$ instead of x and at instead of t and thus $A\mu(ax+b)+B$ satisfies the same M-condition if μ itself satisfies it. QED!

Proof of Theorem 2.2: First we suppose that the M-condition is satisfied for some M for all $\mu \in \mathcal{F}$. So suppose μ is normalized mapping in $\mathcal{F}$. Now we look at the M condition if we let $x = t = 2^{-n}$ for any n. Now notice that since μ is normalized then $\mu(2^{-n})$ is positive. And so

$$\frac{1}{M} \leq \frac{\mu(x+t)-\mu(x)}{\mu(x)-\mu(x-t)},$$
$$\frac{1}{M} \leq \frac{\mu(2^{-n+1})-\mu(2^{-n})}{\mu(2^{-n})},$$
$$\frac{\mu(2^{-n})}{M}+\mu(2^{-n}) \leq \mu(2^{-n+1}),$$
$$\left(1+\frac{1}{M}\right)\mu(2^{-n}) \leq \mu(2^{-n+1}),$$
$$\mu(2^{-n}) \leq \left(\frac{M}{M+1}\right)\mu(2^{-n+1}). \tag{2.1}$$

We know that $\mu(1) = 1$ and so for $n = 1$ from (2.1) we get that

$$\mu(2^{-1}) \leq \left(\frac{M}{M+1}\right)\mu(2^{0}) = \left(\frac{M}{M+1}\right).$$

So we can now use induction on n, going in the positive direction. Suppose that

$$\mu(2^{-n+1}) \leq \left(\frac{M}{M+1}\right)^{n-1}$$

holds, then we get that

$$\begin{aligned}\mu(2^{-n}) &\leq \left(\frac{M}{M+1}\right)\mu(2^{-n+1}) \\ &\leq \left(\frac{M}{M+1}\right)\left(\frac{M}{M+1}\right)^{n-1} \\ &= \left(\frac{M}{M+1}\right)^{n}. \end{aligned} \tag{2.2}$$

Since $\mu(x)$ is increasing, then we have this inequality for all x less than or equal to 2^{-n} as well.

We now combine Lemma 1.1 on page 6 and Lemma 1.2 on page 6 to get

$$-M(M+1)^n \leq \mu(x) \leq (M+1)^n \qquad (\text{for } -2^n \leq x \leq 2^n),$$

or

$$|\mu(x)| \leq M(M+1)^n \qquad (\text{for } x \in [-2^n, 2^n]). \tag{2.3}$$

This means that on the interval $[-2^n, 2^n]$ the normalized $\mu \in \mathcal{F}$ are uniformly bounded, since the M is the same for all μ in the family $\mathcal{F}$. Since any compact set $E \subset \mathbb{R}$ is bounded, we can find n large enough such that all normalized $\mu(x)$ in $\mathcal{F}$ are uniformly bounded on this compact set.

Now we use the fact that $\mathcal{F}$ is closed under composition by affine transformations and so for any a we let $S = \frac{x-\mu(a)}{\mu(a+1)-\mu(a)}$ and let $T = x + a$, then

$$(S\mu T)(x) = \frac{\mu(a+x)-\mu(a)}{\mu(a+1)-\mu(a)}.$$

It is easy to see that $(S\mu T)(0) = 0$ and that $(S\mu T)(1) = 1$, and so $S\mu T$ is normalized. We plug $S\mu T$ into (2.2) and thus for any $0 \leq x \leq 2^{-n}$ we have that

$$\begin{aligned}\frac{\mu(a+x)-\mu(a)}{\mu(a+1)-\mu(a)} &\leq \left(\frac{M}{M+1}\right)^n, \\ \mu(a+x)-\mu(a) &\leq (\mu(a+1)-\mu(a))\left(\frac{M}{M+1}\right)^n. \end{aligned} \tag{2.4}$$

We now wish to show that for any compact set E we have equicontinuity. First we find an N_a such that we have that $E \subset [-2^{N_a}, 2^{N_a}]$, and also such that for every $x \in E$ we have that $x - 1 \in [-2^{N_a}, 2^{N_a}]$ and $x + 1 \in [-2^{N_a}, 2^{N_a}]$.

Suppose that $a \in E$, then by (2.3) we have that $|\mu(a)| \leq M(M+1)^{N_a}$ and also that $|\mu(a+1)| \leq M(M+1)^{N_a}$. Now we wish to show that $\mu(a+1) - \mu(a)$ is bounded for all a in E. So

$$|\mu(a+1) - \mu(a)| \leq |\mu(a+1)| + |\mu(a)| \leq 2M(M+1)^{N_a}.$$

So $\mu(a+1) - \mu(a)$ is bounded for all $a \in E$. Since μ is strictly increasing we have that $\mu(a+1) - \mu(a)$ is positive and so for all $a \in E$ we have

$$0 \leq \mu(a+1) - \mu(a) \leq 2M(M+1)^{N_a}.$$

Now we look at (2.4) and we can see that

$$\mu(a+x) - \mu(a) \leq (\mu(a+1) - \mu(a)) \left(\frac{M}{M+1}\right)^n \leq 2M(M+1)^{N_a} \left(\frac{M}{M+1}\right)^n.$$

Since we can make the right side arbitrarily small by taking large enough n since $\frac{M}{M+1}$ is less than 1 and $2M(M+1)^{N_a}$ is fixed, and since μ is strictly increasing we can see that μ must be continuous. Since this is true for all normalized $\mu \in \mathcal{F}$ then the normalized $\mu \in \mathcal{F}$ are equicontinuous over E.

We now apply the Arzelà-Ascoli theorem (Theorem A.1 on page 60). We notice that on any compact set E the normalized mappings in $\mathcal{F}$ are equicontinuous and also uniformly bounded. So this implies that on E, each sequence of normalized mappings has a subsequence that converges uniformly on E.

So take any infinite set of normalized mappings and take a sequence from this set which converges uniformly over some compact set E. Specifically let $\{\mu_n(x)\}_1^\infty$ converge uniformly (over E) to $\mu(x)$. Since $\mu_n(0) = 0$ and $\mu_n(1) = 1$ for all n, then the limit function $\mu(x)$ is also normalized and $\mu(0) = 0$ and $\mu(1) = 1$. Also a uniform limit of continuous functions is continuous so the limit function $\mu(x)$ is continuous (Theorem A.2 on page 60).

Now we wish to show that $\mu(x)$ satisfies the M-condition. We know that all the $\mu_n(x)$ satisfy the M-condition, all for the same M. Now given an $\epsilon > 0$, we know we can find an N such that for all $n \geq N$, we have that $|\mu_n(x) - \mu(x)| < \epsilon$ for all x. So looking now at the right side of the M condition for $\mu_n(x)$ for $n \geq N$ we have that

$$\begin{aligned}
\frac{\mu_n(x+t) - \mu_n(x)}{\mu_n(x) - \mu_n(x-t)} &\leq M, \\
\mu_n(x+t) - \mu_n(x) &\leq M(\mu_n(x) - \mu_n(x-t)), \\
\mu(x+t) - \mu(x) &\leq M(\mu_n(x) - \mu_n(x-t)) + 2\epsilon, \\
\mu(x+t) - \mu(x) &\leq M(\mu(x) - \mu(x-t) + 2\epsilon) + 2\epsilon, \\
\mu(x+t) - \mu(x) &\leq M(\mu(x) - \mu(x-t)) + M2\epsilon + 2\epsilon, \\
\frac{\mu(x+t) - \mu(x) - M2\epsilon - 2\epsilon}{\mu(x) - \mu(x-t)} &\leq M.
\end{aligned}$$

And since M is fixed and ϵ can get arbitrarily small, we have the right side of the M condition satisfied. The left side follows in a similar way. And thus the limit function also satisfies the M-condition.

If the M-condition holds for any limit of such a sequence then this limit must be strictly monotone. Thus if M-condition is satisfied, then the compactness condition (Definition 2.1 on page 9) holds.

Conversely suppose that the compactness condition holds, then we want to show that this implies the M-condition. Now set

$$\alpha = \inf \mu(-1), \qquad \beta = \sup \mu(-1),$$

where μ ranges over all normalized mappings in $\mathcal{F}$. By the compactness condition there exists a sequence of normalized mappings $\mu_n \in \mathcal{F}$, such that $\mu_n(-1) \to \beta$ and such that the limit of this sequence is a normalized strictly monotone function. This implies that $\beta < 0$. Similarly there exists a sequence of normalized mappings $\mu_n \in \mathcal{F}$, such that $\mu_n(-1) \to \alpha$ and such that the limit of this sequence is a normalized strictly monotone function. This implies that $\alpha > -\infty$.

Consider, for any $\mu \in \mathcal{F}$, the mapping

$$v(x) = \frac{\mu(y+tx) - \mu(y)}{\mu(y+t) - \mu(y)}.$$

It is easy to see that v is a normalized mapping, and it is also in $\mathcal{F}$, since it is just a composition of two affine transformations and the mapping μ, similarly as we did before.

Now if we look at $v(-1)$ we know that it must be between α and β, since that was the inf and the sup of all the values at -1. So we have that

$$\begin{aligned} \alpha \leq \frac{\mu(y-t)-\mu(y)}{\mu(y+t)-\mu(y)} &\leq \beta, \\ -\alpha \geq \frac{\mu(y)-\mu(y-t)}{\mu(y+t)-\mu(y)} &\geq -\beta, \\ -\frac{1}{\alpha} \leq \frac{\mu(y+t)-\mu(y)}{\mu(y)-\mu(y-t)} &\leq -\frac{1}{\beta}, \end{aligned}$$

and if we take $M = \max\left(-\alpha, -\frac{1}{\beta}\right)$ then clearly

$$\frac{1}{M} \leq \frac{\mu(y+t)-\mu(y)}{\mu(y)-\mu(y-t)} \leq M.$$

And so if the compactness condition is satisfied, then the M-condition is also satisfied for all (not just normalized) mappings $\mu \in \mathcal{F}$. QED!

Now Theorem 2.1 on page 8 and Theorem 2.2 on page 9 imply the following corollary [2] which gives a qualitative condition for the boundary correspondence of a quasiconformal mapping.

Corollary 2.1 *A boundary mapping μ can be extended to a quasiconformal mapping of the upper halfplanes if and only if the family of all mappings $S\mu T$ satisfies the compactness condition (Definition 2.1 on page 9)*

Proof: Now this is easy to see. Since we can take our μ and normalize it by use of the affine mappings S and T as defined earlier we get a family of mappings. And if this family of mappings satisfies our compactness condition it will imply that μ will satisfy the M-condition, and thus there exists a quasiconformal map of the upper halfplanes. QED!

2.3 Extensions of the Ahlfors-Beurling Type

Now we come to the actual construction of the extension given a mapping which satisfies the M-condition. For this we use the definition due to Ahlfors [1], which is slightly simpler than the definition in the original paper by Beurling and Ahlfors [2] since that one introduced an extra parameter to minimize the maximal dilatation. Suppose we are given a mapping μ which satisfies the M-condition, we will construct a quasiconformal mapping which maps the upper halfplane to itself with μ as the boundary correspondence. This formula will be given in the two dimensional notation.

Definition 2.2 (Beurling-Ahlfors Extension) *Suppose that μ is a quasisymmetric mapping of the real line onto itself which fixes ∞. Then we define a map $\phi(x, y) = u(x, y) + iv(x, y)$ of the upper halfplane to itself as*

$$\begin{aligned} u(x,y) &= \frac{1}{2y}\int_{-y}^{y} \mu(x+t)dt, \\ v(x,y) &= \frac{1}{2y}\int_{0}^{y} (\mu(x+t) - \mu(x-t))dt. \end{aligned} \tag{2.5}$$

First it is easy to see that when y tends to 0, then $u(x, y)$ tends to $\mu(x)$. It is also easy to see that $v(x, y)$ tends to 0 as y tends to 0. Furthermore $v(x, y)$ is positive when y is positive as $\mu(x + t) - \mu(x - t)$ is always a positive quantity. This means that this function satisfies the boundary condition and maps the upper halfplane to the upper halfplane.

As an example let's use the mapping $\mu(x) = 2x$. Again it is easy to see that this mapping is quasisymmetric with $M = 1$. Computing the $\phi(x, y) = u(x, y) + i\, v(x, y)$ we get

$$\begin{aligned} u(x,y) &= \frac{1}{2y}\int_{-y}^{y} \mu(x+t)dt \\ &= \frac{1}{2y}((x+y)^2 - (x-y)^2) \\ &= 2x, \\ v(x,y) &= \frac{1}{2y}\left(\int_{0}^{y} \mu(x+t)dt - \int_{0}^{y} \mu(x-t)dt\right) \\ &= \frac{1}{2y}\left((x+y)^2 - x^2 + (x-y)^2 - x^2\right) \\ &= y. \end{aligned}$$

Now let's compute the dilatation of this mapping. This is fairly trivial as $u_x = 2$, $u_y = 0$, $v_x = 0$ and $v_y = 1$. This is true everywhere so the dilatation is constant. Looking at the small dilatation, we get by Definition 1.3 on page 3 and by (1.1) that

$$d = \frac{|f_{\overline{z}}|}{|f_z|} = \left|\frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)}\right| = \left|\frac{2 - 1 + i(0+0)}{2 + 1 + i(0-0)}\right| = \frac{1}{3},$$

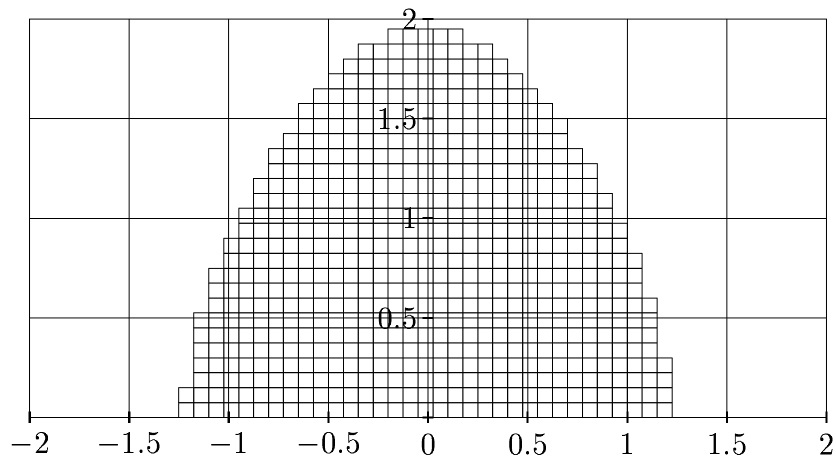
which yields a dilatation of $D = \frac{1+d}{1-d} = 2$. Clearly ϕ is a homeomorphism of the upper halfplane onto itself and so a quasiconformal map as we wanted. It is however not with the best dilatation as we've seen a function $2z$ also satisfies μ as a boundary condition and would have a dilatation of 1. However if we take a look at $u(x,y) + i\,2\,v(x,y)$ this precisely equals $2z$ and so we can improve on the dilatation by multiplying $v(x,y)$ by a constant.

Now for a less trivial example let's use the mapping $\mu(x) = x^3$ which we have shown is quasisymmetric on page 5. Again we compute $\phi(x,y) = u(x,y) + i\,v(x,y)$ and get

$$\begin{aligned} u(x,y) &= \frac{1}{2y}\int_{-y}^{y} \mu(x+t)dt \\ &= \frac{1}{2y}\frac{(x+y)^4 - (x-y)^4}{4} \\ &= x^3 + y^2 x, \\ v(x,y) &= \frac{1}{2y}\left(\int_0^y \mu(x+t)dt - \int_0^y \mu(x-t)dt\right) \\ &= \frac{1}{2y}\frac{(x+y)^4 - x^4 + (x-y)^4 - x^4}{4} \\ &= \frac{3}{2}x^2 y + \frac{1}{4}y^3. \end{aligned}$$

We can see that this mapping is no longer the same as the conformal mapping. In fact since the M for $\mu(x) = x^3$ is quite high, we know that this mapping will have a fairly high maximal dilatation. Figure 2.1 on the next page shows what the extension of $\mu(x) = x^3$ does to a grid in the upper halfplane. It is clear from the graph that the worst dilatation occurs as we approach the point 0. The computer code used to produce this graph, as well as all the other grapha,s is given in Appendix B.

Now we want to show that the mapping (2.5) is in fact quasiconformal for any quasisymmetric mapping μ. So we first need to show that the dilatation of the mapping is bounded. We follow the proof of Ahlfors in [1] page 69. First we need some lemmas concerning bounds on normalized μ.



The above grid is taken to:

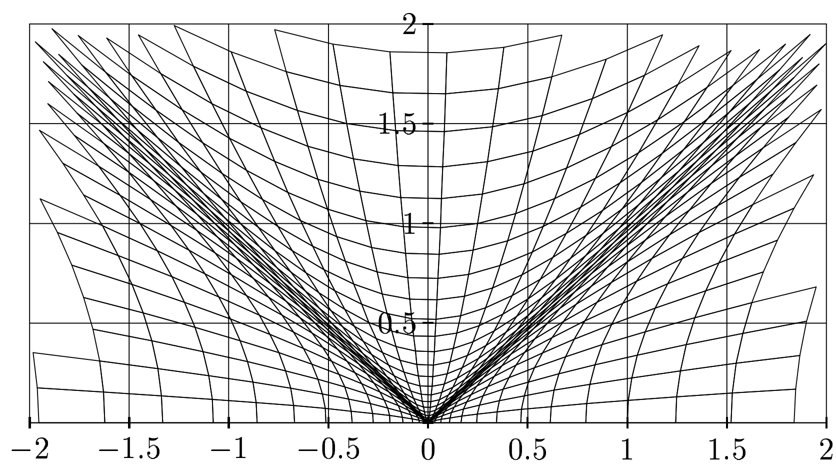


Figure 2.1: Graph of the Beurling-Ahlfors Extension of $\mu(x) = x^3$

Lemma 2.2 *For a normalized M-quasisymmetric μ we have*

$$\frac{1}{2(M+1)} \leq \int_0^1 \mu(t)dt \leq \frac{2M+1}{2(M+1)}.$$

Proof: By plugging $x = \frac{1}{2}$ and $t = \frac{1}{2}$ into the M-condition and noting that μ is normalized, we get

$$\frac{1}{M} \leq \frac{1-\mu(\frac{1}{2})}{\mu(\frac{1}{2})} \leq M,$$

which just becomes

$$\frac{1}{M+1} \leq \mu\left(\frac{1}{2}\right) \leq \frac{M}{M+1}.$$

Now we wish to get upper and lower bounds on $\int_0^1 \mu(t)dt$. By the fact that μ is increasing we get that the μ that would minimize the integral would be 0 on $[0, \frac{1}{2})$ and $\frac{1}{M+1}$ on $[\frac{1}{2}, 1]$. The μ that would maximize the integral would be $\frac{M}{M+1}$ on $[0, \frac{1}{2}]$ and 1 on $(\frac{1}{2}, 1]$. So from this we get the inequality

$$\frac{1}{2}\frac{1}{M+1} \leq \int_0^1 \mu(t)dt \leq \frac{1}{2}\frac{M}{M+1} + \frac{1}{2},$$

and a little bit of elementary algebra proves the lemma. QED!

Lemma 2.3 *For a normalized M-quasisymmetric μ we have*

$$\frac{1}{2(M+1)} \leq \frac{1}{\mu(-1)} \int_{-1}^0 \mu(t)dt \leq \frac{2M+1}{2(M+1)}.$$

Proof: By substitution of variables we can see that

$$\int_{-1}^0 \mu(t)dt = -\int_0^1 -\mu(-t)dt.$$

Now we take the M-condition and plug in $x = 0$ to get

$$\frac{1}{M} \leq \frac{\mu(t)-\mu(0)}{\mu(0)-\mu(-t)} \leq M.$$

Since μ is normalized we immediately get that $M^{-1}\mu(t) \leq -\mu(-t)$ and $-\mu(-t) \leq M\mu(t)$. From this we can conclude

$$\frac{1}{M}\frac{1}{2(M+1)} \leq \frac{1}{M}\int_0^1 \mu(t)dt \leq \int_0^1 -\mu(-t)dt \leq M\int_0^1 \mu(t)dt \leq M\frac{2M+1}{2(M+1)},$$
$$-\frac{1}{M}\frac{1}{2(M+1)} \geq -\int_0^1 -\mu(-t)dt \geq -M\frac{2M+1}{2(M+1)},$$
$$-\frac{1}{M}\frac{1}{2(M+1)} \geq \int_{-1}^0 \mu(t)dt \geq -M\frac{2M+1}{2(M+1)},$$
$$\frac{1}{2(M+1)} \leq \frac{1}{\mu(-t)}\int_{-1}^0 \mu(t)dt \leq \frac{2M+1}{2(M+1)}.$$

QED!

We now rewrite (2.5) to the form

$$\begin{aligned} u(x,y) &= \frac{1}{2y}\int_{x-y}^{x+y} \mu(t)dt, \\ v(x,y) &= \frac{1}{2y}\left(\int_x^{x+y} \mu(t)dt - \int_{x-y}^{x} \mu(t)dt\right), \end{aligned} \tag{2.6}$$

since in this form we can take the partial derivatives easily. Thus the partial derivatives of u and v are

$$\begin{aligned} u_x &= \frac{1}{2y}(\mu(x+y) - \mu(x-y)), \\ u_y &= -\frac{1}{2y^2}\int_{x-y}^{x+y} \mu(t)dt + \frac{1}{2y}(\mu(x+y) + \mu(x-y)), \\ v_x &= \frac{1}{2y}(\mu(x+y) - 2\mu(x) + \mu(x-y)), \\ v_y &= -\frac{1}{2y^2}\left(\int_x^{x+y} \mu(t)dt - \int_{x-y}^{x} \mu(t)dt\right) + \frac{1}{2y}(\mu(x+y) - \mu(x-y)). \end{aligned} \tag{2.7}$$

Lemma 2.4 *Suppose μ is a quasisymmetric mapping and suppose ϕ is the Beurling-Ahlfors extension of μ. If we replace $\mu(x)$ by $\mu_1(x) = A\mu(ax+b) + B$ $(a, A > 0)$ then the Beurling-Ahlfors extension of μ_1 will be $\phi_1(z) = A\phi(az+b) + B$. Furthermore μ_1 satisfies the same M-condition as μ and ϕ_1 has the same maximal dilatation as ϕ.*

Proof: If we replace $\mu(x)$ by $\mu_1(x) = A\mu(ax+b) + B$ $(a, A > 0)$ we still have the same M-condition on this new μ_1 by Lemma 2.1 on page 10. Since multiplying by a constant nor translation changes the maximal dilatation we have that ϕ_1 has the same maximal dilatation as ϕ.

So we still need to show that $\phi_1(z) = A\phi(az+b)+B$. In the two dimensional notation this means showing that $u_1(x,y) = Au(ax+b,ay)+B$ and $v_1(x,y) = Av(ax+b,ay)$. So

$$\begin{aligned} u_1(x,y) &= \frac{1}{2y}\int_{x-y}^{x+y} \mu_1(t)dt \\ &= \frac{1}{2y}\int_{x-y}^{x+y} A\mu(at+b)+Bdt \\ &= A\frac{1}{2y}\int_{x-y}^{x+y} \mu(at+b)dt + B\frac{1}{2y}\int_{x-y}^{x+y} dt \\ &= A\frac{1}{2ay}\frac{1}{a}\int_{ax-ay+b}^{ax+ay+b} \mu(t)dt + B \\ &= Au(ax+b,ay)+B, \end{aligned}$$

and $v_1(x,y) = Av(ax+b,ay)$ can be shown in a similar manner. QED!

Proof of the sufficiency of Theorem 2.1: What Lemma 2.4 buys us is that we can really study the dilatation of the extension of a normalized mapping at a single point and generalize what we find to the dilatation at any point of an extension of any quasisymmetric mapping. Specifically let $\mu_1(x) = A\mu(ax+b)+B$ and pick A and B such that μ_1 is normalized, that is we pick $A = \frac{1}{\mu(a+b)-\mu(b)}$ and $B = \frac{-\mu(b)}{\mu(a+b)-\mu(b)}$. Now $a>0$ and b are arbitrary and so we extend μ_1 to get ϕ_1 and then look at $\phi_1(i) = A\phi(ai+b)+B$. Now A and B do not change the dilatation and a and b can transport us to any point in the plane. So if we can show that the dilatation is bounded at the point i, or (0,1) in the two dimensional notation, we have shown it is bounded anywhere in the upper halfplane.

So from now on we will just study the dilatation of some ϕ at i, and we will assume that the μ that induces this extension is normalized (that is $\mu(1)=1$ and $\mu(0)=0$). So we plug i into our partial derivatives (2.7) to obtain

$$\begin{aligned} u_x(0,1) &= \tfrac{1}{2}(1-\mu(-1)), \\ u_y(0,1) &= -\tfrac{1}{2}\int_{-1}^{1} \mu(t)dt + \tfrac{1}{2}(1+\mu(-1)), \\ v_x(0,1) &= \tfrac{1}{2}(1+\mu(-1)), \\ v_y(0,1) &= -\tfrac{1}{2}\left(\int_0^1 \mu(t)dt - \int_{-1}^0 \mu(t)dt\right) + \tfrac{1}{2}(1-\mu(-1)). \end{aligned}$$

Let us now consider the small dilatation. By Definition 1.3 on page 3 and (1.1) we see that

$$d_\phi = \frac{|\phi_{\overline{z}}|}{|\phi_z|} = \left| \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)} \right|.$$

We could plug in the partial derivatives that we got above to get the small dilatation at i, but this may get too messy and so we will define 3 new variables

$$\begin{aligned} \xi &= 1 - \int_0^1 \mu(t)dt, \\ \beta &= -\mu(-1), \\ \eta &= 1 - \frac{1}{\mu(-1)} \int_{-1}^0 \mu(t)dt, \end{aligned}$$

which also gives

$$\eta\beta = -\mu(-1) + \int_{-1}^0 \mu(t)dt.$$

With this we can write the partial derivatives at i as

$$\begin{aligned} u_x(0,1) &= \tfrac{1}{2}(1+\beta), \qquad & v_x(0,1) &= \tfrac{1}{2}(1-\beta), \\ u_y(0,1) &= \tfrac{1}{2}(\xi - \eta\beta), \qquad & v_y(0,1) &= \tfrac{1}{2}(\xi + \eta\beta), \end{aligned}$$

which gives (and let's call $d_\phi(0,1)$ just d from this point on)

$$\begin{aligned} d = d_\phi(0,1) &= \left| \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)} \right| \\ &= \left| \frac{((1+\beta) - (\xi + \eta\beta)) + i((1-\beta) + (\xi - \eta\beta))}{((1+\beta) + (\xi + \eta\beta)) + i((1-\beta) - (\xi - \eta\beta))} \right| \\ &= \left| \frac{((1-\xi) + \beta(1-\eta)) + i((1+\xi) - \beta(1+\eta))}{((1+\xi) + \beta(1+\eta)) + i((1-\xi) - \beta(1-\eta))} \right|. \end{aligned}$$

Squaring this we get

$$d^2 = \frac{1 + \xi^2 + \beta^2(1+\eta^2) - 2\beta(\xi + \eta)}{1 + \xi^2 + \beta^2(1+\eta^2) + 2\beta(\xi + \eta)},$$

and this in turn gives

$$\frac{1+d^2}{1-d^2} = \frac{1}{2}\left(\frac{1}{\beta}\frac{1+\xi^2}{\xi+\eta} + \beta\frac{1+\eta^2}{\xi+\eta} \right).$$

We need some bounds on our temporary variables β, ξ and η. First let's consider β, and let's use the M-condition with $x = 0$ and $t = -1$. We then get

$$\frac{1}{M} \leq \frac{\mu(-1) - \mu(0)}{\mu(0) - \mu(1)} \leq M,$$
$$\frac{1}{M} \leq -\mu(-1) \leq M,$$
$$\frac{1}{M} \leq \beta \leq M.$$

Then we want to consider ξ. By Lemma 2.2 on page 18 we can see that

$$\frac{1}{2(M+1)} \leq \int_0^1 \mu(t)dt \leq \frac{2M+1}{2(M+1)},$$
$$-\frac{2M+1}{2(M+1)} \leq -\int_0^1 \mu(t)dt \leq -\frac{1}{2(M+1)},$$
$$1 - \frac{2M+1}{2(M+1)} \leq 1 - \int_0^1 \mu(t)dt \leq 1 - \frac{1}{2(M+1)},$$
$$\frac{1}{2(M+1)} \leq 1 - \int_0^1 \mu(t)dt \leq \frac{2M+1}{2(M+1)},$$
$$\frac{1}{2(M+1)} \leq \xi \leq \frac{2M+1}{2(M+1)}.$$

Finally let's consider η. By Lemma 2.3 on page 18 we can see that

$$\frac{1}{2(M+1)} \leq \frac{1}{\mu(-1)} \int_{-1}^0 \mu(t)dt \leq \frac{2M+1}{2(M+1)},$$
$$-\frac{2M+1}{2(M+1)} \leq -\frac{1}{\mu(-1)} \int_{-1}^0 \mu(t)dt \leq -\frac{1}{2(M+1)},$$
$$1 - \frac{2M+1}{2(M+1)} \leq 1 - \frac{1}{\mu(-1)} \int_{-1}^0 \mu(t)dt \leq 1 - \frac{1}{2(M+1)},$$
$$\frac{1}{2(M+1)} \leq 1 - \frac{1}{\mu(-1)} \int_{-1}^0 \mu(t)dt \leq \frac{2M+1}{2(M+1)},$$
$$\frac{1}{2(M+1)} \leq \eta \leq \frac{2M+1}{2(M+1)}.$$

We now have bounds on all three of our variables. We also note that since $0 < \xi < 1$ and $0 < \eta < 1$ we have $\xi^2 < \xi$ and $\eta^2 < \xi$. So

$$\begin{aligned}
\frac{1+d^2}{1-d^2} &= \frac{1}{2}\left(\frac{1}{\beta}\frac{1+\xi^2}{\xi+\eta} + \beta\frac{1+\eta^2}{\xi+\eta}\right) \\
&\leq \frac{1}{2}\left(M\frac{1+\xi^2}{\xi+\eta} + M\frac{1+\eta^2}{\xi+\eta}\right) \\
&= \frac{M}{2}\left(\frac{2+\xi^2+\eta^2}{\xi+\eta}\right) \\
&\leq \frac{M}{2}\left(\frac{2+2\frac{2M+1}{2(M+1)}}{\frac{2}{2(M+1)}}\right) \\
&= \frac{M}{4}\left(4(M+1)+2(2M+1)\right) \\
&= M\left(2M+\frac{3}{2}\right).
\end{aligned}$$

Given that $2d \leq 1 + d^2$ we look at the large dilatation

$$\begin{aligned}
D &= \frac{1+d}{1-d} \\
&= \frac{1+d}{1-d}\frac{1+d}{1+d} \\
&= \frac{1+d^2}{1-d^2} + \frac{2d}{1-d^2} \\
&\leq \frac{1+d^2}{1-d^2} + \frac{1+d^2}{1-d^2} \\
&= 2\frac{1+d^2}{1-d^2} \\
&\leq 2M\left(2M+\frac{3}{2}\right).
\end{aligned}$$

So we have the dilatation (anywhere) bounded by an expression involving only M. Ahlfors gets a nicer bound, but he has to prove another lemma to get it, and in the view of the bounds discussed in Section 4.1 it is irrelevant. The bound on the dilatation also means that the Jacobian must be always positive and thus by the Inverse Function Theorem (Theorem A.3 on page 60) the mapping is locally one-to-one, and has a continuous inverse (locally). What we now need to show is that $|\phi| \to \infty$ as $z \to \infty$. So from (2.6) we have

$$u(x,y) = \frac{1}{2y}\left(\int_x^{x+y}\mu(t)dt + \int_{x-y}^{x}\mu(t)dt\right),$$
$$v(x,y) = \frac{1}{2y}\left(\int_x^{x+y}\mu(t)dt - \int_{x-y}^{x}\mu(t)dt\right).$$

So

$$u(x,y)^2 + v(x,y)^2 = \frac{1}{4y^2}\left[\left(\int_x^{x+y}\mu(t)dt\right)^2 + \left(\int_{x-y}^{x}\mu(t)dt\right)^2\right].$$

If we keep y bounded, then as $x \to \infty$ or as $x \to -\infty$, the integrals go to ∞ or $-\infty$ since μ is monotone and unbounded. So if y is bounded then $u^2 + v^2$ goes to ∞ as z goes to ∞.

We also have

$$\text{for } x \geq 0 \qquad u(x,y)^2 + v(x,y)^2 \geq \frac{1}{4y^2}\left(\int_0^{y}\mu(t)dt\right)^2,$$
$$\text{for } x \leq 0 \qquad u(x,y)^2 + v(x,y)^2 \geq \frac{1}{4y^2}\left(\int_{-y}^{0}\mu(t)dt\right)^2.$$

Now suppose we wish to see what happens if we let $y \to \infty$. If we let $y = 2^{n+1}$, we note that by Lemma 1.1 on page 6 we have that $\int_0^{2^{n+1}}\mu(t)dt \geq 2^n\left(1+\frac{1}{M}\right)^n$ since $\mu(2^n) \geq \left(1+\frac{1}{M}\right)^n$. Using this we have (for $x \geq 0$) that

$$\begin{aligned}
u(x,y)^2 + v(x,y)^2 &\geq \frac{1}{4(2^{n+1})^2}\left(\int_0^{2^{n+1}}\mu(t)dt\right)^2 \\
&\geq \frac{1}{4(2^{n+1})^2}\left(2^n\left(1+\frac{1}{M}\right)^n\right)^2 \\
&\geq \frac{1}{2^{2n}2^4}2^{2n}\left(1+\frac{1}{M}\right)^{2n} \\
&\geq \frac{1}{2^4}\left(1+\frac{1}{M}\right)^{2n},
\end{aligned}$$

which obviously goes to ∞ as n (and thus y) goes to ∞. The case for $x \leq 0$ is similar but we use Lemma 1.2 on page 6.

Now we wish to show that ϕ maps the upper halfplane onto itself. Suppose it doesn't and so there are points in the upper halfplane which are not in the image of ϕ. We could then find a point w in the upper half plane which is on the boundary of the image of

ϕ. Now suppose we take a sequence $\{z_i\}_0^\infty$ such that $\phi(z_i) \to w$. Now since $\phi(z)$ goes to infinity if z goes to infinity, then this means that z_i cannot go to infinity (since w is of finite modulus). This means that the z_i are bounded. Then there must exist a subsequence z_{i_k} which converges to some point z in the upper halfplane. Then we have by continuity that $\phi(z) = w$ and this means that w is in the image of ϕ, but it is also on the boundary. But from this we get a contradiction since the Jacobian of ϕ at z is positive and thus there is an open neighborhood of w in the image of ϕ and so w could not be on the boundary. This means that every point in the upper halfplane is in the image of ϕ and thus ϕ maps the upper halfplane onto itself.

We now wish to show that it is also one to one and thus a homeomorphism. For this we use the Monodromy Theorem (Theorem A.10 on page 63). The upper halfplane H is a topological space and ϕ is a continuous function from H to H, that is locally one to one and thus (H, ϕ) is a covering space of H.

Now suppose that $\phi(a) = \phi(b)$, then suppose we have two paths from 0 to $\phi(a)$, we can make them the same path and thus they are fixed end point (FEP) homotopic (Definition A.5 on page 62). The liftings (Definition A.4 on page 62) of those paths are in the domain of H and by the Monodromy Theorem they have the same endpoint, that is $a = b$. This means that ϕ must be one to one. Which means that ϕ is a homeomorphism. QED!

Ahlfors and Beurling introduce an extra parameter in the definition, to prove the existence of a quasiconformal mapping with the dilatation less than M^2. In fact better estimates for this extension are possible and we will expand on those results in Section 4.1.

CHAPTER 3

THEOREM OF DOUADY AND EARLE

3.1 Conformally Natural Extension

We now come to the second explicit construction of a quasiconformal mapping given the boundary correspondence. This was done by Douady and Earle [4]. In fact what they do is something slightly different. They take a homeomorphism of the unit circle and extend it to a homeomorphism of the whole unit disc in a specific way. This turns out to be a quasiconformal mapping if such a mapping is possible. This extension is also called the barycentric extension as it is defined in terms of "barycenters" of probability measures. We will mostly follow the statement and proof of this theorem from Gardiner and Lakic [7] as it is somewhat easier to follow. We will also only consider the quasiconformal case in the plane. Extending a mapping of the unit circle to a mapping of the whole unit disc is really equivalent to extending a mapping of the real line to a mapping of the upper halfplanes since we can just compose with a conformal map, which does not change the quasiconformality constant, to get one from the other. Now we will also talk about quasisymmetric mappings of the unit circle. These are basically mappings which admit a quasiconformal extension to the unit disc. We can reformulate the M-condition for the unit circle as stated in [7].

Definition 3.1 *A function $\mu(x)$ of the unit circle to itself satisfies the M-condition if there is some M such that for any x and t (where t is not a multiple of 2π),*

$$\frac{1}{M} \leq \frac{|\mu(e^{i(x+t)}) - \mu(e^{i(x)})|}{|\mu(e^{i(x)}) - \mu(e^{i(x-t)})|} \leq M.$$

So let's first fix the notation. Let $D = \{z \in \mathbb{C}; |z| < 1\}$ be the unit disc, let $\overline{D}$ be the closed unit disc. Furthermore let G be the group of all conformal automorphisms of D, and let G_+ be the subgroup consisting of sense preserving automorphisms of D. This is really just the set of Möbius transformations that fixes D. We know in fact that all

$g \in G_+$ are in fact transformations of the form

$$g_a(z) = \lambda \frac{z-a}{1-\overline{a}z} \qquad \text{Where } |\lambda| = 1 \text{ and } |a| < 1. \tag{3.1}$$

Note that whenever we use g_a without specifying λ we will assume that $\lambda = 1$.

Since we are talking about *conformally natural* mappings we should define what this means. It can be defined in general terms in terms of group actions, but we will be interested in the special case of composition with Möbius transformations of the unit disc. So suppose that $\mathcal{H}_{\partial D}$ is the space of all homeomorphisms of the unit circle and $\mathcal{H}_D$ is the space of homeomorphisms of the open unit disc.

Definition 3.2 *A mapping $T : \mathcal{H}_{\partial D} \to \mathcal{H}_D$ is* conformally natural *if and only if for any $\alpha, \beta \in G_+$ and $\mu \in \mathcal{H}_{\partial D}$ we have*

$$T(\alpha \circ \mu \circ \beta) = \alpha \circ T(\mu) \circ \beta.$$

We are now ready to state the theorem of Douady and Earle.

Theorem 3.1 (Douady and Earle) *There is a conformally natural extension of any quasisymmetric homeomorphism $\mu : \partial D \to \partial D$ to a quasiconformal homeomorphism $\phi = E(\mu)$ where $\phi : \bar{D} \to \bar{D}$. The extension $\phi = E(\mu)$ has the following properties:*

1. *The mapping $\phi = E(\mu)$ is conformally natural,*
2. *$E(\text{identity on } \partial D) = \text{identity on } D$, and*
3. *if $\int_{\partial D} \mu(z)dz = 0$, then $\phi(0) = 0$.*

3.2 Conformal Barycenter

Let m denote a probability measure defined on ∂D which has no atoms, that is the distribution function of m has no jump discontinuities. Now we consider a conformally natural way to create a vector field $\xi_m(z)$ which is defined on D and is associated with a probability measure m on ∂D. Suppose that m is nonnegative, has no atoms and that we have

$$\int_{\partial D} dm(t) = 1.$$

We define

$$\xi_m(0) = \int_{\partial D} t\, dm(t).$$

Since we already said that the extension is called barycentric, it may be useful to give some physical interpretation to these functions. We can think of the probability distribution as weight distribution. Then $\xi_m(0)$ is zero only if the center of the disc is truly the barycenter of all this weight. If we'd try to balance the disc at 0 with the weight distributed by m on the boundary it would tilt in the direction of $\arg \xi_m(0)$ as if there was a weight of $|\xi_m(0)|$ in that direction on the boundary. We also note that $|\xi_m(0)| < 1$ if m has no atoms since it could only be one if there was a single point with all the weight on the boundary, but we're assuming no such points exist.

We will need the following definition in order to make new measures out of old ones using mappings of the unit circle.

Definition 3.3 *A* push-forward *of a probability measure m (on ∂D) by the map $g : \partial D \to \partial D$ will be denoted by $g_* m$ and given any $S \subset \partial D$ then*

$$g_* m(S) = m(g^{-1}(S)).$$

What this means is that $g_* m(S)$ is a measure of the set that g would take to S. If g is really a map of the whole unit disc, we will just consider the restriction to the unit circle for the purposes of the push-forward.

Now we wish to take a Möbius transformation that would take some point $w \in D$ to 0 and use it to transport the value of $\xi_m(0)$ to w. This is the transformation g_w as defined in (3.1) with $\lambda = 1$ since we do not want to rotate. If we look at the measure $g_{w*}m$ (the push forward of m by g_w), then this transports the value of $\xi_m(0)$ to another point w, since the old 0 will now be at w for the measure $g_{w*}m$. So we define $\xi_m(w)$ as

$$\xi_m(w) = \frac{\xi_{g_{w*}m}(0)}{g'_w(w)}.$$

Now $g'_w(w) = \frac{1-|w|^2}{(1-\overline{w}w)^2} = \frac{1}{1-|w|^2}$ and so with this we have

$$\begin{aligned} \xi_m(w) &= (1-|w|^2) \int_{\partial D} t\, d\, g_{w*}m(t) \\ &= (1-|w|^2) \int_{\partial D} \frac{t-w}{1-\overline{w}t}\, dm(t). \end{aligned} \tag{3.2}$$

The function ξ_m is real analytic, which obvious from (3.2). Now we need to show that ξ_m has exactly one zero in D.

Lemma 3.1 *For a probability measure m with no atoms the corresponding vector field ξ_m has a unique zero in D*

This zero can be interpreted as the barycenter of the measure m, that is we could balance the disc at this point with the weight distributed according to m.

Definition 3.4 *For every probability measure m on ∂D with no atoms, the corresponding vector field ξ_m has a unique zero in D. We call this zero the* conformal barycenter $B(m)$ *of m.*

Lemma 3.2 *The Jacobian of ξ_m at the point 0 is positive. For a probability measure m with no atoms the corresponding vector field ξ_m has a unique zero in D*

Proof: From (3.2) we get that

$$\begin{aligned}
\xi_m(w) &= (1-|w|^2)\int_{\partial D}\frac{t-w}{1-\overline{w}t}\,dm(t)\\
&= \int_{\partial D}\frac{t-w}{1-\overline{w}t}\,dm(t) + |w|^2\int_{\partial D}\frac{t-w}{1-\overline{w}t}\,dm(t)\\
&= \int_{\partial D}\frac{t-w}{1-\overline{w}t}\,dm(t) + o(w)\\
&= \int_{\partial D}(t-w)(1+\overline{w}t)\,dm(t) + \int_{\partial D}\frac{(t-w)(\overline{w}t)^2}{1-\overline{w}t}\,dm(t) + o(w)\\
&= \int_{\partial D}(t-w)(1+\overline{w}t)\,dm(t) + \overline{w}^2\int_{\partial D}t\frac{t-w}{1-\overline{w}t}\,dm(t) + o(w)\\
&= \int_{\partial D}(t-w)(1+\overline{w}t)\,dm(t) + o(w)\\
&= \int_{\partial D}(1-|w|^2)t + \overline{w}t^2 - w\,dm(t) + o(w)\\
&= (1-|w|^2)\xi_m(0) - w + \overline{w}\int_{\partial D}t^2\,dm(t) + o(w)\\
&= \xi_m(0) - w + \overline{w}\int_{\partial D}t^2\,dm(t) + o(w).
\end{aligned}$$

Now we can easily take the derivatives with respect to w and $\overline{w}$ and evaluate them at 0. First $\xi_m(0)$ is a constant, and the derivative of $o(w)$ evaluated at 0 will just be zero.

So the Jacobian of ξ_m at $w = 0$ is

$$\begin{aligned}
J\xi_m(0) &= |(\xi_m)'_w(0)|^2 - |(\xi_m)'_{\overline{w}}(0)|^2 \\
&= |-1|^2 - |\int_{\partial D} t^2\, dm(t)|^2 \\
&= 1 - \left(\int_{\partial D} t^2\, dm(t)\right)\overline{\left(\int_{\partial D} t^2\, dm(t)\right)} \\
&= 1 - \iint_{\partial D\times\partial D} t^2\overline{s}^2\, dm(t)\times dm(s) \\
&= \iint_{\partial D\times\partial D} 1 - t^2\overline{s}^2\, dm(t)\times dm(s) \\
&= \frac{1}{2}\Big(\iint_{\partial D\times\partial D} |s|^2 - t^2\overline{s}^2\, dm(t)\times dm(s) \\
&\qquad + \iint_{\partial D\times\partial D} |t|^2 - \overline{t}^2 s^2\, dm(t)\times dm(s)\Big) \\
&= \frac{1}{2}\iint_{\partial D\times\partial D} |s|^2 - t^2\overline{s}^2 + |t|^2 - \overline{t}^2 s^2\, dm(t)\times dm(s) \\
&= \frac{1}{2}\iint_{\partial D\times\partial D} |s^2 - t^2|^2\, dm(t)\times dm(s) \\
&> 0.
\end{aligned}$$

QED!

Proof of Lemma 3.1: By Lemma 3.2 on the preceding page we know that the Jacobian at 0 is a positive quantity so now suppose that that $\xi_m(0) = 0$, then since $J\xi_m(0) > 0$ this is an isolated zero of index one by Lemma A.1 on page 64. Now since we can take $\xi_m(w) = \frac{1}{(g_w)'(w)}\xi_{g_{w*}m}(0)$ then any zero will be an isolated zero of index one.

Now since m has no atoms, then there exists an $\alpha > 0$ such that $m(I) < \frac{1}{3}$ whenever an arc I of ∂D has arc length of at most α. So suppose that I is centered at the point 1. Now take r_0 so that the hyperbolic rays going from r_0 to the endpoints of I subtend an angle of $\frac{3\pi}{2}$. That is $g_{r_0}(I)$ has arc length $\frac{3\pi}{2}$. This can be seen in Figure 3.1 on the following page.

Now for $r \geq r_0$ let $C_r = \{w; |w| = r\}$. And for w such that $|w| = r$, let g be a conformal map which takes w to 0, and $\frac{-w}{|w|}$ to 1. If J is the arc from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$, then this

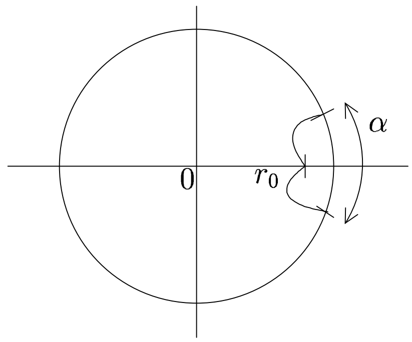


Figure 3.1: Point r_0 with Hyperbolic Rays that Subtend $\frac{3\pi}{2}$

makes $g_*m(J) = m(g^{-1}(J)) \geq \frac{2}{3}$ because g takes the $\partial D - I$ or more to J. Now

$$\begin{aligned} Re\,\xi_{g_*m}(0) &= \int_{\partial D} Re(t)\, dg_*m(t) \\ &= \int_{\partial D - J} Re(t)\, dg_*m(t) + \int_J Re(t)\, dg_*m(t) \\ &\geq \int_{\partial D - J} -1\, dg_*m(t) + \int_J \frac{\sqrt{2}}{2}\, dg_*m(t) \\ &\geq (-1)\frac{1}{3} + \frac{\sqrt{2}}{2}\frac{2}{3} \\ &> 0. \end{aligned}$$

This means that $\xi_{g_*m}(0)$ points into $g(C_r)$ because C_r is mapped onto the positive real halfplane because $\frac{-w}{|w|}$ (which is on the opposite side of 0) goes to 1. However this implies that $\xi_m(w)$ points into C_r because $g(z) = \lambda g_w(z)$ where λ is just a rotation and thus we can again use the definition of $\xi_m(w)$. Since $\xi_m(w)$ points into C_r then it is not 0, and so no zeros are on C_r.

Now suppose we make a manifold W by taking the inside of C_r and removing a small ball around every zero. Now ξ_m is never zero on W nor does it approach zero. So now we can define the mapping $f(z) = \frac{\xi_m(z)}{|\xi_m(z)|}$. This is a smooth mapping from W to the unit circle ∂D. If we assign orientation to boundaries of this manifold, then the internal boundaries are negatively oriented and the outer one (the C_r) is positively oriented. Around each of the internal boundaries f is of degree -1. Since f restricted to the boundaries obviously extends to a mapping of the whole W to ∂D, then by the

Extension Theorem (Theorem A.11 on page 64), we know that the degree of f if restricted to the boundary of W is 0. This means that the degree of f restricted to C_r is the number of zeros of ξ_m inside C_r. It is easy to see that since f always points inside on C_r, it is homotopic to the Gauss map which maps each point to the unit vector pointing to the origin. The degree of the Gauss map is one, and thus there is exactly one zero of ξ_m inside C_r and hence, ξ_m has exactly one zero in D. QED!

3.3 Extensions of the Douady-Earle Type

We can now define the Douady-Earle extension of a quasisymmetric homeomorphism of the unit circle. But first we need to define what is a harmonic measure.

Definition 3.5 *For any Borel set $A \subset \partial D$ we define the harmonic measure η_z of z as*

$$\eta_z(A) = \frac{1}{2\pi} \int_A \frac{1 - |z|^2}{|z - t|^2} |dt|.$$

The harmonic measure really measures the set that A would be taken to if z would be taken 0 by a conformal transformation. An interpretation of this measure that is useful for us is that suppose that we are at z, then $\eta_z(A)$ measures how much A takes up our "field of view". If we want to get the "barycenter" or the center of gravity of such a measure, then we'd want to find a point where no matter where we put A, it takes up the same amount from our "field of view". This point is 0. Now what we are going to do is we are going to push-forward this harmonic measure by our boundary mapping and then get the barycenter of the resulting measure.

Definition 3.6 (The Douady-Earle Extension) *Let μ be a quasisymmetric mapping of ∂D onto itself. We define the extension $\phi = E(\mu)$ on D as the conformal barycenter of the forward push of η_z by μ, that is*

$$\phi(z) = B(\mu_* \eta_z).$$

On ∂D we define $\phi(z) = \mu(z)$.

We now let $p(z,t) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-t|^2}$ to make notation easier. Then $\eta_z(A) = \int_A p(z,t)|dt|$ and so by (3.2) the Douady-Earle extension is the unique zero of the function

$$\begin{aligned} F_{\mu,z}(w) &= (1-|w|^2)\int_{\partial D} \frac{t-w}{1-\overline{w}t}\, d\,\mu_*\eta_z(t) \\ &= (1-|w|^2)\int_{\partial D} \frac{\mu(t)-w}{1-\overline{w}\mu(t)}\, d\,\eta_z(t) \qquad (3.3) \\ &= (1-|w|^2)\int_{\partial D} \frac{\mu(t)-w}{1-\overline{w}\mu(t)} p(z,t)|dt|. \end{aligned}$$

Intuitively what is happening is that μ distributes the weight on the unit circle and then we try to find the barycenter of this weight. Imagine a set of weights on the boundary of the unit circle as the μ. Then suppose we are standing on the unit disc at the point z. We then compare these weights by just judging their size depending on how much of our field of view they take up, and find the barycenter based on this. This is done by the η_z measure, which is the same as first applying to these weights the Möbius transformation that takes z to 0 and then find the barycenter. This can be seen in Figure 3.2.

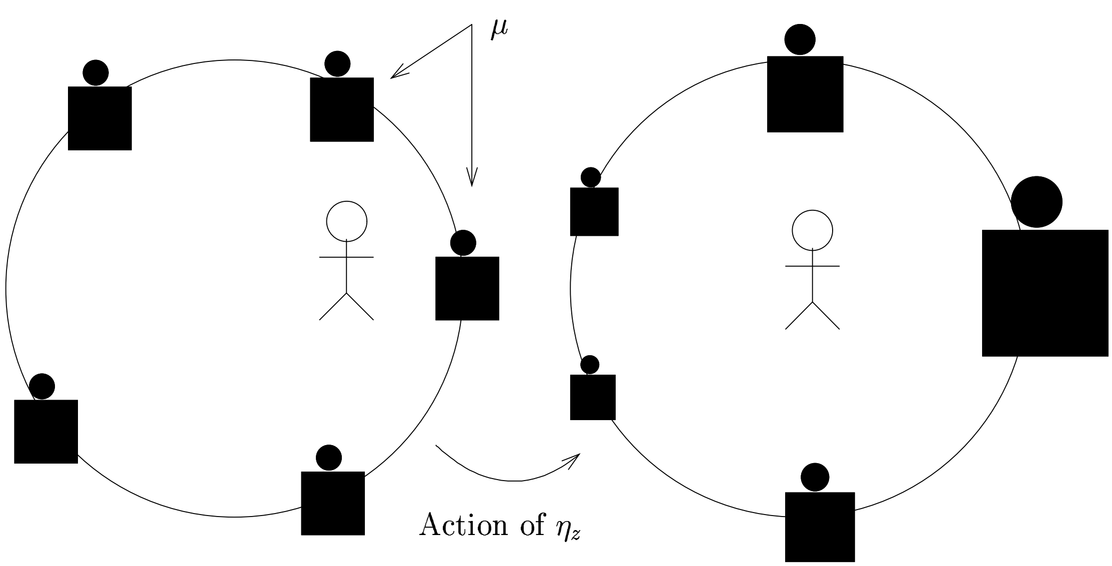


Figure 3.2: Distribution of Weights by μ as Seen Through η_z

As an example we can try to see what happens if we use the x^3 function for our boundary correspondence as we did with the Beurling-Ahlfors extension. This time however we must map it onto the unit circle. So suppose that $S(z)$ is a conformal map which maps the unit circle onto the real line, that is $S(z) = \frac{z+i}{iz+1}$. Now we can consider the function $S^{-1}(S(x)^3)$. The result of this mapping can be seen in Figure 3.3 on the next page. So that we can compare the result to Figure 2.1 on page 17 we map the disk onto the plane in Figure 3.4 on page 36.

Proof of Theorem 3.1: The last condition of the theorem follows easily by looking at $w = 0$ and $z = 0$ in (3.3). We also note that the harmonic measure when looking from $z = 0$ is just the normal measure. So suppose that $\int_{\partial D} \mu(t)\,|dt| = 0$ then

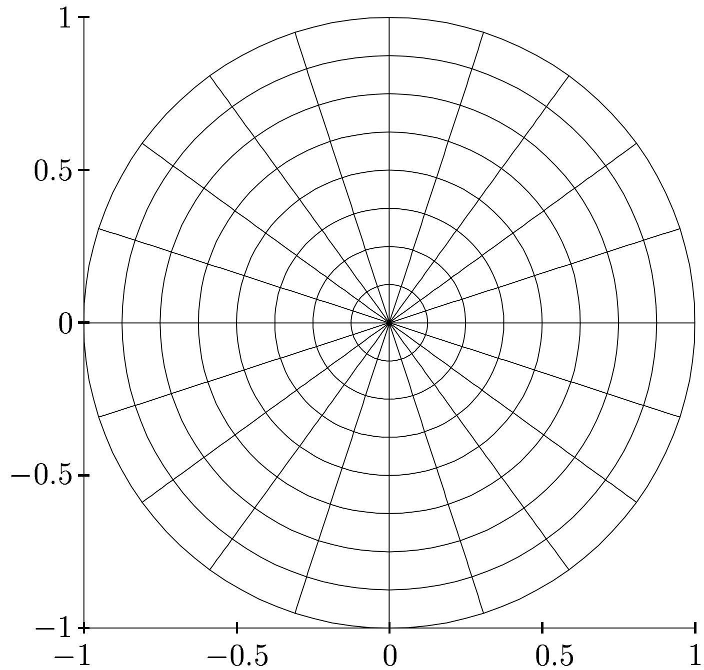
$$\begin{aligned}
F_{\mu,0}(0) &= (1 - |0|^2) \int_{\partial D} \frac{t - 0}{1 - \overline{0}t}\, d\,\mu_*\eta_0(t) \\
&= \int_{\partial D} t\, d\,\mu_*\eta_0(t) \\
&= \int_{\partial D} \mu(t)\, d\,\eta_0(t) \\
&= \int_{\partial D} \mu(t)\, |dt| \\
&= 0.
\end{aligned}$$

And since the zero of $F_{\mu,z}$ is unique, then $\phi(0) = 0$.

Similarly for the second condition. Suppose that μ is the identity and suppose that $w = z$. Notice that $g_{z*}\eta_z$ is just the same as η_0 since η_z is the measure of a set when we take z to 0 conformally, but we're pushing forward by a conformal mapping that takes z to 0 already (so we take 0 to z before applying the measure). So this means we have

$$\begin{aligned}
F_{\mu,z}(z) &= (1 - |z|^2) \int_{\partial D} \frac{t - z}{1 - \overline{z}t}\, d\,\eta_z(t) \\
&= (1 - |z|^2) \int_{\partial D} t\, d\,g_{z*}\eta_z(t) \\
&= (1 - |z|^2) \int_{\partial D} t\, d\,\eta_0(t) \\
&= (1 - |z|^2) \int_{\partial D} t\, |dt| \\
&= 0.
\end{aligned}$$

Next we wish to show that ϕ is conformally natural in the sense that for any two conformal automorphisms α and β of D, we have that $E(\alpha \circ \mu \circ \beta) = \alpha \circ E(\mu) \circ \beta$. We



The above grid is taken to:

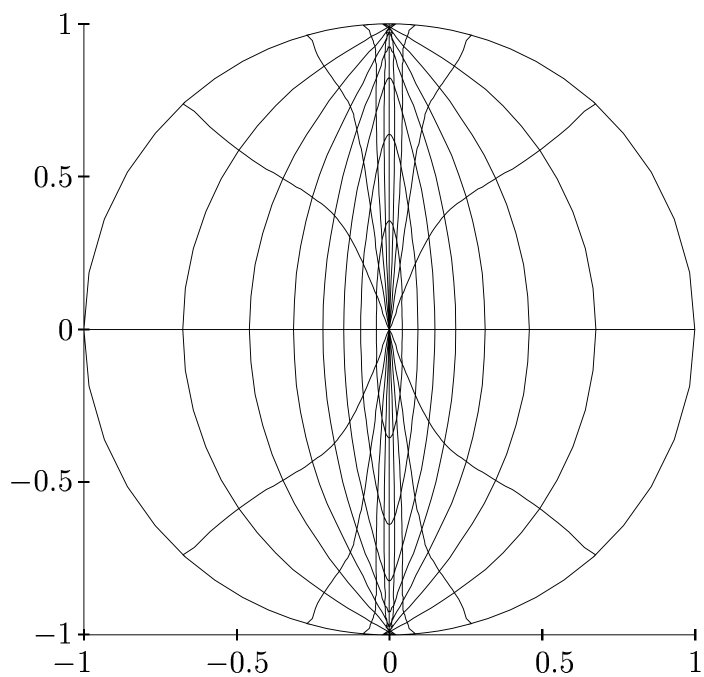
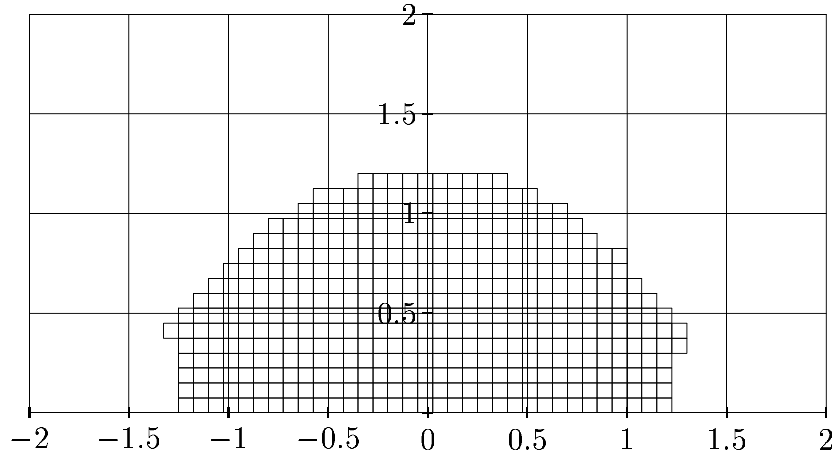


Figure 3.3: Graph of the Douady-Earle Extension of $\mu(z) = S^{-1}(S(z)^3)$



The above grid is taken to:

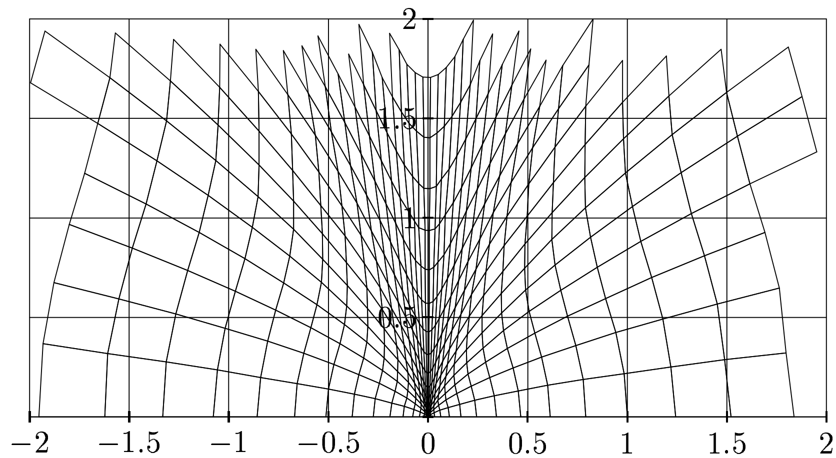


Figure 3.4: Graph of the Douady-Earle Extension in the Plane of $\mu(x) = x^3$

can do this by showing that whenever $F_{\alpha\circ\mu\circ\beta,z}(w)$ has a zero, then $F_{\mu,\beta(z)}(\alpha^{-1}(w))$ also has a zero, and since the zero is unique by Lemma 3.1 on page 29 we have conformal naturality. So suppose that $\alpha(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$ and then

$$\begin{aligned}\left(\frac{\alpha(z)-\alpha(w)}{1-\overline{\alpha(w)}\alpha(z)}\right)\left(e^{i\theta}\frac{1-\overline{a}w}{1-a\overline{w}}\right) &= \left(\frac{e^{i\theta}\frac{z-a}{1-\overline{a}z} - e^{i\theta}\frac{w-a}{1-\overline{a}w}}{1-e^{-i\theta}\frac{\overline{w}-\overline{a}}{1-a\overline{w}}e^{i\theta}\frac{z-a}{1-\overline{a}z}}\right)\left(e^{i\theta}\frac{1-\overline{a}w}{1-a\overline{w}}\right)\\ &= \left(\frac{\frac{z-a}{1-\overline{a}z} - \frac{w-a}{1-\overline{a}w}}{1-\frac{\overline{w}-\overline{a}}{1-a\overline{w}}\frac{z-a}{1-\overline{a}z}}\right)\left(\frac{1-\overline{a}w}{1-a\overline{w}}\right)\\ &= \frac{(z-a)(1-\overline{a}w)-(1-\overline{a}z)(w-a)}{(1-\overline{a}z)(1-a\overline{w})-(\overline{w}-\overline{a})(z-a)}\\ &= \frac{z-a-\overline{a}wz+a\overline{a}w-w+\overline{a}zw-\overline{a}az+a}{1-\overline{a}z-a\overline{w}+\overline{a}a\overline{w}z-\overline{w}z+\overline{a}z+a\overline{w}-\overline{a}a}\\ &= \frac{z-w-\overline{a}az+\overline{a}aw}{1-\overline{w}z+\overline{a}a\overline{w}z-\overline{a}a}\\ &= \frac{(1-\overline{a}a)(z-w)}{(1-\overline{a}a)(1-\overline{w}z)}\\ &= \frac{z-w}{1-\overline{w}z}.\end{aligned}$$

Which means that we can write

$$\frac{\alpha(z)-\alpha(w)}{1-\overline{\alpha(w)}\alpha(z)}q(w) = \frac{z-w}{1-\overline{w}z}, \tag{3.4}$$

where

$$q(w) = e^{i\theta}\frac{1-\overline{a}w}{1-a\overline{w}},$$

which is non-zero at any point on D.

Now we look at $F_{\mu,\beta(z)}$ and using (3.3) and (3.4) we see that

$$\begin{aligned}F_{\mu,\beta(z)}(w) &= (1-|w|^2)\int_{\partial D}\frac{\mu(t)-w}{1-\overline{w}\mu(t)}p(\beta(z),t)|dt|\\ &= (1-|w|^2)q(w)\int_{\partial D}\frac{\alpha(\mu(t))-\alpha(w)}{1-\overline{\alpha(w)}\alpha(\mu(t))}p(\beta(z),t)|dt|\\ &= (1-|w|^2)q(w)\int_{\partial D}\frac{\alpha(\mu(\beta(t)))-\alpha(w)}{1-\overline{\alpha(w)}\alpha(\mu(\beta(t)))}p(z,t)|dt|.\end{aligned}$$

Next we look at $F_{\mu,\beta(z)}(\alpha^{-1}(w))$. When we plug in $\alpha^{-1}(w)$ we get

$$\begin{aligned}F_{\mu,\beta(z)}(\alpha^{-1}(w)) &= (1-|\alpha^{-1}(w)|^2)q(\alpha^{-1}(w))\int_{\partial D}\frac{\alpha(\mu(\beta(t)))-\alpha(\alpha^{-1}(w))}{1-\overline{\alpha(\alpha^{-1}(w))}\alpha(\mu(\beta(t)))}p(z,t)|dt|\\ &= \frac{(1-|\alpha^{-1}(w)|^2)}{1-|w|^2}q(\alpha^{-1}(w))\,(1-|w|^2)\int_{\partial D}\frac{\alpha(\mu(\beta(t)))-w)}{1-\overline{w}\alpha(\mu(\beta(t)))}p(z,t)|dt|\\ &= \frac{(1-|\alpha^{-1}(w)|^2)}{1-|w|^2}q(\alpha^{-1}(w))\,F_{\alpha\circ\mu\circ\beta,z},\end{aligned}$$

and since $\frac{(1-|\alpha^{-1}(w)|^2)}{1-|w|^2}q(\alpha^{-1}(w))$ is never 0 for $w\in D$, we have that $F_{\alpha\circ\mu\circ\beta,z}$ is zero if and only if $F_{\mu,\beta(z)}(\alpha^{-1}(w))$ is zero and thus $\phi = E(\mu)$ is conformally natural.

So now what we need is to show that $E(\mu)=\phi$ is continuous at a point $s\in\partial D$. For this we first define H to be the complex harmonic extension of μ to the unit disc. That is

$$H(z) = \int_{\partial D}\mu(t)\,d\,\eta_z(t). \tag{3.5}$$

By definition of ϕ we can see that $F_{\mu,z}(\phi(z))=0$, since ϕ is defined in terms of the zeros of $F_{\mu,z}$. If we plug $\phi(z)$ into (3.3) we get

$$0 = \int_{\partial D}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t). \tag{3.6}$$

Then combining (3.5) and (3.6) we get

$$\begin{aligned}\phi(z)-H(z) &= \int_{\partial D}\phi(z)-\mu(t)\,d\,\eta_z(t)\\ &= \int_{\partial D}(1-\mu(t)\overline{\phi(z)})\frac{\phi(z)-\mu(t)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t)\\ &= \int_{\partial D}\frac{\phi(z)-\mu(t)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t) - \int_{\partial D}\mu(t)\overline{\phi(z)}\frac{\phi(z)-\mu(t)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t)\\ &= 0 + \int_{\partial D}\mu(t)\overline{\phi(z)}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t).\end{aligned} \tag{3.7}$$

Now we can multiply a zero by anything and still have zero and so (3.6) also yields

$$\begin{aligned}0 &= \mu(s)\overline{\phi(z)}\int_{\partial D}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t)\\ &= \int_{\partial D}\mu(s)\overline{\phi(z)}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t).\end{aligned} \tag{3.8}$$

So now subtracting (3.8) from (3.7) we get

$$\begin{aligned}\phi(z) - H(z) &= \int_{\partial D} \mu(t)\overline{\phi(z)}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t) - \int_{\partial D} \mu(s)\overline{\phi(z)}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t) \\ &= \int_{\partial D} (\mu(t)-\mu(s))\overline{\phi(z)}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\,d\,\eta_z(t).\end{aligned}$$

We note that $|\phi(z)| < 1$ and $\left|\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\right| = 1$ (which is just a Möbius transformation of the unit disk and $\mu(t)$ is on the unit circle) for $z \in D$ and thus we get

$$\begin{aligned}|\phi(z) - H(z)| &\le \int_{\partial D} \left|(\mu(t)-\mu(s))\overline{\phi(z)}\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\right|\,d\,\eta_z(t) \\ &= \int_{\partial D} |\mu(t)-\mu(s)|\,\left|\overline{\phi(z)}\right|\,\left|\frac{\mu(t)-\phi(z)}{1-\mu(t)\overline{\phi(z)}}\right|\,d\,\eta_z(t) \qquad (3.9)\\ &\le \int_{\partial D} |\mu(t)-\mu(s)|\,d\,\eta_z(t).\end{aligned}$$

Since μ is a continuous function then the right hand side of this equation goes to zero as z approaches s, since the closer z is to s the more "weight" is given to t that are close to s when thinking of the integral as a sum. So the left side also goes to zero as z approaches s and thus $\phi(z)$ gets closer to $H(z)$ which is a continuous function and thus ϕ is also continuous at s (this can be seen by the triangle inequality).

We still need to show that ϕ is a real analytic diffeomorphism of the unit disc. Given that we have continuity at the boundary, what is left is to show that ϕ is real analytic and that the Jacobian is nonzero everywhere in D. To make things easier let's first define the function

$$G(z,w) = \frac{F_{\mu,z}(w)}{1-|w|^2}.$$

By Lemma 3.2 on page 29 the Jacobian of $F_{\mu,z}$ is positive, and so the Jacobian of G with respect to w (thus keeping z constant) is also positive. Then since $\phi(z)$ is defined in the terms of the zeros of G then by the Implicit Function Theorem (Theorem A.4 on page 60) we have that $\phi(z)$ is in fact a real analytic function in D. So what is left to show is that the Jacobian is positive everywhere on D.

The Implicit Function Theorem also gives us a formula for the derivative of $\phi(z)$ in terms of the derivatives of G. Note that taking the determinant of a matrix and its negative is the same given that the matrix is of an even size. So given that our determinant matrices that come from the Implicit Function Theorem are all 2 by 2, we see

$$\begin{aligned} D_z G(z,\phi(z)) + D_w G(z,\phi(z)) \cdot D\phi(z) = 0, \\ D_z G(z,\phi(z)) &= -D_w G(z,\phi(z)) \cdot D\phi(z), \\ J_z G(z,\phi(z)) &= J_w G(z,\phi(z)) \cdot J\phi(z), \\ J\phi(z) &= \frac{J_z G(z,\phi(z))}{J_w G(z,\phi(z))}. \end{aligned}$$

Now we note that $J_w G(z,w) > 0$ by Lemma 3.2 on page 29 (else we couldn't really do the above division). So all that is left to prove is that $J_z G(z,w) > 0$, but we note that if we treat w as a constant, then $G(z,w)$ is really just the Harmonic extension of $g_w \circ \mu$, that is

$$G(z,w) = \int_{\partial D} (g_w \circ \mu)(t)\, d\,\eta_z(t).$$

By Theorem A.9 on page 62 we know a harmonic extension of a continuous function that maps the unit circle to itself homeomorphically is a diffeomorphism with a positive Jacobian, if we assume this is a sense preserving map. This means that $J_z G(z,w) > 0$ and in turn $J\phi(z) > 0$. This means that since $\phi(z)$ maps the circle to itself homeomorphically and the Jacobian is positive everywhere in D, then $\phi(z)$ must be a diffeomorphism of the disk to itself.

Next we need to show that the dilatation of the extension is uniformly bounded. For this we will need a lemma about the dependence of $E(\mu) = \phi$ on μ. The proof of this is different from both [7] and [4]. It is a more elementary δ-ϵ proof, but is substantially longer.

Lemma 3.3 *The Douady-Earle extension $E(\mu) = \phi$ is a continuous mapping with respect to the sup norm.*

Proof: First let's see what we wish to prove. Suppose we are given an $\epsilon > 0$ then we wish to find a $\delta > 0$ such that $\sup_{z\in D} |E(\mu_0) - E(\mu)| < \epsilon$ whenever $\sup_{z\in\partial D} |\mu_0 - \mu| < \delta$.

So let's pick $\gamma = \frac{\epsilon}{3}$ and consider for now μ and μ_0 where $\sup_{z\in\partial D} |\mu_0 - \mu| < \gamma$. Close to the boundary we have shown in (3.9) that $\phi(z)$ is very close to $H(z)$, the harmonic extension of μ. Furthermore it is easy to see from that equation, that the closer z gets to the boundary, the closer $H(z)$ and $\phi(z)$ are. So we can find a circle $|z| = r_0 < 1$ such that $|\phi(z) - H(z)| < \frac{\epsilon}{3}$ for all $|z| = r_0$ (we can do this as a circle is a compact set and we'll just get close enough to the boundary as needed). Also if H_0 is the harmonic extension of μ_0 and $\phi_0 = E(\mu_0)$ then we can also pick r_0 such that $|\phi_0(z) - H_0(z)| < \frac{\epsilon}{3}$ for all $|z| = r_0$. In fact since as we said $H(z)$ gets closer to $\phi(z)$ as z gets closer to the boundary then we know that $|\phi_0(z) - H_0(z)| < \frac{\epsilon}{3}$ and $|\phi(z) - H(z)| < \frac{\epsilon}{3}$ for all $1 \geq |z| \geq r_0$. So now since $\sup_{z\in\partial D} |\mu_0 - \mu| < \gamma = \frac{\epsilon}{3}$ we have

$$\begin{aligned}
|H_0(z) - H(z)| &= \left|\int_{\partial D} \mu_0(t)d\,\eta_z(t) - \int_{\partial D} \mu(t)d\,\eta_z(t)\right| \\
&= \left|\int_{\partial D} \mu_0(t) - \mu(t)d\,\eta_z(t)\right| \\
&\leq \int_{\partial D} |\mu_0(t) - \mu(t)|d\,\eta_z(t) \\
&< \int_{\partial D} \gamma d\,\eta_z(t) \\
&= \gamma = \frac{\epsilon}{3}.
\end{aligned}$$

Then for all z such that $1 \geq |z| \geq r_0$ we have

$$\begin{aligned}
|\phi_0(z) - \phi(z)| &= |\phi_0(z) - H_0(z) + H_0(z) - H(z) + H(z) - \phi(z)| \\
&\leq |\phi_0(z) - H_0(z)| + |H_0(z) - H(z)| + |H(z) - \phi(z)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

We are half way done. Let's now concentrate on the set $|z| \leq r_0$. This is a compact set and thus $\phi(z)$ and $\phi_0(z)$ are continuous and thus uniformly continuous on this set. This means that they both achieve their maximum on this set and thus suppose that $|\phi(z)| \leq r_1$ and $|\phi_0(z)| \leq r_1$ for some $r_1 < 1$. Now let's consider the function $F_{\mu,z}$ where $|z| \leq r_0$ and $|w| \leq r_1$. Now pick any z_0 such that $|z_0| \leq r_0$. Then we have the functions $F_{\mu_0,z_0}(w)$ and $F_{\mu,z_0}(w)$ are continuous (for $|w| \leq r_1$) and in fact 1-1 since the Jacobian is positive by Lemma 3.2 on page 29. This means these functions have an inverse and by the Inverse Function Theorem (Theorem A.3 on page 60) this inverse is continuous.

Since we are dealing with a compact set (continuous functions map compact sets to compact sets) the inverse is in fact uniformly continuous. Now suppose that w_0 is such that $F_{\mu_0,z_0}(w_0) = 0$ (we know $|w_0| \le r_1$ because of how we picked r_1). Now

$$\begin{aligned}
|F_{\mu,z_0}(w_0)| &= |F_{\mu,z_0}(w_0) - F_{\mu_0,z_0}(w_0)| \\
&= \left| \int_{\partial D} \frac{(1-|w_0|^2)(\mu(t)-w_0)}{1-\overline{w_0}\mu(t)} d\,\eta_{z_0}(t) - \int_{\partial D} \frac{(1-|w_0|^2)(\mu_0(t)-w_0)}{1-\overline{w_0}\mu_0(t)} d\,\eta_{z_0}(t) \right| \\
&= \left| \int_{\partial D} \frac{(1-|w_0|^2)^2(\mu_0(t)-\mu(t))}{(1-\overline{w_0}\mu(t))(1-\overline{w_0}\mu_0(t))} d\,\eta_{z_0}(t) \right| \\
&\le \int_{\partial D} |\mu_0(t)-\mu(t)| \frac{|1-|w_0|^2|^2}{|(1-\overline{w_0}\mu(t))(1-\overline{w_0}\mu_0(t))|} d\,\eta_{z_0}(t)
\end{aligned}$$

then by reverse triangle inequality

$$\le \int_{\partial D} |\mu_0(t)-\mu(t)| \frac{|1-|w_0|^2|^2}{(1-|\overline{w_0}\mu(t)|)(1-|\overline{w_0}\mu_0(t)|)} d\,\eta_{z_0}(t)$$

and now since $|\mu(t)| = 1$ and $|\mu_0(t)| = 1$

$$\begin{aligned}
&\le \int_{\partial D} |\mu_0(t)-\mu(t)| \frac{|1-|w_0|^2|^2}{(1-|\overline{w_0}|)^2} d\,\eta_{z_0}(t) \\
&= (\sup_{z\in\partial D} |\mu_0(z)-\mu(z)|) \int_{\partial D} \frac{|1-|w_0|^2|^2}{(1-|\overline{w_0}|)^2} d\,\eta_{z_0}(t) \\
&\le (\sup_{z\in\partial D} |\mu_0(z)-\mu(z)|) \int_{\partial D} 2^2 d\,\eta_{z_0}(t) \\
&= 4(\sup_{z\in\partial D} |\mu_0(z)-\mu(z)|).
\end{aligned}$$

So how close $F_{\mu,z_0}(w_0)$ is to zero depends only on the maximum difference of the μ and μ_0. Since the inverse of F_{μ,z_0} is uniformly continuous, we can pick a $\sigma(z_0) > 0$ (this σ depends on z_0) such that whenever $\sup_{z\in\partial D} |\mu_0(z) - \mu(z)| < \sigma(z_0)$ we have that $|w - w_0| < \epsilon$ (and so $|\phi(z_0) - \phi_0(z_0)| < \epsilon$). We could further pick $\sigma(z_0)$ such that $\sigma(z_0) < \gamma$ (where γ is as it was defined above). Now $\sigma(z_0)$ is a continuous function defined on a compact set and it thus attains a minimum, and this minimum is not zero because $\sigma(z_0) > 0$ for all z_0 on this compact set. Let's call this minimum just σ.

Now we are finished with the proof. We can pick $\delta = \min\{\sigma, \gamma\}$ and we get that for all $z \in D$ we have $|\phi(z) - \phi_0(z)| < \epsilon$. QED!

We are now ready to finish the proof of the Douady-Earle extension.

Continuation of the proof of Theorem 3.1: Because a rotation around 0 does not change the dilatation anywhere in the disk, we can just study such μ that fix a single point, say -1. Now suppose that we have α and β such that α and β are conformal maps of D onto D and fix -1, and we pick α such that $\alpha \circ \mu \circ \beta$ fixes i, -1 and 1. Now since $\alpha \circ E(\mu) \circ \beta = E(\alpha \circ \mu \circ \beta)$ and since α and β are conformal the maximal dilatation of those two functions is the same. If we consider the conformal map that takes -1 to infinity and the unit disc to the upper half plane, then β corresponds to the affine transformations from Definition 2.1 on page 9, and then the $\alpha \circ \mu$ corresponds to normalized quasisymmetric maps of the real line onto itself. Suppose we call S the family of such normalized μ that admit a K-quasiconformal extension. That is all of the elements of S satisfy the same M-condition. By Theorem 2.2 on page 9 we have that S is a compact set.

If we take $E(\mu)(z) = \phi(z)$ as a function of both z and μ we now know that this is continuous function. Now on any compact set within D such as $|z| \leq r < 1$ we have that the partial derivative of $E(\mu)(z)$ with respect to z exists, is continuous as a function of z and so the sequence (given n large enough)

$$f_n(z,\mu) = \frac{E(\mu)(z) - E(\mu)(z + 1/n)}{1/n}$$

is a uniformly convergent sequence. The uniform limit of continuous functions is continuous by Theorem A.2 on page 60 and so $\frac{\partial E(\mu)}{\partial z}$ is a continuous function in both μ and z (for $z \in D$ and not on the boundary of course). Same can be done for $\overline{z}$.

Now because $E(\mu)(z)$ is a real analytic diffeomorphism as a function of z, we have that

$$\frac{|\frac{\partial E(\mu)}{\partial \overline{z}}(0)|}{|\frac{\partial E(\mu)}{\partial z}(0)|} < 1.$$

What we want to show is that

$$\sup_{\mu \in S} \left\{ \frac{|\frac{\partial E(\mu)}{\partial \overline{z}}(0)|}{|\frac{\partial E(\mu)}{\partial z}(0)|} \right\} < 1.$$

If we can show this, then since the β was arbitrary we could transport any point to 0, pick an appropriate α and still be in S. So we have the dilatation bounded for all

$z \in D$, and have the same bound for all μ satisfying the same M-condition, since all the elements of S satisfy the same M-condition. We have already shown that the partial derivatives of $E(\mu)(z)$ with respect to z and $\overline{z}$ are continuous functions of μ. Since μ are on a compact set, then the supremum is achieved and must thus be less than 1. This means that the Douady-Earle extension $E(\mu)$ of a homeomorphism μ of the unit circle that admits a quasiconformal extension is quasiconformal and that the in fact by Lemma 2.1 on page 10 the quasiconformal constant K of the extension depends only on the quasisymmetricity constant of μ (since all of S satisfies the same M-condition). QED!

CHAPTER 4

COMPARISON OF THE EXTENSIONS

4.1 Maximal Dilatation of the Extensions

Now that we have both the Beurling-Ahlfors and Douady-Earle extensions defined we wish to compare their maximal dilatations. Suppose that we have a boundary correspondence function $\mu(x)$ and suppose that we define

$$K(\mu) = \inf\{K : \mu \text{ has a } K\text{-quasiconformal extension}\}.$$

Also suppose that $K^{BA}(\mu)$ is the maximal dilatation of the Beurling-Ahlfors extension, and that $K^{DE}(\mu)$ is the maximal dilatation of the Douady-Earle extension. By [4] (using results of Ahlfors [1] and Lehtinen [11]) we have

$$K^{BA}(\mu) < \frac{1}{8}e^{\pi K(\mu)},$$

while by [4] we have that

$$K^{DE}(\mu) \leq Ae^{B\,K(\mu)} \qquad \text{for some positive } A \text{ and } B \qquad A < 4 \times 10^8 \quad B < 35.$$

This means we have better bounds for the maximal dilatation of the Beurling-Ahlfors extension.

Now given the M-condition on the boundary we can get a bound on the dilatation purely based on the M. There is in fact a linear bound found by Lehtinen [10] which is $K^{BA}(\mu) \leq 2M$. We can also alter the definition of the extension by adding an extra parameter r, then

$$\phi_r(x, y) = u(x, y) + i\,r\,v(x, y),$$

where $u(x, y)$ and $v(x, y)$ are the same as in the original definition.

Let's call the maximum dilatation of this modified extension K_r^{BA}. We then note that with $r = 1$ we have the same exact extension as before and so $K_1^{BA} = K^{BA}$. With this new definition, Beurling and Ahlfors [2] prove that there exists an r depending only on M such that $K_r^{BA} < M^2$. Lehtinen also proves in [11] that there exists an r depending only on M such that $K_r^{BA} < 2M - 1$. In the same paper Lehtinen constructs an M-quasisymmetric μ such that $K_r^{BA} > \frac{3}{2}M$ for every r.

4.2 Conformal Naturality

The Beurling-Ahlfors extension is easier to define and construct and furthermore we have very nice bounds on its dilatation. However it lacks the conformal naturality property of the Douady-Earle extension. The Beurling-Ahlfors extension has something somewhat similar in fact. Suppose that $BA(\mu) = \phi$ is the Beurling-Ahlfors extension, then by Lemma 2.4 on page 19 we have that if α and β are Möbius transformations that fix infinity (that is they are of the form $az+b$), then we have $\alpha \circ BA(\mu) \circ \beta = BA(\alpha \circ \mu \circ \beta)$. Note that the restriction on $a > 0$ is only in our original lemma since we are looking at sense preserving mappings and we could very well just allow $a < 0$ as well. But the restriction that α and β fix a single point cannot be removed. In fact μ has to fix the same point for the extension to work.

If we wish to look at things in the same context as with the Douady-Earle extensions, that is in the unit disc, then we let $S(z)$ be the Möbius mapping from the real line onto the unit circle bringing ∞ to i. Then we could define the Beurling-Ahlfors extension on a disk as $BA_D(\mu) = S \circ BA(S^{-1} \circ \mu \circ S) \circ S^{-1}$. That is of course if μ fixes i, which means that $S^{-1} \circ \mu \circ S$ fixes ∞, since that is a requirement of the Beurling-Ahlfors extension. Now if α and β are Möbius transformations that fix i, then we have $\alpha \circ BA_D(\mu) \circ \beta = BA_D(\alpha \circ \mu \circ \beta)$. Which is almost conformal naturality, but not quite.

So for the purpose of being able to study the behavior of the extension by just looking at a single point, this is enough. In fact we have used this property when proving that the Beurling-Ahlfors extension is quasisymmetric since we only needed to look at the dilatation at the point i. However if what we need is for the extension operator to commute with composition with any conformal map, then we cannot use the Beurling-Ahlfors extension.

CHAPTER 5

APPLICATIONS

5.1 Absolute Continuity on the Boundary

In this section we will construct a quasisymmetric function of the real line to itself which is not absolutely continuous (See Definition A.1 on page 61). Then by Theorem 2.1 on page 8 we conclude that there exists a quasiconformal mapping of the upper halfplane to itself with such a boundary. Furthermore we will show that we can have the maximal dilatation of that extension as close to 1 as we like. This result was proved in [2]. We will in fact construct a purely singular function (See Definition A.2 on page 61).

Theorem 5.1 *There exists a quasiconformal mapping ϕ of the halfplane to itself whose boundary correspondence is given by a completely singular function μ which satisfies the M-condition for M arbitrarily close to 1. And the maximal dilatation of phi is also arbitrarily close to 1.*

Proof: Given any $M > 1$ we shall take a strictly increasing sequence of numbers m_ν such that $1 < m_\nu < M$. And we will take a fixed number λ such that $0 < \lambda < \frac{m_1-1}{m_1+1} < 1$. We will use the notation $M(\mu)$ to mean the smallest M for which μ satisfies the M-condition. Now we will construct a sequence of quasisymmetric functions $\{\mu_\nu\}$ normalized by $\mu_\nu(0) = 0$ and $\mu_\nu(2\pi) = 2\pi$ and such that they satisfy $M(\mu_\nu) \leq m_\nu < M$. Then by Theorem 2.2 on page 9 we have that this sequence has a subsequence that converges to a quasisymmetric mapping μ such that $M(\mu) \leq M$. We will show that this mapping μ is purely singular that is, $\mu'(x) = 0$ for almost all x, and μ is not constant. Such a mapping is not absolutely continuous by application of Theorem A.7 on page 61.

We will construct a strictly increasing sequence of positive integers $\{n_i\}_1^\infty$ and define μ_ν by

$$\mu_\nu(x) = \int_0^x \prod_{i=1}^{\nu}(1 + \lambda \cos n_i s)ds.$$

It is easy to see (by fundamental theorem of calculus) that

$$\mu_{\nu+1}(x) = \int_0^x (1 + \lambda \cos n_{\nu+1}s)\mu'_\nu(s)ds.$$

We will use the notation $\mu_\nu(\omega)$ and $\mu(\omega)$ for the set functions on the set ω. That is by how much does μ_ν or μ increase on the set ω. If ω is an interval, say $\omega = [a, b]$ then $\mu(\omega) = \mu(b) - \mu(a)$, and in terms of the integral definition of μ_ν we can just take the integral over ω.

So suppose ω is an interval, then we have

$$\begin{aligned} 1 - \lambda &= (1 - \lambda)\left(\frac{\int_\omega \mu'_\nu(s)ds}{\int_\omega \mu'_\nu(s)ds}\right) \\ &\leq \frac{\int_\omega (1 + \lambda \cos n_{\nu+1}s)\mu'_\nu(s)ds}{\int_\omega \mu'_\nu(s)ds} \\ &= \frac{\mu_{\nu+1}(\omega)}{\mu_\nu(\omega)} \\ &= \frac{\int_\omega (1 + \lambda \cos n_{\nu+1}s)\mu'_\nu(s)ds}{\int_\omega \mu'_\nu(s)ds} \\ &\leq (1 + \lambda)\left(\frac{\int_\omega \mu'_\nu(s)ds}{\int_\omega \mu'_\nu(s)ds}\right) \\ &= 1 + \lambda. \end{aligned} \tag{5.1}$$

And now if we have a pair of intervals ω and ω', then using (5.1) we get

$$\frac{1 - \lambda}{1 + \lambda} \leq \frac{\mu_{\nu+1}(\omega)}{\mu_{\nu+1}(\omega')} \bigg/ \frac{\mu_\nu(\omega)}{\mu_\nu(\omega')} \leq \frac{1 + \lambda}{1 - \lambda}. \tag{5.2}$$

Now suppose that $\omega = [x, x + t]$ and $\omega' = [x - t, x]$, which are of equal length, then for μ_1 we get immediately that

$$\frac{1 - \lambda}{1 + \lambda} \leq \frac{\int_\omega 1 + \lambda \cos n_1 s\, ds}{\int_{\omega'} 1 + \lambda \cos n_1 s\, ds} = \frac{\mu_1(\omega)}{\mu_1(\omega')} \leq \frac{1 + \lambda}{1 - \lambda},$$

which is just the M-condition for μ_1. So the M-condition is satisfied with $M(\mu_1) \leq \frac{1+\lambda}{1-\lambda} < m_1$. This means that n_1 can be arbitrary.

We will proceed inductively. So suppose we have already determined $n_1, n_2, \cdots, n_\nu$. Then suppose that $N_\nu = \sum_{i=1}^\nu n_i$, and we take that $n_{\nu+1} > N_\nu$. From this we can conclude that

$$\int_0^{2\pi} \mu'_\nu(s) \cos n_{\nu+1}s\, ds = 0, \tag{5.3}$$

which is easily seen by the fact that all the different cos terms are orthogonal to each other. In fact this is true for any interval of length 2π, not just $[0, 2\pi]$. Now by the Riemann-Lebesgue Theorem (Theorem A.6 on page 61) we have that

$$\int_0^x \mu'_\nu(s) \cos n_{\nu+1} s \, ds \to 0,$$

at least pointwise as $n_{\nu+1} \to \infty$. Now this is also uniform in x. To see that first define $\rho_n(x) = \int_0^x \mu'_\nu(s) \cos ns \, ds$. We can see that $\rho(0) = 0$ and $\rho(2\pi) = 0$ and also $\rho(x + 2\pi) = \rho(x) + 2\pi$ by (5.3), so we only need to look at $x \in [0, 2\pi]$, and we wish to show that $\rho_n(x) \to 0$ uniformly as $n \to 0$. First note that if C is a constant, then

$$\int_a^b C \cos(ns) ds = \frac{C}{n} \int_{na}^{nb} \cos(s) ds \to 0$$

at the same rate no matter what a and b are as the integral of a cosine over any interval is no larger than 1 in absolute value because of the oscillations. So if $\psi(x)$ is a step function then we know

$$\int_0^x \psi(s) \cos(ns) ds \to 0$$

uniformly for all x. So suppose we are given an $\epsilon > 0$, then by Theorem A.5 on page 61 we know that we can find a step function $\psi(x)$ such that $\int_0^{2\pi} |\psi(s) - \mu'_\nu(s)| ds < \frac{\epsilon}{2}$. Also we can find an N such that for all $n \geq N$ we have that $\left| \int_0^x \psi(s) \cos(ns) ds \right| < \frac{\epsilon}{2}$. Thus for $n \geq N$,

$$\begin{aligned}
|\rho_n(x)| &= \left| \int_0^x \mu'_\nu(s) \cos ns \, ds \right| \\
&= \left| \int_0^x (\psi(s) + \mu'_\nu(s) - \psi(s)) \cos ns \, ds \right| \\
&\leq \left| \int_0^x \psi(s) \cos ns \, ds \right| + \int_0^x |(\mu'_\nu(s) - \psi(s)) \cos ns| \, ds \\
&\leq \left| \int_0^x \psi(s) \cos ns \, ds \right| + \int_0^x |\mu'_\nu(s) - \psi(s)| \, ds \\
&\leq \left| \int_0^x \psi(s) \cos ns \, ds \right| + \int_0^{2\pi} |\mu'_\nu(s) - \psi(s)| \, ds \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

So $\rho_n(x) \to 0$ uniformly as $n \to \infty$. This means that $\mu_{\nu+1}$ tends uniformly to μ_ν, since

$$\mu_{\nu+1}(x) = \int_0^x \mu'_\nu(s)(1 + \lambda \cos n_{\nu+1} s) \, ds = \int_0^x \mu'_\nu(s) \, ds + \lambda \int_0^x \mu'_\nu(s) \cos n_{\nu+1} s \, ds.$$

We proceed by induction since n_1 is arbitrary and we have seen that $M(\mu_1) < m_1$. So assume that $n_1, \cdots, n_\nu$ have been picked, and that we have $M(\mu_i) \leq m_i$ for $1 \leq i \leq \nu$. Let ω and ω' be two intervals of the form $[x, x+t]$ and $[x-t, x]$. Since $\mu_\nu(x)$ is a smooth analytic function, and thus the left and right derivative are equal at every point, we know that

$$\begin{aligned}
\lim_{t\to 0^+} \frac{\mu_\nu(\omega)}{\mu_\nu(\omega')} &= \lim_{t\to 0^+} \frac{\mu_\nu(x+t) - \mu_\nu(x)}{\mu_\nu(x) - \mu_\nu(x-t)} \\
&= \lim_{t\to 0^+} \frac{\mu_\nu(x+t) - \mu_\nu(x)}{t} \frac{t}{\mu_\nu(x) - \mu_\nu(x-t)} \\
&= 1.
\end{aligned}$$

The function $\mu'_\nu(x)$ is uniformly continuous, because it is just a trigonometric polynomial, and so we can pick $\delta_\nu > 0$ such that whenever $t < \delta_\nu$ we have

$$\frac{1+\lambda}{1-\lambda}\frac{1}{m_{\nu+1}} \leq \frac{\int_\omega \mu'_\nu(s)\, ds}{\int_{\omega'} \mu'_\nu(s)\, ds} = \frac{\mu_\nu(\omega)}{\mu_\nu(\omega')} \leq \frac{1-\lambda}{1+\lambda} m_{\nu+1}.$$

Combining this with (5.2) we get, as long as $t < \delta_\nu$,

$$\frac{1}{m_{\nu+1}} \leq \frac{\mu_{\nu+1}(\omega)}{\mu_{\nu+1}(\omega')} \leq m_{\nu+1}. \tag{5.4}$$

Now since $\mu_{\nu+1}$ tends uniformly to μ_ν and since $M(\mu_\nu) \leq m_\nu < m_{\nu+1}$ we can choose $n_{\nu+1}$ large enough such that (5.4) is true for all t, and so we have the M-condition satisfied and $M(\mu_{\nu+1}) \leq m_{\nu+1}$. Now we could also always choose $n_{\nu+1}$ even larger to make sure that for all x we have that

$$|\mu_{\nu+1}(x) - \mu_\nu(x)| < \frac{1}{\nu^2 N_\nu}, \tag{5.5}$$

which we can do since $\mu_{\nu+1}$ tends uniformly to μ_ν. And so we shall pick $n_{\nu+1}$ large enough such that both of the above conditions are satisfied.

So now we have a sequence of normalized quasisymmetric mappings all of which satisfy the M-condition for our initial $M > 1$. By Theorem 2.2 on page 9 we know that the limit of this sequence is also a quasisymmetric mapping which satisfies the M-condition for the same M. We call this limit function μ and so what is left to show is that this limit function is purely singular. We note that since (5.3) is true for all ν, then

$\mu(x+2\pi) = \mu(x)+2\pi$, which really means that we only have to worry about the interval $[0, 2\pi]$. If we can show that μ is purely singular on this interval then it is purely singular everywhere.

First we write a related function in terms of the complex Fourier series

$$g(x) = \log(1 + \lambda \cos x) = \sum_{k=-\infty}^{\infty} \gamma_k \, e^{ikx},$$

where

$$\sum_{k=-\infty}^{\infty} |\gamma_k| < +\infty,$$

because $g(x)$ is continuous and twice differentiable on the interval $[0, 2\pi]$ and $g(0) = g(2\pi)$, so the periodic extension is continuous. Also note that

$$\gamma_0 = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + \lambda \cos s) ds < 0,$$

so let's from now on define $a = -\gamma_0$, and notice that $a > 0$.

Since the sum $\sum_{k=-\infty}^{\infty} |\gamma_k|$ converges to some positive number we can pick a q such that

$$\sum_{|k|>q} |\gamma_k| < \frac{a}{2},$$

which gives us

$$g(x) < -\frac{a}{2} + \sum_{1 \le |k| \le q} \gamma_k \, e^{ikx}.$$

Now if we take the log of $\mu_\nu'(x)$ we notice that it's a sum of the g's with different arguments. Specifically

$$\begin{aligned} \log \mu_\nu'(x) &= \sum_{j=1}^{\nu} g(n_j x) \\ &< -\frac{a\nu}{2} + \sum_{j=1}^{\nu} \left(\sum_{1 \le |k| \le q} \gamma_k \, e^{ikn_j x} \right). \end{aligned}$$

We can thus write

$$\log \mu_\nu'(x) < -\frac{a\nu}{2} + h_\nu(x), \tag{5.6}$$

where we define

$$h_\nu(x) = \sum_{j=1}^{\nu}\left(\sum_{1\le|k|\le q}\gamma_k\, e^{ikn_jx}\right).$$

We can see that the Fourier expansion of $h_\nu(x)$ will by definition contain at most $2q\nu$ different terms, and in fact each coefficient will be less than $S = \sum_{k=-\infty}^{\infty}|\gamma_k|$. Now if we look at the Fourier series for $h_\nu^2(x)$, it in fact has the same number of terms as the Fourier series for $h_\nu(x)$, but they are bounded by S^2. So if we integrate term by term we can see that

$$\frac{1}{2\pi}\int_0^{2\pi} h_\nu^2(s)ds \le 2q\nu S^2.$$

Now the set where $h_\nu^2(x) > \frac{a^2\nu^2}{16} > 0$ (only for $x \in (0, 2\pi)$) must be of measure less than $\frac{64\pi qS^2}{a^2\nu}$, else the integral would be larger than it is. This is the set where $h_\nu(x) > \frac{a\nu}{4}$ and if we exponentiate both sides of (5.6) and define E_ν as the set where

$$\mu_\nu'(x) > e^{-\frac{a\nu}{4}},$$

we can see that

$$m(E_\nu) < \frac{64\pi qS^2}{a^2\nu} = \frac{64\pi qS^2}{a^2}\,\frac{1}{\nu}.$$

We notice that q, S and a only depend on the definition of $g(x)$ and thus all of them depend on the choice of λ only. So if we let $c = \frac{64\pi qS^2}{a^2}$, we can see that $m(E_\nu) < \frac{c}{\nu}$, where c is a constant that depends only on λ. This means that as $\nu \to \infty$ we have $m(E_\nu) \to 0$.

Now we wish to show that $\mu(E_\nu) \to 2\pi$. By $\mu(E_\nu)$ we mean that the amount that the function μ grows on the set E_ν. Now since $\mu_\nu'(x)$ is a trigonometric polynomial of degree at most N_ν, which we can see since if we look at the Fourier series of $\mu_\nu'(x)$ all the terms higher than N_ν will be zero since $\mu_\nu'(x)$ will be orthogonal to sin's and cos's with higher degree by definition of $\mu_\nu'(x)$. This means that E_ν is in fact at most N_ν arcs when x is considered on the unit circle. Now by (5.5) we get that for all x

$$|\mu(x) - \mu_\nu(x)| \le \sum_{k=1}^{\infty}|\mu_{\nu+1}(x) - \mu_\nu(x)| < \sum_{k=1}^{\infty}\frac{1}{\nu^2 N_\nu} = \frac{1}{(\nu-1)N_\nu}.$$

Now E_ν is at most N_ν arcs so let's call them $A_{\nu,k}$. Also μ and μ_ν are increasing functions which satisfy $\mu(x + 2\pi) = \mu(x) + 2\pi$ and $\mu_\nu(x + 2\pi) = \mu_\nu(x) + 2\pi$ and so for each arc $A_{\nu,k}$ which goes from α_k to β_k we get that

$$|\mu(A_{\nu,k}) - \mu_\nu(A_{\nu,k})| \leq |\mu(\alpha_k) - \mu_\nu(\alpha_k)| + |\mu(\beta_k) - \mu_\nu(\beta_k)| < \frac{2}{(\nu-1)N_\nu},$$

which implies

$$|\mu(E_\nu) - \mu_\nu(E_\nu)| \leq \sum_{k=1}^{N_\nu} |\mu(A_{\nu,k}) - \mu_\nu(A_{\nu,k})| < \frac{2}{\nu - 1}.$$

Now this means that as $\nu \to \infty$ then $\mu(E_\nu) \to \mu_\nu(E_\nu)$. Now since μ_ν is an increasing, analytic, continuous function and the derivative outside of E_ν gets smaller and smaller, and since $\mu_\nu([0, 2\pi]) = 2\pi$ we must have that $\mu_\nu(E_\nu) \to 2\pi$. And this means that $\mu(E_\nu) \to 2\pi$. Since $m(E_\nu) \to 0$ we can find a subsequence $\{E_{\nu_k}\}$ such that we have $m(E_{\nu_k}) < \frac{1}{2^k}$. Then by the subadditivity of Lebesgue measure we have that if we pick any positive integer j,

$$m\left(\cup_{k=j}^{\infty} E_{\nu_k}\right) \leq \sum_{k=j}^{\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}},$$

which means that $m\left(\cup_j^\infty E_{\nu_k}\right) \to 0$ as $j \to \infty$. And this is a sequence of nested sets. This means that there exists a set Z of measure 0 that is the limit (intersection) of the $\cup_j^\infty E_{\nu_k}$. Now since $\mu(E_{\nu_k}) \to 2\pi$ we must have that $\mu(Z) = 2\pi$. So we have a set Z, where $m(Z) = 0$ and $\mu(Z) = 2\pi$. Since μ cannot grow any more than 2π on the interval $[0, 2\pi]$, this means that outside of Z, μ must be constant (on a set of measure 2π) and thus have derivative 0 (if the derivative exists). Since by Theorem A.8 on page 62 increasing functions have derivative defined almost everywhere, then the derivative must be 0 on a set of measure 2π and thus μ is purely singular. QED!

Now to have some sort of a feel of how such a function looks we look at the graphs of first few elements of the sequence $\{\mu_\nu(x)\}$, with $n_1 = 1$, $n_\nu = N_{\nu-1} + 1$ and with $\lambda = \frac{1}{2}$. In Figure 5.1 on the next page, we can see μ_1, μ_2, μ_3 and μ_{20} with those parameters. It is also interesting to see how the Beurling-Ahlfors extension looks like. Figure 5.2 on page 55 gives the Beurling-Ahlfors extension of μ_{20}. It should be noted that the n_ν's we use do not follow all the conditions we set in the proof, however this should only affect the maximal dilatation and the rate of convergence. From several trial tests it seems that for our choice of n_ν the convergence is actually very fast already. It should be noted that the sharp edges in the graph of the extension are there because not enough points are computed to produce completely smooth lines and not because the extension has such sharp edges.

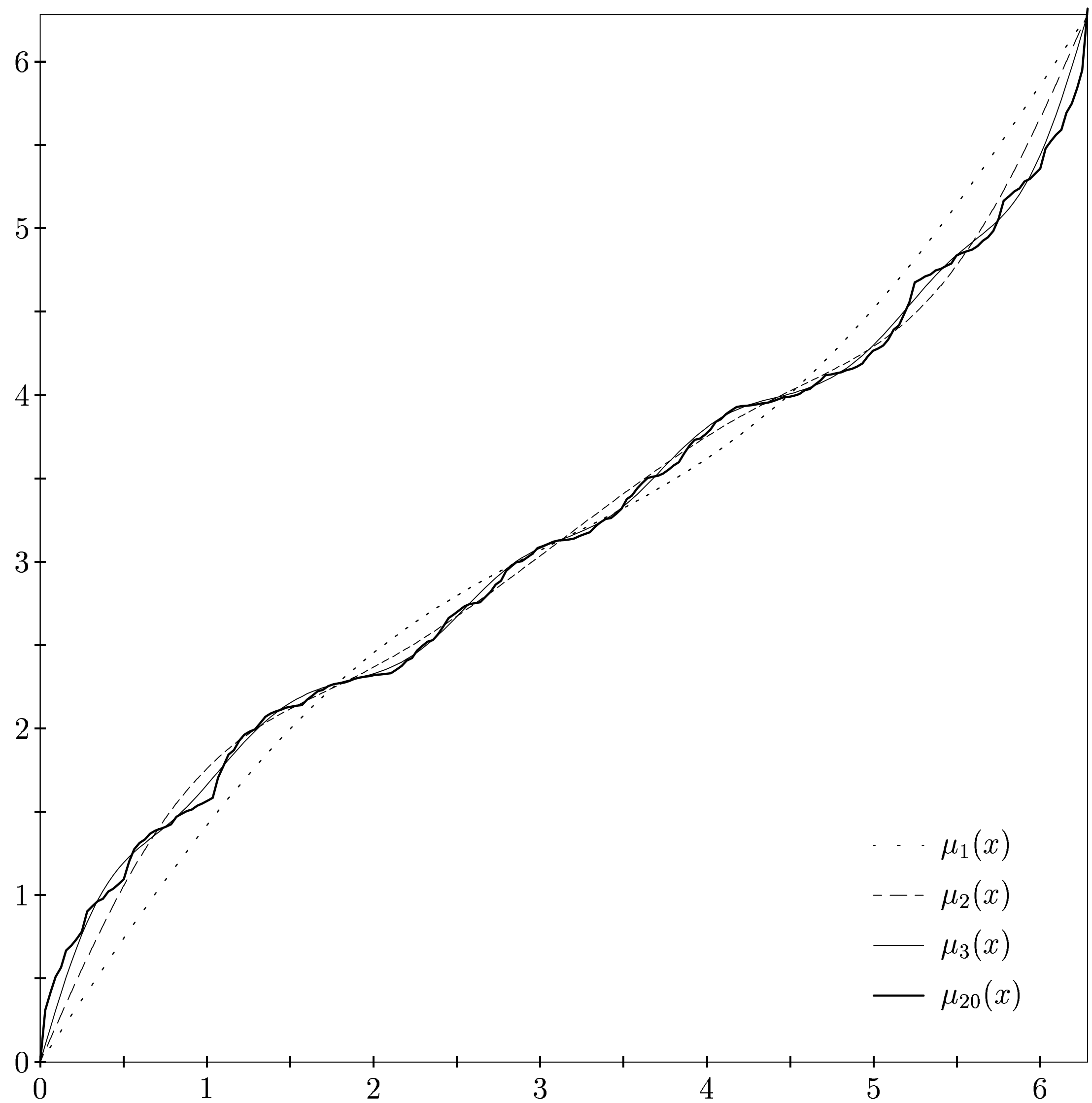
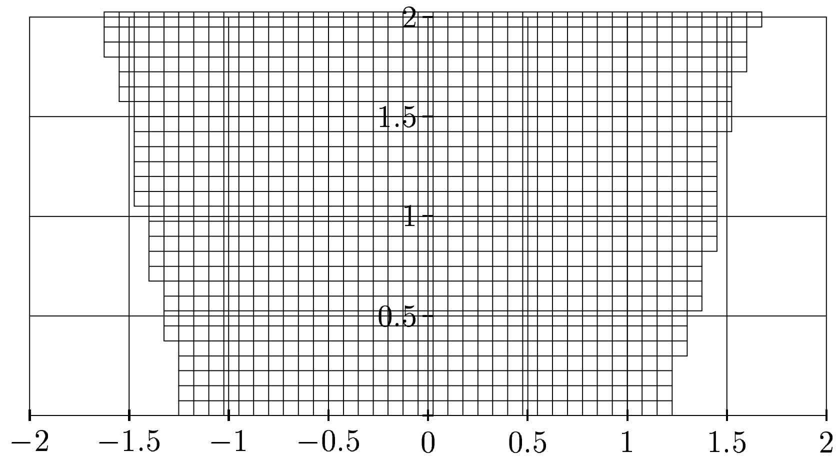


Figure 5.1: Graph of $\mu_1(x)$, $\mu_2(x)$, $\mu_3(x)$ and $\mu_{20}(x)$



The above grid is taken to:

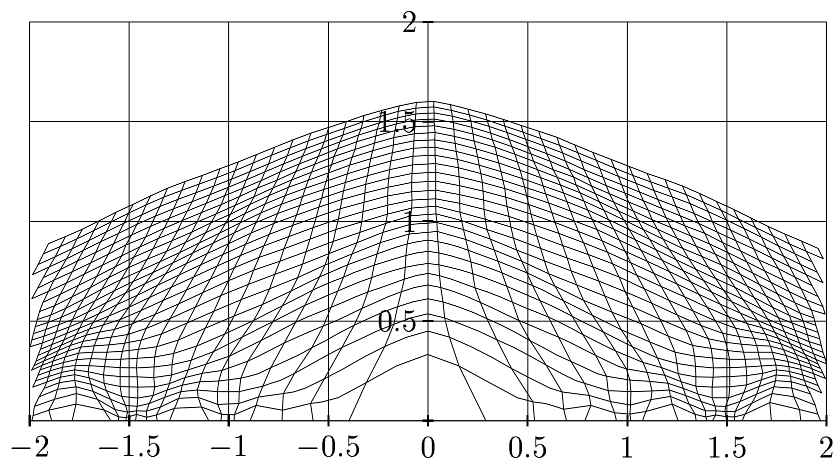


Figure 5.2: Graph of the Beurling-Ahlfors Extension of $\mu_{20}(x)$

5.2 Application to Teichmüller Spaces

Some applications require the conformal naturality property of the Douady-Earle extension. As a small example we will give a proof of an alternative definition for a *Teichmüller space* as done by Douady and Earle [4]. This application requires the fact that for any quasisymmetric map of the unit circle there exists a conformally natural quasiconformal extension to the unit disc.

So let M be the open unit ball in $L^\infty(D, \mathbb{C})$. Now for each $\sigma \in M$ there is a unique quasiconformal map f^σ of $\bar{D}$ onto itself that fixes the points 1, i, and -1 and satisfies the Beltrami differential equation

$$f^\sigma_{\bar{z}}(z) = \sigma(z) f^\sigma_z(z),$$

and let μ^σ be the restriction of f^σ to the unit circle.

Remember that G is the group of conformal maps of the unit disc to itself. Now we let Γ be a Fuchsian group, that is, a subgroup of G which is discrete (that is, any convergent sequence in the group eventually becomes just the repetition of one element).

We can define

$$M(\Gamma) = \{\sigma \in M; f^\sigma \circ \gamma \circ (f^\sigma)^{-1} \in G \text{ for all } \gamma \in \Gamma\}. \tag{5.7}$$

Douady and Earle [4] start with a different definition but prove that (5.7) is equivalent, and it is what we need for the proof. Intuitively this is all the complex dilatations, such that when we compose the associated quasiconformal mapping with a member of Γ as above we get a conformal map. Also let $\mathcal{H}(\partial D)$ be the space of all homeomorphisms of the unit circle. We can now define the Teichmüller space.

Definition 5.1 *We define the Teichmüller space for the Fuchsian group* Γ *as*

$$T(\Gamma) = \{\mu \in \mathcal{H}(\partial D); \mu = \mu^\sigma \textit{ for some } \sigma \in M(\Gamma)\}.$$

So a Teichmüller space is really all the quasisymmetric maps of the unit circle such that there is a quasiconformal extension to the whole disk whose complex dilatation is in $M(\Gamma)$. If we denote by 1 the trivial subgroup of G, then we can see that $M(1) = M$

by definition and then $T(1)$ is the space of all quasisymmetric maps of the unit circle. That is, all the maps that allow a quasiconformal extension.

Now if $\mu \in T(\Gamma)$, then we will denote by $\tilde{\mu}$ the quasiconformal extension to $\bar{D}$ such that μ is the boundary of this extension and $\tilde{\mu} = f^\sigma$ for some $\sigma \in M(\Gamma)$, which we know can be done by the definition of $T(\Gamma)$.

The following is a theorem due to Tukia [16], with a proof from Douady and Earle [4]. If we let $E(\mu)$ be the Douady-Earle extension of μ (or any conformally natural extension in fact), we get

Theorem 5.2 *Given a Fuchsian group* Γ *we have*

$$T(\Gamma) = \{\mu \in T(1); \tilde{\mu} \circ \gamma \circ \tilde{\mu}^{-1} \in G \text{ for all } \gamma \in \Gamma\}.$$

So this gives an alternative definition of the Teichmüller space, one that is similar to our definition of $M(\Gamma)$. It says that if we extend the μ to the disk, then if we compose this as above with an element of Γ we get back a conformal mapping.

Proof: First define $S = \{\mu \in T(1); \tilde{\mu} \circ \gamma \circ \tilde{\mu}^{-1} \in G$ for all $\gamma \in \Gamma\}$. Now by (5.7) we have that if $\sigma \in M(\Gamma)$ then $\mu^\sigma \in S$, since $\tilde{\mu} = f^\sigma$. This means that $T(\Gamma) \subset S$.

Conversely suppose that $\mu \in S$, this means that for some all $\gamma \in \Gamma$ we have $\mu \circ \gamma = g \circ \mu$ where $g \in G$. Now we can extend both sides since they are equal and get $E(\mu \circ \gamma) = E(g \circ \mu)$, by conformal naturality we get $E(\mu) \circ \gamma = g \circ E(\mu)$ or

$$E(\mu) \circ \gamma \circ E(\mu)^{-1} = g \in G.$$

So we have to prove that $E(\mu) = f^\sigma$ for some $\sigma \in M(\Gamma)$. Now by Theorem 3.1 on page 27, we have that $E(\mu)$ is quasiconformal and thus $E(\mu) = f^\sigma$ where $\sigma \in M$. Now by definition of the complex dilatation we can write

$$\sigma = \frac{E(\mu)_{\overline{z}}}{E(\mu)_z}.$$

Now since $f^\sigma = E(\mu)$ and since $\mu = \mu^\sigma \in S$ we have $f^\sigma \circ \gamma \circ (f^\sigma)^{-1} \in G$ for all $\gamma \in \Gamma$ and so $\sigma \in M(\Gamma)$ by (5.7) and this means that $S \subset T(\Gamma)$. Thus $S = T(\Gamma)$. QED!

For further discussion of Teichmüller spaces see Gardiner and Lakic [7].

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APPENDIX A

THEOREMS USED

THEOREMS USED

We need a few basic theorems and results and so we list them here. First the Arzelà-Ascoli theorem with statement due to [6].

Theorem A.1 (Arzelà-Ascoli) *Let E be a compact subset of $\mathbb{C}$, and let $\mathcal{F}$ be a family of continuous complex-valued functions on E that is uniformly bounded. Then the following are equivalent:*

1. *The family $\mathcal{F}$ is equicontinuous at each point of E.*
2. *Each sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly on E.*

Specifically we will use this result to show that there always exists a uniformly converging subsequence in any sequence of functions in such a family. Of course this is also true for real valued functions as well, and this is what we will use.

We also need a few basic calculus results. The following three theorems are all due to [5].

Theorem A.2 *Suppose that $\{f_n : D \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to the function $f : D \to \mathbb{R}$. Then the limit function $f : D \to \mathbb{R}$ is also continuous.*

Theorem A.3 (Inverse Function Theorem) *Let O be an open subset of the plane $\mathbb{R}^2$ and suppose the mapping $f : O \to \mathbb{R}^2$ is continuously differentiable. Let (x_0, y_0) be a point in O at which the derivative matrix $Df(x_0, y_0)$ is invertible (the Jacobian is not zero). Then there is a neighborhood U of the point (x_0, y_0) and a neighborhood V of its image $f(x_0, y_0)$ such that $f : U \to V$ is one-to-one and onto. Furthermore the inverse mapping $f^{-1} : V \to U$ is also continuously differentiable.*

Theorem A.4 (Implicit Function Theorem) *Let n and k be positive integers, let $\mathcal{O}$ be an open subset of $\mathbb{R}^{n+k}$, and suppose that the mapping $f : \mathcal{O} \to \mathbb{R}^k$ is continuously differentiable. At the point (x_0, y_0) (where the first coordinate is n dimensional and the second k dimensional) in $\mathcal{O}$, suppose that $f(x_0, y_0) = 0$ and that the Jacobian with respect to y_0 is non zero. Then there exists a number $r > 0$ and a continuously differentiable $g : B(x_0; r) \to \mathbb{R}^k$ where $B(x_0; r)$ is an open ball around x_0 of radius r such that*

1. $f(x, g(x)) = 0$ *for all* $x \in B(x_0; r)$
2. *whenever* $\|x - x_0\| < r$, $\|y - y_0\| < r$, *and* $f(x, y) = 0$ *then* $y = g(x)$.
3. *and lastly* $D_x f(x, g(x)) + D_y f(x, g(x)) \cdot Dg(x) = 0$ *for all* $x \in B(x_0; r)$

Next we need some theorems from the Lebesgue integral theory. The next two theorems come from Royden [14]. First is a result about step functions.

Theorem A.5 *Suppose* f *is integrable over* E*, then given any* $\epsilon > 0$ *there exists a step function* ψ *such that*

$$\int_E |f - \psi| < \epsilon.$$

And the second is the Riemann-Lebesgue Theorem.

Theorem A.6 (Riemann-Lebesgue) *Suppose* f *is an integrable function on* $(-\infty, \infty)$ *then*

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0.$$

Specifically, we can multiply $f(x)$ by any characteristic function and the above becomes true for any subset of $(-\infty, \infty)$.

We also need the definitions for absolutely continuous functions and singular functions. These come from [14].

Definition A.1 *A real-valued function* f *defined on* $[a, b]$ *is said to be absolutely continuous on* $[a, b]$ *if, given* $\epsilon > 0$*, there is a* $\delta > 0$ *such that*

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

for every finite collection $\{(x_i, x_i')\}$ *of non overlapping intervals with*

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Definition A.2 *A monotone, non-constant, function* f *on* $[a, b]$ *is called singular if* $f'(x) = 0$ *almost everywhere.*

So if we can show that a function has derivative 0 almost everywhere on $[a, b]$, and $f(a) \neq f(b)$ then we have shown that it is a purely singular function.

Theorem A.7 *If* f *is absolutely continuous on* $[a, b]$ *and* $f'(x) = 0$ *almost everywhere, then* f *is constant.*

Now by the above theorem we know that a singular function cannot be absolutely continuous. Also another useful result from Royden about monotone functions is the following theorem.

Theorem A.8 *Let f be an increasing real-valued function on $[a, b]$ then f is differentiable almost everywhere.*

There is also a need for some results about harmonic maps of a disk. The following theorem comes from [15].

Theorem A.9 (Rado) *Suppose $\Omega \subset \mathbb{R}^2$ is a convex domain with a smooth boundary $\partial\Omega$, and suppose D is the unit disc. Given any homeomorphism $\mu : \partial D \to \partial\Omega$, there exists a unique harmonic map $u : D \to \Omega$ such that $u = \mu$ on ∂D and u is a diffeomorphism.*

Specifically we will take Ω to be the unit disc as well, and we will use this theorem, to show that the harmonic extension of the unit disc is unique and secondly that the Jacobian must be non-zero (since it is a diffeomorphism).

We need some results from topology as well. First we will need the Abstract Monodromy Theorem. For this we need a few definitions. All of this comes from [3]

Definition A.3 *Suppose Ω is a topological space, then a* covering space *of Ω is a pair (X, ρ) where X is a connected topological space and ρ is a continuous function of X onto Ω such that for each $\omega \in \Omega$ there is a neighborhood Δ of ω such that each component of $\rho^{-1}(\Delta)$ is open and ρ maps each of these components homeomorphically onto Δ.*

Definition A.4 *Let (X, ρ) be a covering space of Ω, and let γ be a path in Ω. A path $\tilde{\gamma}$ in X is called a* lifting *of γ if $\rho \circ \tilde{\gamma} = \gamma$*

Definition A.5 *Two paths $\gamma : [0, 1] \to X$ and $\sigma : [0, 1] \to X$ in a topological space X with $\gamma(0) = \sigma(0) = a$ and $\gamma(1) = \sigma(1) = b$ are* fixed end point homotopic *or* FEP homotopic *if there exists a continuous map $\Gamma : [0, 1] \times [0, 1] \to X$ such that*

$$\Gamma(s, 0) = \gamma(s), \qquad \Gamma(s, 1) = \sigma(s),$$
$$\Gamma(0, t) = a, \qquad \Gamma(1, t) = b.$$

Where $0 \leq s, t \leq 1$.

Now we can state the (abstract) Monodromy Theorem.

Theorem A.10 (Monodromy Theorem) *Let (X, ρ) be a covering space of Ω and let γ and σ be two paths in Ω with the same initial and final points. Let $\tilde{\gamma}$ and $\tilde{\sigma}$ be paths in X with the same initial points such that $\tilde{\gamma}$ and $\tilde{\sigma}$ are liftings of γ and σ respectively. If γ is FEP homotopic to σ in Ω then $\tilde{\gamma}$ and $\tilde{\sigma}$ have the same final points and are FEP homotopic in X.*

We also need some results from differential topology. The following theorems and definitions come from [8].

Definition A.6 *Two functions, $f_0 : X \to Y$ and $f_1 : X \to Y$ are* homotopic *if there exists a smooth map $F : X \times [0,1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.*

Definition A.7 *If X is a subset of $\mathbb{R}^N$ and it is locally diffeomorphic to $\mathbb{R}^k$, then X is called a k-*dimensional manifold. *Furthermore if X is locally diffeomorphic to the halfspace H^k such that each point in X possesses a neighborhood diffeomorphic to an open set in H^k then X is called a k-*dimensional manifold with boundary. *The points that are mapped to the boundary of H^k are the boundary, ∂X of X.*

Now we need to define the degree of a mapping f. First we define $sign\, df_x$, that is the sign of the derivative of f at the *regular point* x, that is a point where the derivative matrix is non-singular, as +1 if the derivative matrix preserves orientation or -1 if it reverses orientation. Next we define

$$deg(f; y) = \sum_{x \in f^{-1}(y)} sign\, df_x.$$

In fact this is the same for all regular points. And so

Definition A.8 *The* degree *of a mapping is the $deg(f; y)$ for some regular value y*

Next we need to define the *index* of a zero of a smooth vector field on a manifold.

Definition A.9 *Suppose M is a manifold and $\vec{v} : M \to \mathbb{R}^n$. And suppose that $\vec{x} = 0$ is an isolated zero for some $x \in M$. Then if we take a sphere around x (call it S_ϵ) then the index of the zero is the degree of the map $\frac{\vec{v}(x)}{|\vec{v}(x)|}$ restricted to S_ϵ.*

In two dimensions, the index is really the number times that the direction rotates completely counterclockwise as we move counterclockwise around the zero. We also need this very useful Lemma about indexes of zeros of vectorfields defined over manifolds as given in [12].

Lemma A.1 *Suppose $\vec{v}$ is a vectorfield and $\vec{v}(x)$ is a zero where the derivative matrix is non singular (the Jacobian is non-zero). Then the index of this zero is +1 or -1 depending on if the Jacobian is positive or negative respectively.*

Now we can state a generalized version of the *Hopf Degree Theorem* as stated in [8]. The Hopf theorem is a particular instance of the following theorem. S^k is the boundary of the $k+1$ dimensional ball, that is $S^k = \{x \in \mathbb{R}^{k+1} : |x| = 1\}$.

Theorem A.11 (Extension Theorem) *Let W be a compact, connected, oriented $k+1$ dimensional manifold with boundary, and let $f : \partial W \to S^k$ be a smooth map. f extends to a globally defined map $F : W \to S^k$, with $\partial F = f$, if and only if the degree of f is zero.*

In particular we are interested in this theorem in the plane where S^1 is the unit circle. We also need the Hopf Degree Theorem itself.

Theorem A.12 (Hopf Degree Theorem) *Two maps of a compact, connected, oriented k-manifold X into S^k are homotopic if and only if they have the same degree.*

APPENDIX B

COMPUTER CODE USED

COMPUTER CODE USED

All of the figures used have been produced with the ePix program [9] version 0.8.9. This program takes C++ code using special calls and produces eepic format files for direct inclusion in LaTeX documents. The website for this program has extensive documentation on its use. Where normal real numbers are needed, the *double* type is used, and where a complex number is needed, the ePix *pair* type is used. Unfortunately the ePix division of *pair* values does not seem to be standard complex number division and so I have implemented a "div" routine in places where this is needed. If more than one picture is needed to be produced from the same code, I use environment variables to tell the program which plot to produce. For example in the figures for graphing the extensions of the upper half plane to itself, the code produces either the grid, or where the grid is taken with the mapping depending on if the PLOTGRID environment variable is set.

It should be noted that some routines, such as the "integral" routine I have implemented once and reused everywhere where they were needed.

B.1 Computer Code to Produce Figure 2.1

For Figure 2.1 on page 17 we need to plot the Beurling-Ahlfors extension to the upper half plane of the quasisymmetric mapping $\mu(x) = x^3$.

To plot the Beurling-Ahlfors extension we need to calculate integrals and so the midpoint rule with 500 equally spaced rectangles for each calculation is used. Since the function that we deal with is smooth enough, this produces very accurate results. When run with the explicit formula calculated for $\mu(x) = x^3$ the output was indistinguishable from the output gotten with the numerical approximation of the integral. To change the function being used as μ all that is needed is to change the definition of the "mu" function in the code. The code is run first with the environment variable PLOTGRID set to produce the source grid and then without to produce the resulting graph.

What follows is the C++ source code for generating the plot of Figure 2.1 with ePix.

```
#include "epix.h"
using namespace std;
using namespace ePiX;

#define SIZE 2
#define LABELGRID 0.5
#define GRID 0.075

static double
integral (double (* func) (double), double a, double b, int p)
{
  double sum = 0.0;
  double len = b-a;
  double em = len/p;
  int i;

  for (i = 0; i < p; i++) {
    sum += (func (a + (len*i)/p + em/2) * em);
  }

  return sum;
}

static double
mu (double x)
{
  // mu is x^3
  return x*x*x;
}

static void
function (double x, double y, double *ox, double *oy)
{
  if (y == 0.0) {
    *ox = mu (x);
    *oy = 0.0;
  } else {
    *ox = (1 / (2*y)) * integral (mu, x-y, x+y, 500);
    *oy = (1 / (2*y)) * (integral (mu, x, x+y, 500) -
                         integral (mu, x-y, x, 500));
  }
}
```

```
static int
out_of_bounds (double x, double y)
{
  return (x < - SIZE || x > SIZE || y < 0.0 || y > SIZE);
}

int
main()
{
  double x,y;

  bounding_box (P(-SIZE,0), P(SIZE, SIZE));
  picture (P(300, 150));
  unitlength ("1pt");

  begin ();

  grid (2*SIZE/LABELGRID, SIZE/LABELGRID);
  h_axis (P(-SIZE,0), P(SIZE,0), 2*SIZE/LABELGRID);
  v_axis (P(0,0), P(0,SIZE), SIZE/LABELGRID);
  h_axis_labels (P(-SIZE,0), P(SIZE,0), 2*SIZE/LABELGRID, P(0,-10), c);
  v_axis_labels (P(0,LABELGRID), P(0,SIZE),
                 SIZE/LABELGRID-1, P(-4,0), l);

  for (x = -SIZE; x <= SIZE; x += GRID) {
    for (y = 0; y <= SIZE; y += GRID) {
      double f1x, f1y;
      double f2x, f2y;
      double f3x, f3y;
      double f4x, f4y;
      function (x, y, &f1x, &f1y);
      function (x + GRID, y, &f2x, &f2y);
      function (x + GRID, y + GRID, &f3x, &f3y);
      function (x, y + GRID, &f4x, &f4y);
      if (out_of_bounds (f1x, f1y) ||
          out_of_bounds (f2x, f2y) ||
          out_of_bounds (f3x, f3y) ||
          out_of_bounds (f4x, f4y))
        continue;
      // Switch for pre/post function grid
      if (getenv("PLOTGRID")) {
        line(P(x,y), P(x+GRID,y));
        line(P(x+GRID,y), P(x+GRID,y+GRID));
```

```
        line(P(x+GRID,y+GRID), P(x,y+GRID));
        line(P(x,y+GRID), P(x,y));
      } else {
        line(P(f1x,f1y), P(f2x,f2y));
        line(P(f2x,f2y), P(f3x,f3y));
        line(P(f3x,f3y), P(f4x,f4y));
        line(P(f4x,f4y), P(f1x,f1y));
      }
    }
  }

  end ();

  return 0;
}
```

B.2 Computer Code to Produce Figure 3.3 and Figure 3.4

For Figures 3.3 on page 35 and 3.4 on page 36 we must plot the Douady-Earle extension of x^3 when conformally mapped onto the unit disc.

Since the Douady-Earle extension $E(\mu) = \phi$ is defined implicitly in terms of the vector field

$$F_{z_0}(w) = (1 - |w|^2) \int_{\partial D} \frac{\mu(t) - w}{1 - \overline{w}\mu(t)} p(z_0, t)|dt|,$$

we need to find the zeros of F_{z_0}. Fortunately F_{z_0} will in fact point in the direction of the zero. And so the algorithm to find the value of ϕ at z_0 is to take $w_0 = 0$ as our initial value and then recursively set $w_{n+1} = w_n + F_{z_0}(w_n)$, until the length of the vector returned by F_{z_0} is within some tolerance. This is a fairly fast algorithm to find the zeros and it is clear that this is correct from the proof of the Douady-Earle extension. The body of the function F_{z_0} is again calculated by the midpoint rule, again with 100 rectangles. Increasing both the zero tolerance and the number of rectangles did not produce significantly different plots. It should be noted that if we increase the zero tolerance we must also increase the precision of the integral or the code will loop indefinitely for some z_0. To change the function used for μ, it is only needed to change the function "mu" in the code below.

As for Figure 2.1, when this code is run with PLOTGRID environment variable set, just the grid before being mapped will be produced.

For Figure 3.3 the code is run without any other environment set, and produces a plot of a map from the unit disc to the unit disc. For Figure 3.4 the code is run with RECTANGULAR environment variable set which maps the extension to the upper half plane, such that we get a plot which is similar to that of Figure 2.1 and we can compare the two extensions.

What follows is the C++ source code for generating the plot of Figure 3.3 and 3.4 with ePix.

```
#include <math.h>
#include "epix.h"

using namespace std;
using namespace ePiX;

#define SIZE 2
#define LABELGRID 0.5
#define GRID 0.075

#define conj(c) P((c).x1,-(c).x2)
#define modsq(c) ((c).x1*(c).x1 + (c).x2*(c).x2)

// Unfortunately ePix complex division is broken in the version I used
static pair
div (pair x, pair y)
{
  return P((x.x1*y.x1 + x.x2*y.x2)/(y.x1*y.x1 + y.x2*y.x2),
           (y.x1*x.x2 - x.x1*y.x2)/(y.x1*y.x1 + y.x2*y.x2));
}

// Mobius transform to map the circle onto real line
static pair
S(pair z)
{
  return div((z+P(0,1)),(P(0,1)*z+P(1,0)));
}

static pair
Sinv(pair z)
```

```
{
  return div((z-P(0,1)),(-P(0,1)*z+P(1,0)));
}

static pair
mu (pair t)
{
  // Our mu is the S^-1(S(t)^3)
  pair x = S(t);
  return Sinv(x*x*x);
}

static pair
F (pair z, pair w)
{
  pair sum, t;
  pair modsqz = P(modsq(z),0);
  pair conjw = conj(w);
  pair P1 = P(1,0);
  int i;

#define PARTS 100

  sum = P(0,0);
  for (i = 0; i < PARTS; i++) {
    pair body;

    t = polar (1.0,(i*2*M_PI)/PARTS);

    sum += ( div ((mu(t)-w) , (P1-conjw*mu(t))))
            *
           ( div ((P1-modsqz) , (P(modsq(z-t),0))));
  }

  return sum * P((1-modsq(w)) / PARTS,0);
}

static pair
phi (pair z)
{
  int i,j;
  pair coord = P(0,0);
  pair val = F(z,coord);
```

```
  while (modsq(val) > 0.00001) {
    coord += val;
    // Make sure we are within the circle
    if (modsq(coord) >= 1) {
      coord = coord * P(1.0/sqrt(modsq(coord)),0);
    }
    val = F(z,coord);
  }

  return coord;
}

static int do_grid = 0;

static double phi_mod = 0.0;
static pair
phi_r (double r)
{
  if (do_grid)
    return polar (phi_mod, r);
  if (truncate (phi_mod - 1.0) == 0.0)
    return mu (polar (phi_mod, r));
  else
    return phi (polar (phi_mod, r));
}

static double phi_angle = 0.0;
static pair
phi_ray (double m)
{
  if (do_grid)
    return polar (m, phi_angle);
  if (truncate (m - 1.0) == 0.0)
    return mu (polar (m, phi_angle));
  else
    return phi (polar (m, phi_angle));
}

static void
run_plot (void)
{
  int i;
```

```
  for (i = 1; i <= 8; i++) {
    phi_mod = (1.0*i)/8.0;
    plot (phi_r, 0.0, 2*M_PI, 100);
  }
  for (i = 1; i <= 20; i++) {
    phi_angle = (2*M_PI*i)/20.0;
    plot (phi_ray, 0.0, 1.0, 50);
  }
}

static void
function (double x, double y, double *ox, double *oy)
{
  pair w;
  if (truncate (y) == 0.0) {
    w = S(mu (Sinv(P(x,y))));
  } else {
    w = S(phi (Sinv(P(x,y))));
  }
  *ox = w.x1;
  *oy = w.x2;
  // if *oy is less than 0, let's just make it 0, it's an error
  if (*oy < 0.0)
    *oy = 0.0;
}

static int
out_of_bounds (double x, double y)
{
  return (x < - SIZE || x > SIZE || y < 0.0 || y > SIZE);
}

void
rectangular_plot ()
{
  double x,y;

  bounding_box (P(-SIZE,0), P(SIZE, SIZE));
  picture (P(300, 150));
  unitlength ("1pt");

  begin ();
```

```
  grid (2*SIZE/LABELGRID, SIZE/LABELGRID);
  h_axis (P(-SIZE,0), P(SIZE,0), 2*SIZE/LABELGRID);
  v_axis (P(0,0), P(0,SIZE), SIZE/LABELGRID);
  h_axis_labels (P(-SIZE,0), P(SIZE,0), 2*SIZE/LABELGRID, P(0,-10), c);
  v_axis_labels (P(0,LABELGRID), P(0,SIZE),
                 SIZE/LABELGRID-1, P(-4,0), l);

  for (x = -SIZE; x <= SIZE; x += GRID) {
    for (y = 0; y <= SIZE; y += GRID) {
      double f1x, f1y;
      double f2x, f2y;
      double f3x, f3y;
      double f4x, f4y;
      function (x, y, &f1x, &f1y);
      function (x + GRID, y, &f2x, &f2y);
      function (x + GRID, y + GRID, &f3x, &f3y);
      function (x, y + GRID, &f4x, &f4y);
      if (out_of_bounds (f1x, f1y) ||
          out_of_bounds (f2x, f2y) ||
          out_of_bounds (f3x, f3y) ||
          out_of_bounds (f4x, f4y))
        continue;
      // Switch for pre/post function grid
      if (getenv("PLOTGRID")) {
        line(P(x,y), P(x+GRID,y));
        line(P(x+GRID,y), P(x+GRID,y+GRID));
        line(P(x+GRID,y+GRID), P(x,y+GRID));
        line(P(x,y+GRID), P(x,y));
      } else {
        line(P(f1x,f1y), P(f2x,f2y));
        line(P(f2x,f2y), P(f3x,f3y));
        line(P(f3x,f3y), P(f4x,f4y));
        line(P(f4x,f4y), P(f1x,f1y));
      }
    }
  }

  end ();
}

void
circular_plot ()
```

```
{
  bounding_box (P(-1,-1), P(1, 1));
  picture (P(230, 230));
  unitlength ("1pt");

  begin ();

  h_axis (P(-1,-1), P(1,-1), 4);
  v_axis (P(-1,-1), P(-1,1), 4);
  h_axis_labels (P(-1,-1), P(1,-1), 4, P(-12,-14));
  v_axis_labels (P(-1,-1), P(-1,1), 4, P(-4,0), 1);

  if (getenv ("PLOTGRID"))
    do_grid = 1;
  run_plot ();

  end ();
}

int
main ()
{
  if (getenv ("RECTANGULAR"))
    rectangular_plot ();
  else
    circular_plot ();

  return 0;
}
```

B.3 Computer Code to Produce Figure 5.1

For Figure 5.1 on page 54 we need to plot

$$\mu_\nu(x) = \int_0^x \prod_{i=1}^{\nu}(1 + \lambda \cos n_i x)dx,$$

for several different ν's. We do this for $\nu = 1$, 2, 3 and 20. The λ is set to $\frac{1}{2}$. By experimentation we can see that the curve is more pronounced when λ is larger, which will also mean that the M will also be larger. For n_ν we use $n_1 = 1$ and $n_\nu = N_{\nu-1} + 1$. To calculate the integral more efficiently code computes the integral $\int_{x_{i-1}}^{x_i} \mu'_\nu(x)dx$, where

x_i are the points used for the graph, then this is added to the value computed for x_{i-1}. The integral is again computed using the midpoint rule with 1000 rectangles used for each of the steps. I computed the n_ν's and put them in as an array, so they don't have to be recomputed all the time.

What follows is the code used by ePix for Figure 5.1.

```
#include "epix.h"
using namespace std;
using namespace ePiX;

#define LAMBDA 0.5
#define RECTS 1000

static double
integral (double (* func) (double), double a, double b, int p)
{
  double sum = 0.0;
  double len = b-a;
  double em = len/p;
  int i;

  for (i = 0; i < p; i++) {
    sum += (func (a + (len*i)/p + em/2) * em);
  }

  return sum;
}

static int nu;
static double value;
static double last_x;

static int n[] = {1,2,5,11,23,47,95,191,383,767,1535,3071,6143,12287,
                  24575,49151,98303,196607,393215,786431};


static double
mu_nu_prime (double x)
{
  int i;
  double prod = 1.0;
```

```
  for (i = 0; i < nu; i++) {
    prod *= 1+LAMBDA*std::cos(n[i]*x);
  }

  return prod;
}

static double
mu_nu (double x)
{
  if (x == last_x)
    return value;
  value = value + integral (mu_nu_prime, last_x, x, RECTS);
  last_x = x;
  return value;
}

int
main()
{
  bounding_box (P(0,0), P(2*M_PI,2*M_PI));
  picture (P(400, 400));
  unitlength ("1pt");

  begin ();

  line (P(0,y_max),P(x_max,y_max));
  line (P(x_max,0),P(x_max,y_max));

  line (P(0,0),P(x_max,0));
  line (P(0,0),P(0,y_max));
  h_axis (P(x_min,0), P(floor(x_max),0), 12);
  v_axis (P(0, y_min), P(0, floor(y_max)), 12);
  h_axis_labels (P(0,0), P(floor(x_max),0), 6, P(0,-10), c);
  v_axis_labels (P(0,0), P(0,floor(y_max)), 6, P(-4,0), l);

  dotted ();
  nu = 1;
  value = 0.0;
  last_x = 0.0;
  clipplot (mu_nu, x_min, x_max, 100);

  line (P(5,1.3), P(5.3,1.3));
```

```
  label (P(5.4,1), P(0,0), "$\\mu_1(x)$", r);

  dashed ();
  nu = 2;
  value = 0.0;
  last_x = 0.0;
  clipplot (mu_nu, x_min, x_max, 100);

  line (P(5,1), P(5.3,1));
  label (P(5.4,1), P(0,0), "$\\mu_2(x)$", r);

  solid ();
  nu = 3;
  value = 0.0;
  last_x = 0.0;
  clipplot (mu_nu, x_min, x_max, 200);

  line (P(5,0.7), P(5.3,0.7));
  label (P(5.4,0.7), P(0,0), "$\\mu_3(x)$", r);

  bold ();
  nu = 20;
  value = 0.0;
  last_x = 0.0;
  clipplot (mu_nu, x_min, x_max, 200);

  line (P(5,0.4), P(5.3,0.4));
  label (P(5.4,0.4), P(0,0), "$\\mu_{20}(x)$", r);

  end ();

  return 0;
}
```

B.4 Computer Code to Produce Figure 5.2

To produce Figure 5.2 on page 55 we must do a similar plot as for Figure 2.1, but with the difference that we use the

$$\mu_{20}(x) = \int_0^x \prod_{i=1}^{20} (1 + \lambda \cos n_i x) dx,$$

as our μ function. The parameters for μ_{20} are set the same as for Figure 5.1.

Unfortunately since too many rectangles are used to plot this extension, and this would make LaTeX end with an error, the code had to be modified to be more optimal with drawing lines. So care is taken to not draw lines twice over each other. Also since at some points the distortion of the μ_{20} function made the extension look too "blocky" since only the endpoints of the rectangles are calculated, the code is modified to add two more points if the rectangle width is too big.

Also the function implementation as computed for Figure 5.1 could not be used directly, since recomputing the integral for each point takes too long. The first optimization that is done is to notice that $\mu(x+2\pi) = \mu(x)+2\pi$, and so we only need to compute the μ on the interval $[0, 2\pi]$. Furthermore the code first computes the function value for 10000 different equally spaced points, and then when it needs to compute it for a specific point it picks the closest point from the left and computes the integral just from this point to the x that we want to use. For the initial precomputation, the midpoint rule with 100 rectangles is used for each of the 10000 intervals and for the final integral, the midpoint rule with 10 rectangles is used. Then for the Beurling-Ahlfors extension integrals we use the midpoint rule with 100 rectangles. Even with these optimizations the graph of the extension takes quite long to produce. It is my opinion that the computation time could be further reduced by proper "tuning" of the parameters and using better numerical approximation methods for the integrals. Also a 4 times speedup could be achieved by just optimizing the plotting routine which in fact calculates each point 4 times.

Similarly as before, when this code is run with PLOTGRID environment variable set, just the grid will be produced. What follows is the C++ code used by ePix for producing Figure 5.2.

```
#include "epix.h"
using namespace std;
using namespace ePiX;

#define SIZE 2
#define LABELGRID 0.5
#define GRID 0.075
#define NU 20
#define RECTS 100
```

```
#define LAMBDA 0.5

static double
integral (double (* func) (double), double a, double b, int p)
{
  double sum = 0.0;
  double len = b-a;
  double em = len/p;
  int i;

  if (len == 0.0)
    return 0.0;

  for (i = 0; i < p; i++) {
    sum += (func (a + (len*i)/p + em/2) * em);
  }

  return sum;
}

static int n[] = {1,2,5,11,23,47,95,191,383,767,1535,3071,6143,12287,
                  24575,49151,98303,196607,393215,786431};

static double
mu_prime (double x)
{
  int i;
  double prod = 1.0;

  for (i = 0; i < NU; i++) {
    prod *= 1+LAMBDA*std::cos(n[i]*x);
  }

  return prod;
}

#define CACHE_SIZE 10000

static int cache_inited = 0;
static double fcache[CACHE_SIZE];

static void
init_cache (void)
```

```
{
  int i;
  double x, lx;
  fcache[0] = 0.0;
  lx = 0.0;
  for (i = 1; i < CACHE_SIZE; i++) {
    x = (i*2*M_PI)/CACHE_SIZE;
    fcache[i] = integral (mu_prime, lx, x, RECTS) + fcache[i-1];
    lx = x;
  }
  cache_inited = 1;
}

static double
mu (double x)
{
  int i = 0;
  int c;
  if ( ! cache_inited)
    init_cache ();
  while (x > 2*M_PI) {
    i++;
    x -= 2*M_PI;
  }
  while (x < 0.0) {
    i--;
    x += 2*M_PI;
  }
  c = (int)((x*CACHE_SIZE)/(2*M_PI));

  return fcache[c] + integral (mu_prime, (c*2*M_PI)/CACHE_SIZE, x, 10)
                   + i*2*M_PI;
}

static void
function (double x, double y, double *ox, double *oy)
{
  if (y == 0.0) {
    *ox = mu (x);
    *oy = 0.0;
  } else {
    *ox = (1 / (2*y)) * integral (mu, x-y, x+y, 100);
    *oy = (1 / (2*y)) * (integral (mu, x, x+y, 100) -
```

```
                         integral (mu, x-y, x, 100));
  }
}

static int
out_of_bounds (double x, double y)
{
  return (x < - SIZE || x > SIZE || y < 0.0 || y > SIZE);
}

int
main()
{
  double x,y;

  bounding_box (P(-SIZE,0), P(SIZE, SIZE));
  picture (P(300, 150));
  unitlength ("1pt");

  begin ();

  grid (2*SIZE/LABELGRID, SIZE/LABELGRID);
  h_axis (P(-SIZE,0), P(SIZE,0), 2*SIZE/LABELGRID);
  v_axis (P(0,0), P(0,SIZE), SIZE/LABELGRID);
  h_axis_labels (P(-SIZE,0), P(SIZE,0), 2*SIZE/LABELGRID, P(0,-10), c);
  v_axis_labels (P(0,LABELGRID), P(0,SIZE),
                 SIZE/LABELGRID-1, P(-4,0), l);

  for (x = -SIZE; x <= SIZE; x += GRID) {
    int firstrow = 1;
    for (y = 0; y <= SIZE; y += GRID) {
      double f1x, f1y;
      double f2x, f2y;
      double f3x, f3y;
      double f4x, f4y;
      function (x, y, &f1x, &f1y);
      function (x + GRID, y, &f2x, &f2y);
      function (x + GRID, y + GRID, &f3x, &f3y);
      function (x, y + GRID, &f4x, &f4y);
      if ( ! out_of_bounds (f1x, f1y) &&
           ! out_of_bounds (f4x, f4y) &&
          (out_of_bounds (f2x, f2y) ||
           out_of_bounds (f3x, f3y))) {
```

```
        if (getenv("PLOTGRID")) {
          line(P(x,y+GRID), P(x,y));
        } else {
          line(P(f4x,f4y), P(f1x,f1y));
        }
      }
      if (out_of_bounds (f1x, f1y) ||
          out_of_bounds (f2x, f2y) ||
          out_of_bounds (f3x, f3y) ||
          out_of_bounds (f4x, f4y))
        continue;
      // Switch for pre/post function grid
      if (getenv("PLOTGRID")) {
        if (firstrow)
          line(P(x,y), P(x+GRID,y));
        line(P(x+GRID,y+GRID), P(x,y+GRID));
        line(P(x,y+GRID), P(x,y));
      } else {
        if (firstrow)
          line(P(f1x,f1y), P(f2x,f2y));
        if (abs(f3x-f4x) > 0.15) {
          double ix, iy, iix, iiy;
          function (x + GRID/3.0, y + GRID, &iix, &iiy);
          function (x + 2.0*GRID/3.0, y + GRID, &ix, &iy);
          line(P(f3x,f3y), P(ix,iy));
          line(P(ix,iy), P(iix,iiy));
          line(P(iix,iiy), P(f4x,f4y));
        } else {
          line(P(f3x,f3y), P(f4x,f4y));
        }
        line(P(f4x,f4y), P(f1x,f1y));
      }
      firstrow = 0;
    }
  }

  end ();

  return 0;
}
```

ABSTRACT

ABSTRACT

Suppose that we have a homeomorphism of the real line onto itself, when and how can we extend this to a quasiconformal homeomorphism of the upper half plane to itself? In general we can consider any singly connected domain. First I will give a theorem by Beurling and Ahlfors that there exists a quasiconformal extension if and only if the boundary homeomorphism is a quasisymmetric mapping, that is when the boundary homeomorphism is the equivalent of a quasiconformal map in one dimension. The sufficiency part of the theorem is proved by an explicit construction of an extension. Later Douady and Earle proved, again by explicit construction, that we can in fact make a conformally natural extension. That is we can compose the boundary homeomorphism with two conformal automorphisms and then extend or first extend and then compose and we will in fact end up with the same map. We can then compare these two constructions and look at some applications.